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ON THE SEIDEL – NEWTON METHOD FOR SOLVING QUASILINEAR OPERATOR EQUATIONS

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1. INTRODUCTION

Various differential and integral equations can be reduced to the following operator form :

$$Ax + F(x) = 0, \tag{1.1}$$

where $A \in L(X, Y)$ is a bounded linear Fredholm operator (of index zero); $F: X \to Y$ is a nonlinear operator, and X and Y are two Banach spaces.

By using the degree theory, one can obtain existence theorems for the equations (1.1) (see [1,3]). This equation may be solved by projection methods [4] or by a special iterative method [5].

Another iterative method, combining Seidel's and Newton's methods is given in [6]. For a discussion of the Seidel — Newton method see also [10].

Since the operator A is a Fredholm operator we can write X and Y as direct sums :

$X = X_1 \oplus X_2$; $Y = Y_1 \oplus Y_2$, where $X_2 = Ker\ A$ and $Y_1 = R(A)$

It is well-known (see [1,2]) that X_2 has finite dimension, Y_1 is closed, $\dim X_2 = \operatorname{codim} Y_1 = m < +\infty$, and the restriction $\widehat{A}$ of A to X_1 has a bounded inverse.

Denote by P the bounded linear projection, satisfying $PY = Y_1$; $PY_2 = 0$. We shall solve (1.1) by the Seidel — Newton method, i. e. : Knowing the n — th approximation (the o—th approximation x_0 is supposed to be given), we construct the (n + 1)—th approximation by the formulas :

$$U_{n+1} = -\widehat{A}^{-1} PF(x_n) , \tag{1.2a}$$

$$\tilde{x}_n = u_{n+1} + v_n , \tag{1.2b}$$

$$v_{n+1} = v_n - [QF'(\tilde{x}_n)]^{-1}_{X_2} QF(\tilde{x}_n), \tag{1.2c}$$

$$x_{n+1} = u_{n+1} + v_{n+1}, \tag{1.2d}$$

where $Q = I - P$, I — the identity operator in Y, $U_{n+1} \in X_1$ and $v_{n+1} \in X_2$

Thus, instead of finding the exact solution of the infinite-dimensional nonlinear equation (1.1), in each step we have to solve the linear equation for $u_{n+1} \in X_1$ and the linear finite — dimensional equation for $v_{n+1} \in X_2$

The limiting cases of (1.2a — 1.2d) are: $X = X_2$ $(A = 0)$, and $X = X_1$ (A is invertible) can be considered by Newton's and Picard's methods, respectively.

2. CONVERGENCE THEOREMS

THEOREM 2.1. *Let F be continuously differentiable (in the Fréchet sense) in an open set, including the closed ball S with center at $\mathbf{x}_0$ and radius $\mathbf{r} > 0$, and for all $\mathbf{x}, \mathbf{y} \in S$;*
$\| PF'(x) \| \leqslant \alpha$; $\| QF'(x) \| \leqslant \beta$; $\| QF'(x) - QF'(y) \| \leqslant \rho(\| x - y \|)$,
where $\rho : [0, \infty) \to [0, \infty)$ is a continuous, nondecreasing function, and $\rho(0) = 0$.

Further, suppose that the restriction of the derivative $QF'(x)$ to X_2 has a uniformly bounded inverse:

$$\| [QF'(x)]^{-1}_{X_2} \| \leqslant \gamma (x \in S).$$

If α is sufficiently small and the initial approximation x_0 is good enough, so that:

$$q = 2\alpha\beta\gamma \| \widehat{A}^{-1} \| + \gamma \int_0^1 \rho(\delta t) dt < 1, \tag{2.1}$$

$$2\delta(1 - q)^{-1} < r, \tag{2.2}$$

where δ is defined by the formula:

$$\delta = \beta\gamma \| \widehat{A}^{-1} \| \; \| Ax_0 + PF(x_0) \| + \gamma \| QF(x_0) \|, \tag{2.3}$$

then the sequence $\{x_n\}$, constructed accordings to (1.2a)—(1.2d) converges to a solution x^* of (1.1) and

$$\| x_n - x^* \| \leqslant rq^n \; (n \geqslant 0). \tag{2.4}$$

This theorem is a special case of Theorem 5.1, which will be discussed in §5.

Remark 2.1. Theorem 2.1 remains valid, when $r = +\infty$ (in this case, (2.2) automatically hold), and we have

$$\| x_n - x^* \| = \leqslant 2\delta(1 - q)^{-1} q^n \; (n \geqslant 0). \tag{2.5}$$

Remark 2.2. Theorem 2.1 was proved in [6] under the assumption that $QF'(x)$ is Lipschitz continuous. Note that, Theorem 2.1 holds in the more general case, when $QF'(x)$ is Hölder continuous:

$$\| QF'(x) - QF'(y) \| \leqslant L \| x - y \|^{\omega} . \ (0 < \omega \leqslant 1)$$

In both cases, putting $\rho(t) = Lt^{\omega}$, we may apply Theorem 2.1.

From theorem 2.1, we obtain the following results:

COROLLARY 2.1. *Le E be continuously differentiable in an open set, including a closed ball S with center at x_0 and radius $r > 0$, and for all $x, y \in S$:*

$$\| PF'(x) \| \leqslant \alpha; \quad \| QF'(x) \| \leqslant \beta; \quad \| QF'(x) - QF'(y) \| \leqslant \rho(\| x - y \|),$$

where: $\rho : [0, \infty) \to [0, \infty)$ is a continuous, nondecreasing function, and $\rho(0) = 0$.

