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«TRACES» OF FUNCTIONS FROM SOBOLEV-ORLICZ CLASSES OF INFINITE ORDER.

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This paper is devoted to the theory of «traces» in the Sobolev — Orlicz classes of infinite order. The questions to be treated here are closely related here are closely related to the subject matter of the author's papers [1, 2, 3], where criteria were obtained for the nontriviality of Sobolev—Ozlicz classes and spaces of infinite order and the homogeneous Dirichlet problem was studied for the equations with arbitrary non-linearities

$$\sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} D^{\alpha} A_{\alpha} (x, D^{\gamma} u) = h(x), x \in \Omega, \ |\gamma| \leqslant |\alpha|, \tag{0.1}$$

$$D^{\omega} u \big|_{\partial\Omega} = 0, \ |\omega| = 0,1,... \tag{0.2}$$

In order to study the inhomogeneous Dirichlet problem, namely to solve equation (0.1) under the boundary conditions

$$D^{\omega} u \big|_{\partial\Omega} = f_{\omega}(x'), \ x' \in \partial\Omega, \ |\omega| = 0.1,... \tag{0.3}$$

we note the following.

As in the case of equations of finite order, it turns out that, in going over from the homogeneous boundary conditions (0.2)' to the inhomogeneous conditions (0.3), one does not encounter fundamental difficulties as long as there is available a method for silving the homogeneous problem, and the boundary values $f_{\omega}(x')$, $|\omega| = 0,1,...,$ can be extended into the interior of the domain as functions of the corresponding energy spaces. Thus the study of the inhomogeneous problem (0.1), (0.3) rests on the theory of traces of functions from the Sobolev—Orlicz classes of infinite order.

It is the aim of the present paper to discuss. We first give necessary and sufficient trace conditions for the case of an arbitrary domain $\Omega \wedge R^n$. These conditions, being universal, are not easily verifiable in our opinion. Therefore it seems natural to seek, in conjunction with the trace criterion, sufficient but easily verifiable conditions on $f_\omega(x')$, $|\omega| = 0,1,\ldots$, under which there exists an extension from the Sobolev—Orlicz class of infinite order. This question will be solved for the case where $\dim \Omega = 1$.

Note that the method we shall use in this paper has been worked out by Dubinskii in [4] for the traces of functions from Sobolev spaces of infinite order.

1. TRACE CRITERION

In [1, 3] we obtained a criterion for nontriviality of the Sobolev—Orlicz classes

$$\overset{0}{W}{}^{\infty} \mathcal{L} \{\Phi_\alpha, \Omega\} \equiv \{ u(x) : u(x) \in \overset{\infty}{C_0} (\Omega),$$
$$\rho^{\infty}(u) \equiv \sum_{|\alpha|=0} \int_{\Omega} \Phi_\alpha (D^\alpha u(x))\, dx < +\infty\}$$

where $\Phi_\alpha : \mathbf{R}^1 \to \mathbf{R}^1$ are N—functions [6,7], α—multiindices of differentiation, $D^\alpha = \partial^{\alpha_1} / \partial x_1^{\alpha_1} \ldots \partial^{\alpha_n} / \partial x_n^{\alpha_n}$, $\Omega \subset \mathbf{R}^n$ $n \geqslant 1$. is a region whose boundary is denoted by $\partial\Omega$, $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$. In this section, assuming that the classes $\overset{0}{W}{}^{\infty} \mathcal{L} \{\Phi_\alpha, \Omega\}$ are nontrivial we consider the following question:

What must the $f_\omega(x')$, $x' \in \partial\Omega$, $|\omega| = 0,1,\ldots$ be in order that there exists a function $u(x) \ni C^\infty(\Omega)$, such that $D^\omega u \mid \partial\Omega = f_\omega(x')$, $|\omega| = 0,1,\ldots$, and, in addition, $\rho^\infty(u) < +\infty$? We say in this case that the functions $f_\omega(x')$, $|\omega| = 0,1,\ldots$ are the traces on $\partial\Omega$ of a function $u(x)$ from the class

$$W^\infty \mathcal{L}\{\varphi_\alpha, \Omega\} \equiv \{U(x) \in C^\infty(\Omega),$$

$$\rho^\infty(u) \equiv \sum_{|\alpha|=0}^{\infty} \int_{\Omega} \varphi_\alpha(D^\alpha u(x))\, dx < +\infty\},$$

and that $U(x)$ itself is an extension of a trace in the just mentioned class.

Before stating the trace criterion, let us introduce the following definition:

DEFINITION 1.1. [8] *The domain* $\Omega \subset \mathbf{R}^n$ *is called admissible if for any* $p \geqslant 1$ *the imbedding*

$$W_p^1(\Omega) \to L_{p_*}(\Omega),\ p^* = \frac{np}{(n-p)},$$

is valid.