If $\gamma_0 \rho(r) < 1$ and (2.1), (2.2) hold, where

$\gamma = \gamma_0 (1 - \rho(r)\gamma_0)^{-1}$, $\| [QF'(x_0)]^{-1}_{X_2} \| \leqslant \gamma_0$ and δ is *defined by (2.3), then the* $\{x_n\}$ *converges to a solution x^* of (1.1) and the estimation (2.4) holds.*

COROLLARY 2.2. *Let F be twice continuously differentiable in the closed ball S with center x_0 and radius $r > 0$, and for every $x \in S$:*

$$\| PF'(x) \| \leqslant \alpha \quad ; \quad \| QF'(x) \| \leqslant \beta \quad ; \quad \| QF''(x) \| \leqslant L.$$

If $Lr\gamma_0 < 1$ where $\| [QF'(x_0)]^{-1}_{X_2} \| \leqslant \gamma_0$ and

(2.1), (2.2) hold with $\gamma = \gamma_0 (1 - Lr\gamma_0)^{-1}$ and δ is defined by (2.3), then the conclusion of theorem 2.1 holds.

We end this section with results on the local convergence of (1.2a — 1.2d).

THEOREM 2.2 ([6]). *Let F be continuously differentiable in an open neighborhood of a solution x^* of (1.1). Then the restriction of the derivative $QF'(x^*)$ to X_2* has a bounded inverse, and

$$2 \| PF'(x^*) \| \; \| QF(x^*) \| \; \| [QF'(x^*)]^{-1}_{X_2} \| \; \| \widehat{A}^{-1} \| < 1. \tag{2.6}$$

If the initial approximation x_0 is sufficiently close to x^*, then the sequence $\{x_n\}$, constructed by (1.2a — 1.2d) converges to x^* and the estimation:

$$\| x_n - x^* \| \leqslant cq^n$$

holds, where $c > 0$ and $q \in [0,1]$ are constants, independent of n.

COROLLARY 2.3. *Let F be continuously differentiable in an open neighborhood of a solution x^* of (1.1). If the restriction of the derivative $QF'(x^*)$ to X_2 has a bounded inverse, and $PF'(x^*) = 0$, then x^* is a point of attraction (see [7]) of (1.2a — 1.2d).*

3. THE HILBERT SPACE CASE.

In this section, the previous results are applied to equations in a Hilbert space.

Let us consider a nonlinear equation:

$$x = Kx + F(x) \tag{3.1}$$

in a real separable Hilbert space H, where $K: H \rightarrow H$ is a linear, self—adjoint and completely continuous operator, and $E: H \rightarrow H$ is a non-linear operator.

By the Hilbert-Schmidt theorem, there is an orthonormal basic e_i of eigen-vectors of K in H. We may assume that, the corresponding eigenvalues satisfy

$$\lambda_1 = \lambda_2 = ... = \lambda_m = 1; \lambda_i \neq 1 \ (i > m).$$

According to Fredholm's theorem (see [2,8]) $A = I - K$ is a canonical Fredholm operator, and $m < + \infty$. Clearly, the Seidel-Newton method in this case may be written as follows:

$$u_{n+1} = \sum_{i>m} \frac{1}{1-\lambda_i}(F(x_n), l_i)l_i, \tag{3.2a}$$

$$x'_n = u_{n+1} + \sum_{i=1}^{m} \xi_i^{(n)} e_i, \tag{3.2b}$$

$$\sum_{i=1}^{m} (F(x'_n)e_{i,} e_j)(\xi_i^{n+1} - \xi_i^{(n)}) = -(F(x'_n)e_j), \tag{3.2c}$$

(j = 1, 2.,.,m)

$$x_{n+1} = u_{n-1} + \sum_{i+1}^{m} \xi_i^{(n+1)} e_i, \tag{3.2d}$$

Applying Theorem 2.1 to equation (3.1) yields the following:

THEOREM 3. 1: *Let F be continuously differentiable in open set, including the closed ball S (with center at x^0 and radius $r > 0$) Assume that for every $x \in S$, the matrix (a_{ij}), where $a_{ij} = F'(\,(x)\, e_i, l_j)$ (i, j = 1, 2,..., m), has a nonzero determinant; and that*

$$\frac{1}{d(x)} \left\{ \sum_{ij=1}^{m} | A_{ij}(x) |^2 \right\}^{1/2} \leqslant \gamma \qquad (x \in S), \tag{3.3}$$

where, *$A_{ij}(x)$ is the algebraic complement of a_{ij}. Further, assume that $\| F'(x(\| \leqslant \alpha$, $\| E'(x) - F'(y) \| \leqslant \rho(\| x - y \|)$ for all $x, y \in S$, where $\rho(t)$ is a nonnegative nondecreasing continuous function, and. $\rho(0) = 0$.*

If α, r and x^0 satisfly

$$q = 2\alpha^2 \gamma \omega + \gamma \int_0^1 \rho(\delta t) dt < 1; \ 2\delta(1 - q)^{-1} < r$$

where $\omega \geqslant \sup_{i>m} | 1 - \lambda_i |^{-1}$ *and* δ *is defined by*

$$\delta = \alpha\gamma\omega \| Ax_0 + F(x_0) - \sum_{i=1}^{m} (x_0), e_i) e_i , \| + \gamma \| \sum_{i=1}^{m} (F(x_0), e_i) e_i \| ,$$

then the sequence $\{ x_n \}$, *constructed by* (3. 2a — 3 . 2d) *converges to a solution* x^* *of* (3.1) *and* (2,4) *holds.*

4. EQUATIONS WITH A SMALL PARAMETER

The previous results can be applied to equations with a small parameter $\varepsilon > 0$:

$$Ax + \varepsilon F(x) = 0 \tag{4.1}$$

COROLLARY 4.1 *Let* F *be continuously differentiable in the closed ball* S *with center at* x_0 and *radius* $r > 0$, *and for every* $x,y \in S$:

$$\| F'(x) \| \leqslant \alpha ; \ \| QF'(x) - QF'(y) \| \leqslant \rho(\| x - y \|),$$

where ρ *is a non-negative, non-decreasing, continuous function, and* $\rho(0) = 0$.

Suppose that, the restriction of $QF'(x)$ to X_2 has a uniformly bounded inverse:

$$=[QF'(x)]^{-1}_{X_2} \| \leqslant \gamma \ (x \in S).$$

If the initial approximation x_0 satisfies the following conditions:

$$q_0 + \gamma \int_0^1 \rho(\delta_0 t)\, dt < 1 : 2\delta_0 (1-q_0)^{-1} < r,$$

where:

$$\delta_0 \ \| \ \alpha\gamma \| Q \| \ \| \ \widehat{A}^{-1} \| \ \| Ax_0 \| \ + \gamma \ \| \ QF(x_0) \| ,$$

then for a sufficiently small $\varepsilon > 0$, the sequence $\{ x_n \}$ constructed according to the formulas:

$$u_{n+1} = - \varepsilon \widehat{A}^{-1} PF(x_n), \tag{4.2a}$$

$$x'_n = u_{n+1} + v_n , \tag{4.2b}$$

$$v_{n+1} = V_n - \left[QF'(x'_n) \right]^{-1}_{X_2} QF(x'_n ,) \tag{4.2c}$$

$$x_{n+1} = U_{n+1} + V_{n+1} , \tag{4.2d}$$

converges to a solution $x^*(\varepsilon)$ of (4.1) and the estimation

$$\| x_n - x^*(\varepsilon) \| \leqslant C q^n (\varepsilon)$$

holds with $C > 0$ and $q(\varepsilon) \in (0, 1.)$

COROLLARY 4. 2. *Let F be continuously differentiable in an open set U, and the restriction of QF' (x) to X_2 has a uniformly bounded inverse:*

$$\| [QF'(x)]^{-1}_{x2} \| \leqslant \gamma \, ; \; \| F'(x) \| \leqslant \alpha \quad (x \in U).$$

If for every sufficiently small $\varepsilon > 0$, the equation (4.1) has a solution $x(\varepsilon) \in U$, then there is a number $\varepsilon_0 > 0$ such that, for any $\varepsilon \in (0, \varepsilon_0)$, $x(\varepsilon)$ is a point attraction of (4.2a — 4. 2d).

5. MODIFICATION OF THE SEIDEL — NETWON METHOD

Consider the following modification of the Seidel—Newton method

$$\hat{A} u_{n+1} = - PF(x_n), \qquad (5.1a)$$

$$x'_n = u_{n+1} + v_n, \qquad (5.1b)$$

$$v_{n+1} = v_n - \widehat{M}^{-1}(x'_n) QF(x'_n), \qquad (5.1c)$$

$$x_{n+1} = u_{n+1} + v_{n+1}, \qquad (5.1d)$$

where $M : X \to L(X, Y_2)$ and $\widehat{M}(x)$ is a restriction of $M(x)$ to X_2.

THEOREM 5. 2. *Assume that $F: X \to Y$ is continuously differentiable in an open set, including the closed ball, S with center at x_0 and radius $r > 0$, and for all $x, y \in S$:*

$$\| PF'(x) \| \leqslant \alpha \, ; \; \| QF'(\tau) \| \leqslant \beta \, ; \; \| QF'(x) - QF'(y) \| \leqslant \rho(\| x - y \|),$$

where $\rho : [0, \infty) \to [0, \infty)$ is a non-decreasing continuous function, and $\rho(0) = 0$.

Further, suppose that, the restriction $\widehat{M}(x)$ of $M(x)$ to X_2 has a uniformly bounded inverse:

$$\| \widehat{M}^{-1}(x) \| \leqslant \gamma \quad (x \in S)$$

and

$$\| M(x) - QF'(x) \| \leqslant \varepsilon \; (x \in S)$$

If α, ε and the initial approximation x_0 satisfy the following conditions:

$$q = 2\alpha(\varepsilon + \beta)\gamma \| \widehat{A}^{-1} \| + \gamma\left(\varepsilon + \int_0^1 \rho(\delta t)\, dt\right) < 1,$$

$$2\delta (1 - q)^{-1} < \gamma$$

where

$$\delta = (\varepsilon + \beta)\gamma \| \widehat{A}^{-1} \| \, \| Ax_0 + PF(x_0) \| + \gamma \| QF(x_0) \|,$$

then the sequence $\{x_n\}$ *, constructed by (5.1a — 5.1d) converge to a solution of (1.1) and the estimation (2.4) holds.*

Proof: For $n \geqslant 0$, let us denote:

$h_n = x_{n+1} - x_n$; $\lambda_n = u_{n+1} - u_n = x'_n - x_n$; $\mu_n = v_{n+1} - v_n = x_{n+1} - x'_n$

Assume that x_n, $x'_n \in S$ for every $n \geqslant 0$. Then

$$\lambda_n = -\widehat{A}^{-1}\left(\int_0^1 PF'(x_{n-1} + th_{n-1})\, h_{n-1}\, dt\right),$$

hence,

$$\|\lambda_n\| \leqslant \alpha \|\widehat{A}^{-1}\| (\|\lambda_{n-1}\| + \|\mu_{n-1}\|). \tag{5.2}$$

Further, $\|\mu_n\| = \|\widehat{M}^{-1}(x'_n) QF(x'_n)\| \leqslant \gamma \|QF(x_n)\| \leqslant$

$\leqslant \gamma \|QF(x_n)\| + \beta\gamma \|\lambda_n\|$.

But

$\|QF(x_n)\| = \|QF(x_n) - QF(x'_{n-1}) - M(x'_{n-1})\mu_{n-1}\| =$

$= \|\int_0^1 QF'(x'_{n-1} + t\mu_{n-1})\mu_{n-1}\, dt - M(x'_{n-1})\mu'_{n-1}\| \leqslant$

$\leqslant \int_0^1 \|QF'(x'_{n-1} + t\mu_{n-1}(-QF')x'_{n-1})\| \; \|\mu_{n-1}\|\, dt +$

$+ \|QF'(x'_{n-1}) - M(x'_{n-1})\| \; \|\mu_{n-1}\| \leqslant \left\{\int_0^1 \rho(t\|\mu_{n-1}\|)\, dt + \varepsilon\right\} \|\mu_{n-1}\|$.

Therefore

$$\|\mu_n\| \leqslant \beta\gamma \|\lambda_n\| + \gamma\left\{\int_0^1 \rho(t\|\mu_{n-1}\|)\, dt + \varepsilon\right\} \|\mu_{n-1}\| \tag{5.3}$$

Next we prove by induction the following relations:

$$x_n, x'_n \in S; \; \|\lambda_n\| \leqslant \delta q^n; \; \|\mu_n\| \leqslant \delta q^n (n \geqslant 0). \tag{5.4}$$

By assumption, $x_o \in S$. Observe that,

$$1 = \|\widehat{M}^{-1}(x_o)\widehat{M}(x_o)\| \leqslant \gamma \|\widehat{M}(x_o)\| \leqslant \gamma \|M(x_o)\| \leqslant$$

$$\leqslant \gamma(\|M(x_o) - QF'(x_o)\| + \|QF'(x_o)\|) \leqslant \gamma(\varepsilon + \beta)$$

Then we have:

$\|\lambda_o\| \leqslant \|\widehat{A}^{-1}\| \; \|Ax_o + PF(x_o)\| \leqslant (\varepsilon + \beta)\, \gamma \|\widehat{A}^{-1}\| \; \|Ax_o + PF(x_o)\| < \delta$

Since $\| x'_o - x_o \| = \| \lambda_o \| < \delta < r$, it follows that $x'_o \in S$.

Furthermore, $\| \mu_o \| = \| v_1 - v_o \| = \| \widehat{M}^{-1}(x'_o) QF(x'_o) \| \leqslant \gamma \| QF(x'_o) \| \leqslant$

$\leqslant \gamma \| QF(x_o) \| + \beta\gamma \| x_o - x'_o \| \leqslant (\varepsilon + \beta)\gamma \| \widehat{A}^{-1} \| \, \| Ax_o + PF(x_o) \| +$

$\gamma \| QF(x_o) \| = \delta$.

Now assume that $x_k, x'_k \in S$ and that (5.4) holds f $k \leqslant n$,

$$\text{Then } \| x_{n+1} - x_o \| \leqslant \sum_{k=o}^{n} \| x_{k+1} - x_k \| \leqslant \sum_{k=o}^{n} (\| \lambda_k \| + \| \mu_k \|) \leqslant$$

$$\leqslant 2\delta \sum_{k=o}^{n} q^k < 2\delta(1-q)^{-1} < r,$$

which shows that $x_{n+1} \in S$.

Since $x_n, x'_n \in S$, we have from (5.2):

$$\| \lambda_{n+1} \| \leqslant 2\alpha \| \widehat{A}^{-1} \| \, \delta q^n \leqslant 2\alpha(\varepsilon + \beta)\gamma \| \widehat{A}^{-1} \| \, \delta q^n < \delta q^{n+1}$$

$$\| x'_{n+1} - x_o \| \leqslant \| x'_{n+1} - x_{n+1} \| + \| x_{n+1} - x_o \| \leqslant$$

$$\delta q^{n+1} + 2\delta (1 + q + \dots + q^n) < 2\sigma (1 - q)^{-1} < r.$$

It follows from the induction assumption and the monotonity of ρ (t) that

$$\| \mu_{n+1} \| \leqslant \beta\gamma \| \lambda_{n+1} \| + \gamma \left\{ \int_0^1 \rho (t \| \mu_n \|) \, dt + \varepsilon \right\} \| \mu_n \| \leqslant$$

$$\leqslant 2\alpha (\varepsilon + \beta) \gamma \| \widehat{A}^{-1} \| \, \delta q^n + \gamma \left\{ \int_0^1 \rho (t \, \delta q^n) \, dt + \varepsilon \right\} \delta q^n <$$

$$< \delta q^n \left\{ 2\alpha (\varepsilon + \beta) \gamma \| \widehat{A}^{-n} \| + \gamma \left[\int_0^1 \rho (t \, \delta^n) \, dt + \varepsilon \right] \right\} = \delta q^{n+1}.$$

Hence $x_n, x'_n \in S$ for all $n \geqslant 0$, and so (5.4) holds. Note that (5.4) implies

$$\| x_{n+m} - x_n \| \leqslant \| x_{n+m} - x_{n+m-1} \| +$$
$$\| x_{n+m-1} - x_{n+m-2} \| + \dots + \| x_{n+1} - x_n \| \leqslant$$
$$\leqslant 2\delta (q^{n+m-1} + \dots + q^n) < 2\delta (1-q)^{-1} q^n < r q^n. \tag{5.5}$$

Hence $\{ x_n \}$ is a Cauchy sequence. Let x^* be a limit of $\{ x_n \}$. Clearly, x^* is a solution of (1.1). The error estimation (2, 4) follows (5.5) when $m \to \infty$.

Note that Theorem 2. 1 is a special case of Theorem 5. 1 when $\varepsilon = 0$ $(M(x) = QF'(x)$.

We next consider the simplified Seidel-Newton method :

$$\widehat{A} u_{n+1} = - PF(x_n) \quad (5.6a)$$

$$x' = u_{n+1} + v_n \quad (5.6b)$$

$$v_{n+1} = v_n - [QF'(x_0)]^{-1}_{X_2} QF(x'_n) \quad (5.6c)$$

$$x_{n+1} = u_{n+1} + v_{n+1} \quad (5.6d)$$

THEOREM 5.2. *Let F be continuously differentiable in an open set, including the closed ball S with center at x_0 and radius $r > 0$, and for all $x, y \in S$*

$$\| PF'(x) \| \leqslant \alpha ; \| QF'(x) \| \leqslant \beta ; \| QF'(x) - QF'(y) \| \leqslant \varepsilon \| x - y \|^{\omega} (0 < \omega \leqslant 1).$$

If $\quad \| [QF'(x_0)]^{-1}_{X_2} \| \leqslant \gamma_0 \quad$ and

$$q = 2\alpha(3r^{\omega} + \beta)\gamma_0 \| \widehat{A}^{-1} \| + \gamma_0 \varepsilon (r^{\omega} + \delta^{\omega}/(1+\omega)) < 1,$$

$$2\delta(1-q)^{-1} < r$$

where $\quad \delta = (3r^{\omega} + \beta)\gamma_0 \| \widehat{A}^{-1} \| \; \| Ax_0 + PFx_0) \| + \gamma_0 \| QF(x_0) \|$,
Then the sequence $\{x_n\}$, *constructed by (5.6a — 5.6d) converges to a solution x^* of (1.1) and (2.4) holds.*

This theorem follows directly from Theorem 5. *1*, if we put $M(x) = QF'(x_0)$ and $\rho(t = \varepsilon t^{\omega})$The above observation suggests that we may consider equation (*1.1*) eve when F is not differetiabie. In that case, we use the following algorithm :

$$\widehat{A} u_{n+1} + PF(x_n) = 0 \quad (5.7a)$$

$$x'_n = u_{n+1} + v_n \quad (5.7b)$$

$$v_{n+1} = v_n - [QG'(x'_n)]^{-1}_{X2} QF(x'_n) \quad (5.7c)$$

$$x_{n+1} = u_{n+1} + v_{n+1} \quad (5.7d)$$

where G is a continuously differentiable operator and GF approximates QE.

The proof of Theorem 5.1 can easily be modified to yield the following result :

THEOREM 5.5 *Let the operator G be continuously differentiable in the closed ball S with center at x_0 and radius $r < 0$, and for all $x, y \in S$:*

$$\| PF(x) - PF(y) \| \leqslant \alpha \| x - y \| ; \quad \| QG'(x) - QG'(y) \| \leqslant \rho(\| x - y \|)$$

$$\| QG_1(x) - QG_1(y) \| \leqslant \varepsilon \| x - y \|$$

where ρ is a non-negative non-decreasing continuous function $\rho(o) = 0$ and

$$G_1 = F - G.$$

Further, suppose that $\| QG'(x) \| \leqslant \beta$ and the restriction of $QE'(x)$ to X_2 has a uniformly bounded inverse.

$$\| [Q\,G'(x)]_{X_2}^{-1} \| \leqslant \gamma \ (x \in S)$$

If α and ε are sufficiently small and x_0 is good enough, so that:

$$q = 2\alpha\beta\gamma \| \widehat{A}^{-1} \| + \gamma \Big\} \varepsilon + \int_0^1 \rho(6t)\,dt \Big\} < 1 ; \ 26(1 - q)^{-1} < r$$

where $\delta = \beta\gamma \| \widehat{A}^{-1} \| \, \| Ax_0 + PF(x_0) \| + \gamma \| QF(x_0) \|$,
then the sequence $\{x_n\}$, constructed dy (5. 7a—5. 7b) is convergent and (2.4) holds.

6. PERIODIC BOUNDARY-VALUE PROBLEMS

Consider the following periodic boundary-value problem:

$$x^n + f(t, x, \dot{x}, \dots, x^{(n)}) = 0 \qquad (0 < t < 1) \tag{6.1a}$$

$$x^{(j)}(0) = x^{(j)}(1) \ (j = 0, 1, 2, \dots, n - 1) \tag{6.1b}$$

Problem (6.1a — 6.1b) may be reduced to the form (1.1) by introducing the following spaces and operators:

$$X = \{x \in C^n [0,1] : x^{(j)}(0) = x^{(j)}(1) \ (j = 0, 1, \dots, n - 1)\}$$

$$\| x \| = \sum_{i=0}^{n} \max_{0 \leqslant t \leqslant 1} | x^{(j)}(t) | \ ; \ Y = C[0,1] ; \ \| y \| = \max_{0 \leqslant t \leqslant 1} | y(t) | .$$

$$X_1 = \Big\} x \in X : \int_0^1 x(s)\,ds = 0 \Big\} \quad Y_1 = \Big\} y \in Y : \int_0^1 y(s)\,ds = 0 \Big\}$$

$$X_2 = Y_2 = \{ const \} \ ; \ Ax = x^{(n)} \ ; \ F(x) = f(t, x, \dot{x}, \dots, x^{(n)}).$$

LEMMA 6.1. *$A : X \to Y$ is a bounded linear Fredholm operator with $Ker A = X_2$, $R(A) = Y_1$ and $X = X_1 \oplus X_2$; $Y = Y_1 \otimes Y_2$. Moreover the restriction $\widehat{A}$ of A to X_1 has a bounded inverse;*

$$\| \widehat{A}^{-1} \| \leqslant \omega \equiv 1 + \sum_{k=1}^{n} \left\{ \max_{0 \leqslant t \leqslant 1} \int_0^1 \left| \frac{\partial^k G(t,s)}{\partial t^k} \right| ds \right\}$$

where $G(t, s)$ is the Green's of the following problem:

$$\begin{cases} W^{(n+1)} = 0 \\ W(0) = W(1) = 0 \\ W^{(j)} = W^{(j)}(1) \quad (j = 1, 2, \dots, n-1). \end{cases} \tag{6.2}$$

Set $Qy = \int_0^1 y(s)\,ds$; $Py = y - Q y$. Clearly P and Q are bounded linear projectors $P: Y \to Y_1$; $Q: Y \to Y_2$ and $\|P\| \leqslant 2$; $\|Q\| \leqslant 1$

LEMMA 6.2. *Suppose that the function $f(t, \xi_0, \xi_1, \dots, \xi_n)$ is continuous in t and continuously differentiable in the remaining variables, and that for all pairs $(t,\xi), (t, \xi') \in I$*

$I \equiv \{ t, \xi_0 \xi_1, \dots, \xi_n) : 0 \leqslant t \leqslant 1; |\xi_i| \leqslant r \,(i = 0,1,\dots, n)\}$

$$\left| \frac{\partial f}{\partial \xi_i} (t, \xi) - \frac{\partial f}{\partial \xi_i} (t, \xi') \right| \leqslant L \sum_{j=0}^{n} |\xi_j - \xi'_j|$$

$$\left| \frac{\partial f}{\partial \xi_i} (t, \xi) \right| \leqslant \alpha \quad (i = 0,1,\dots, n)$$

Further, assume that $\frac{\partial f}{\partial \xi_0} (t, \xi) \geqslant a(t)$ for every $(t, \xi) \in I$ where the function $a(t)$ is such that $\int_0^1 a(s)\,ds \equiv \gamma^{-1} > 0$ Then $F(x) = f(t, x\,\dot{x}, \dots, x^{(n)})$ is continuously differentiable in the closed ball $S \equiv \{ x \in X : |||x||| \leqslant r \}$ and for all $x,y \in S$ $\|PF'(x)\| \leqslant 2\alpha$; $\|QF'(x)\| \leqslant \alpha$; $\|QF'(x) - QF'(y)\| \leqslant L\, |||x - y|||$ Moreover, the restriction of $QF(x)$ to X_2 has a uniformly bounded inverse:

$$\| [QF'(x)]_{X2}^{-1} \| \leqslant \gamma \quad (x \in S)$$

The proofs of Lemmas 6.1, 6.2 for the case $n = 2$ may be found in [6]. Using Lemmas 6.1, 6.2 and Theorem 2.1, we can now prove the following:

THEOREM 6.1. *Suppose that the conditions of Lemma 6.2 hold. If*

$$q = 4\alpha^2 \gamma\omega + L\gamma\delta / 2 < 1; \qquad 2\delta(1 - q)^{-1} < r - |||x_0|||$$

where $\delta = \alpha\gamma\omega \max_t | x_0^{(n)} + f(t, x_0, \ldots, x_0^{(n)}) - \int_0^1 f(s, x_0(s), \ldots, x_0^{(n)}(s)) \, ds | +$

$$+ \gamma \left| \int_0^1 f(s, x_0(s), \ldots, x_0^{(n)}) \, ds \right|$$

Then the sequence $\{x_k\}$, *constructed according to the formulas*

$$y_k(t) = \int_0^1 f(s, x_k, \dot{x}_k, \ldots, x_k^{(n)}) \, ds - f(t) \, x_k, \dot{x}_k, \ldots, x_k^{(n)})$$

$$u_{k+1}(t) = \int_0^1 \frac{\partial G}{\partial t}(ts) \, y_k(s) \, ds$$

$$v_{k+1} = v_k - \frac{\int_0^1 f(s, u_{k+1} + v_k, \dot{u}_{k+1}, \ldots, u_{k+1}^{(n)}) \, ds}{\int_0^1 \frac{\partial f}{\partial \xi_0}(s, u_{k+1} + v_k, \dot{u}_{k+1}, \ldots u_{k+1}^{(n)}) \, ds}$$

$$x_{k+1}(t) = u(t) + v_{k+1}$$

converges to a solution of (6. 1a — 6. 1b), and the estimation (2.4) holds.

7. NON-LINEAR NEUMANN PROBLEMS

Consider the non-linear Neumann problem:

$$\Delta u = f(x, u, D^\alpha u) \quad (x \in G) \tag{7.1a}$$

$$\frac{\partial u}{\partial n} = 0 \quad (x \in \sigma G) \tag{7.1b}$$

where $G \subset R^N$ is a bounded connected open set with as mooth boundary ∂G, and α is a multi-index $1 \leqslant |\alpha| \leqslant 2$.

By $|\cdot|$ we mean the absolute value or the Euclidean norm on R^m, for some $0 \leqslant m \leqslant N(N+3)/2$. Further, the inner product on R^m is denoted by $\xi \cdot \eta = \sum_{i=1}^{m} \xi_i \eta_i$, and the inner product on $L_2(G)$ by $(.,.)$.

Define

$$X = \{u \in W_2^2(G) : \frac{\partial u}{\partial n} = 0 \, (x \in \partial G)\}$$

$$\| u \| = \{ \int_G \sum_{|\alpha| \leq 2} | D_u^\alpha |^2 dx \}^{1/2} ; Y = L_2 (G)$$

$$\| y \| = (\int_G | y |^2 dx)^{1/2} ; X_1 = \{ u \in X : \int_G u (x) \, dx = 0 \}$$

$$Y_1 = \{ y \in Y : \int_G y (x) \, dx = 0 \} ; X_2 = Y_2 \equiv \{ const \}$$

$$Au = - \Delta u ; F (u) = f (x, u, D_u^\alpha)$$

Then the problem (7. 1a — 7. 1b) can be reduced to an operator equation of the form (1.1).

An interesting discussion on existence and uniqueness theorems for the linear Neumann problems is given in [9]. A proof of the following facts may be found in [6].

LEMMA 7.1. *$A : X \to Y$ is a bounded linear Fredholm operator with Ker $A = X_2$; $R(A) = Y_1$, and*

$$X = X_1 \oplus X_2 \; ; Y = Y_1 \oplus Y_2$$

Moreover, the restriction $\widehat{A}$ of A to X_1 has a bounded inverse with the norm β.

Set $Q_y = \int_G y (x) \, dx$; $Py = y - Qy$. Clearly, P and Q are linear bounded projectors $\| Q \| \leqslant \mu$; $\| P \| \leqslant 1 + \mu$ where $\mu = mes \, (G)$.

LEMMA 7. 2. *Suppose the function $f(x, u, \xi)$ satisfies the conditions:*

a) $f : G \times R^1 \times R^m \to R^1$; $f, \dfrac{\partial f}{\partial u}, \dfrac{\partial f}{\partial \xi}$

are jointly continuous for $x \in G$; $| u | < \infty$ *and* $| \xi | = (\sum_{i=1}^n | \xi_i |^2)^{1/2} < + \infty$

b) $\left| \dfrac{\partial f}{\partial u} (x, u, \xi) \right| \leqslant a$; $\left| \dfrac{\partial f}{\partial \xi} (x, u, \xi) \right| \leqslant a$

$$\left| \frac{\partial f}{\partial \xi} (x, u, \xi) - \frac{\partial f}{\partial \xi} (x, u' \xi') \right| \leqslant L (| u - u' |^2 + | \xi - \xi' |^2)^{1/2}$$

$$\left| \frac{\partial f}{\partial u} (x, u, \xi) - \frac{\partial f}{\partial u} (x, u' \xi') \right| \leqslant L (| u - u' |^2 + | \xi - \xi' |^2)^{1/2}$$

$\forall \, x \in G \; \forall \; | u |, | u' |, | \xi |, | \xi' | < \infty$

c) $\dfrac{\partial f}{\partial u} (x, u, \xi) \geqslant g(x) \quad \int_G g(x) \, dx = \gamma^{-1} > 0$

$(x \in G : \; | u | , \; | \xi | < \infty)$

Then the operator $F(u) = f(x, u, D_u^\alpha)$ is continuously differentiable on X, and for all $u, u' \in X$:

$$\| QF'(u) - QF'(u') \| \leqslant \sqrt{2\mu}\ L \| u - u' \|$$

$$\| F'(u) \| \leqslant \sqrt{2}\ a\ ; \quad \| [QF'(u)]_{X2}^{-1} \| \leqslant \gamma$$

From Lemmas 7. 1 — 7. 2, Theorem 2.1 and Remark 2.1 we get the following

THEOREM 7.1 *Assume that the conditions a) — c) hold.*

If $\quad q = 4a^2 \mu (1 + \mu) \beta\gamma + \sqrt{2\mu}\ \gamma L\delta/2 < 1$

where $\delta = \sqrt{2}\ \mu a \beta \gamma \| \Delta u_0 - f(x, u_0, D^\alpha u_0) + \int_G f(x, u_0, D^\alpha u_0)\, dx \| +$

$$+ \sqrt{\mu}\ \gamma \| \int_G f(x, u_0, D^\alpha u_0)\, dx \| .$$

then the sequence $\{u_k\}$, *constructed as follows*;

$$\begin{cases} \Delta w_{k+1} = f(x, u_k, D^\alpha u_k) - \int_\Omega f(x, u_k, D^\alpha u_k) dx \\ \dfrac{\partial w_{k+1}}{\partial n} = 0 \ (x \in \partial G) \end{cases}$$

$$v_{k+1} = v_k - \frac{\int_G f(x, w_{k+1} + v_k, D^\alpha w_{k+1})\, dx}{\int \frac{\partial f}{\partial u}(x, w_{k+1} + v_k, D^\alpha w_{k+1}\, dx)}$$

$$u_{k+1} = w_{k+1} + v_{k+1}$$

converges to a solution of (7.1a — 7.1b) and the error estimation (2.5) *holds.*

8. NON-LINEAR INTEGRAL EQUATIONS

We shall consider an integral equation of the form :

$$x(t) = \int_0^1 K(t,s)\, x(s)\, ds + \int_0^1 f(t, s, x(s))\, ds \tag{8.1}$$

where the kernel $K(t, s)$ satisfies the Hilbert — Schmidt condition:

$$K(t, s) = K(s, t) \qquad (s, t \in [0,1])$$

$$\int_0^1 \int_0^1 | K(t, s) |^2\, ds dt < +\infty$$

Set $Tx = \int_0^1 K(t, s)\, x(s)\, ds$; $\qquad F(x) = \int_0^1 f(t, s, x(s)\, ds$;

$H = L_2[0,1]$. Then T is a linear, self-adjoint and completely continuous operator and we can apply the results of Section 3.

Assume that, the eigenvectors $\{e_i\}_{i=1}^{\infty}$ of T form an orthonormal basic in $L_2[0,1]$ and the corresponding eigenvalues are such that :

$$\lambda_1 = \lambda_2 = \ldots = \lambda_m = 1 \; ; \; \lambda_i \neq 1 \; (i > m)$$

The general scheme (3 . 2a — 3 . 2d) leads to the following iterative process :

$$u_{n+1} = \int_0^1 K(t,s)\, u_{n+1}(s)\, ds + \int_0^1 f(t, s, x_n(s))\, ds -$$

$$- \sum_{i=1}^{m} \left\{ \int_0^1 \int_0^1 f(\tau, s, x_n(s))\, e_i(\tau)\, ds\, d\tau \right\} e_i(t) \qquad (8.2a)$$

$$x'_n = u_{n+1} + \sum_{i=1}^{m} \xi_i^{(n)} e_i \qquad (8.2b)$$

$$\sum_{i=1}^{m} \left\{ \int_0^1 \int_0^1 \frac{\partial f}{\partial x}(t, s, x'_n(s))\, e_j(t)\, ds\, dt \right\} \left(\xi_i^{(n+1)} - \xi_i^{(n)} \right) =$$

$$= - \int_0^1 \int_0^1 f(t, s, x'_n(S))\, e_j(t)\, ds\, dt \qquad (8.2c)$$

$$(j = 1, 2, \ldots, m)$$

$$x_{n+1}(t) = v_{n+1}(t) + \sum_{i=1}^{m} \xi_i^{(n+1)} e_i(t) \qquad (8.2d)$$

As an application of Theorem 3.1 to non-linear integral equations we can state :

THEOREM 8. 1 *Let $f(t, s\, x)$ be continuous in t, s and twice continuously differentiable in x, and for all $t, s \in [0,1]$ $|x| < \infty$: $\left| \frac{\partial f}{\partial x}(t, s, x) \right| \leqslant a(t,s)$;*

$$\left| \frac{\partial^2 f}{\partial x^2}(t, s, x) \right| \leqslant L$$

with $\alpha = \left\{ \int_0^1 \int_0^1 |a(t,s)|^2 ds\, dt \right\}^{1/2} < +\infty$

Further, suppose that the matrix

$$(a_{ij}) = \left(\int_0^1 \int_0^1 \frac{\partial f}{\partial x}(t, s, x_0(s) e_i(s))\, e_j(t)\, ds\, dt \right) \quad (i,j = 1,2,\ldots,m)$$

has non-zero determinant d with $\frac{1}{d}\left\{\sum_{ij=1}^{m} | A_{ij} |^2\right\}^{1/2} \leqslant \gamma_0$

where A_{ij} is the algebraic complement of a_{ij} ;

If there is a number $r>0$, such that $Lr\gamma_0<1$
$q = 2\alpha^2 \gamma \omega \delta + L \gamma \delta/2 < 1$; $2\delta (1-q)^{-1} < r$, where $\gamma = \gamma_0 (1 - L r \gamma_0)^{-1}$; $\omega \geqslant \sup_{i\geqslant m} | 1 - \lambda_i |^{-1}$, and $\delta = \alpha \gamma \omega \| Ax_0 + PF(x_0) \| + \gamma \| QF(x_0) \|$, then the sequence $\{x_n\}$ constructed according to (8.2a—8.2d) is convergent and the error estimation holds. (2.4)

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REFERENCES

1. L. Nirenberg. *Topics in non-linear functional analysis*, Russ.transl., Moscow, 1977.
2. S G. Krein, *Linear equations in Banach spaces*, Moscow 1971.
3. P. M. Fitzpactrick, *Existence results for equations involving noncompact perturbation of Fredholm mappings with applications to dfferential equations.* J. Math. Anal. App. 1978, 66, N⁰1, 151—178.
4. G.W. Reddien, *Approximation methods and alternative* problems, J. Math. Anal. App. 1977, 60, N⁰1, 139—149.
5. Pham Ky Anh, *On an approximation method for solving quasilinear operator equations*, Dopl. Akad. Nauk SSSR, 1980, 250, No2, 291—295, Soviet Math. Dokl,21, N⁰1 (1980) 75—79, (in Russian).
6. Pham Ky Anh, *Dissertation*, Voronez 1979, (in Russian).
7. J. M. Ortega, W.C. Fheinboldt, *Iterative solution of non-linear equations in several variables*, Acad. press, New York, 1970.
8. L. V. Kantorovich, G. P. Akilov, *Functional analysis*, 2nd Rev, ed. Moscow, 1977.
9. S. G. Mihlin, *Linear partial differentiat equations*, Wyss. Skola Moscow, 1977.
10. A. B. Pevnyl, *On the convergence of the Newton—Seidel method.* Izv. Vuzov Matematika, 1978, 187, N⁰1, 89—90.