We denote by $W_{\varphi_\alpha}^\alpha(\Omega)$ the following set of functions

$$W_{\varphi_\alpha}^\alpha(\Omega) \equiv \{u(x) : D^\alpha u \in \mathcal{L}\{\varphi_\alpha, \Omega\}\},$$

where
$$\mathcal{L}\{\varphi_\alpha, \Omega\} \equiv \{u(x) : \int_\Omega \varphi_\alpha(u(x))dx < +\infty\}.$$

Let N be an arbitrary positive integer. Suppose that for any N the boundary values $f_\omega(x')$, $|\omega| \leqslant N-1$ admit an extension into the interior of Ω from the class

$$W_N(\Omega) \equiv \bigcap_{|\alpha| \leqslant N} W_{\varphi_\alpha}^\alpha(\Omega),$$

i.e. there exists a function $u(x)$ satisfying the conditions

$$D^\omega u|_{\partial\Omega} = f_\omega(x'),\ x' \in \partial\Omega,\ |\omega| = 0,1,\ldots,$$

$$\rho^N(u) \equiv \sum_{|\alpha|=0}^{N} \rho(D^\alpha u, \varphi_\alpha) < +\infty.$$

THEOREM 1.1. *Let the domain* Ω *be admissible. A family of boundary functions* $f_\omega(x')$, $x' \in \partial\Omega$, $|\omega| = 0,1,\ldots$ *is the trace of a function* $u(x) \in W^\infty \mathcal{L}\{\varphi_\alpha, \Omega\}$ *if and only if the following conditions are satisfied:*

1) *For any* $N = 1,2\ldots$ *the functions* $f_\omega(x')$, $x' \in \partial\Omega$, $|\omega| \leqslant N-1$ *admit an extension in the class,* $W_N(\Omega)$;

2) *There exists a sequence of extensions* $u_N \in W_N(\Omega)$ *such that*

$$\rho^N(u_N) \leqslant Const,\ for\ all\ N.$$

Proof. In fact, if a family of functions $f_\omega(x')$, $x' \in \partial\Omega$, $|\omega| = 0,1,\ldots$ is the trace of a function $u(x)$in $W^\infty \mathcal{L}\{\varphi_\alpha, \Omega\}$, then for any $N = 1,2,\ldots$ the system of functions $f_\omega(x')$, $x' \in \partial\Omega$, $|\omega| \leqslant N-1$, is the trace of $u(x)$ in $W_N(\Omega)$. In this connection, clearly

$$\rho^N(u_N) \leqslant \rho^N \leqslant \rho^\infty(u) < +\infty.$$

Conversely, if conditions 1) and 2) of the theorem are satisfied, then by imbedding W_N (Ω) in C^k (Ω) (k depends on N (theorem 3. 9, p. 71 [8])) and using the diagonal process we obtain that there exists, a subsequence of the sequence $U_N(x)$ which converges to a function $U(x) \in C^\infty(\Omega)$ locally uniformly together with all of its derivatives. It is evident that U(x) satisfies the conditions.

$$D^{\omega}_{U}\Big|_{\partial\Omega} = f_{\omega}(\mathbf{x}'), x' \in \partial\Omega, |\omega| = 0,1,...$$

Also, condition 2) implies that $\rho^\infty(U) \leqslant$ const, and hence $U(x)$ is the desired extension. The theorem is proved.

Remark. As noted before, the verification of the conditions of Theorem 1. 1. is difficult. Therefore in the seque. we shall seek very simple sufficient trace conditions more suitable for practical use in the case dim $\Omega = 1$, i. e $\Omega = (0, a) \subset \mathbf{R}^1$.

2. SUFFICIENT TRACE CONDITIONS.

We consider the following classes of functions defined on $\Omega = (o, a) \subset \mathbf{R}^1$.

$$\overset{0}{W}{}^\infty \mathcal{L}\{\varphi_n, (0, a)\} \equiv \{U(x) \in C^\infty_0 (0, a), \rho^\infty(U) < +\infty\}.$$

From the results of [1, 3] it follows that the class $\overset{0}{W}{}^\infty \mathcal{L}\{\varphi_n, (o, a)\}$ is nontrivial if and only if the sequence M_n $\{\varphi_n^{-1}$ n$(1/a)$, if $\varphi_n \not\equiv 0$; $+\infty$, if $\varphi_n \equiv 0\}$, $n = 0,1,...$ defines a nonquasianalytic Hadamard class (see [5]). This is equivalent to saying that the sequemce M_n satisfies the condition

$$\lim_{n\to\infty} M_n^{1/n} = +\infty, \quad \sum_{n=0}^{\infty} M_n^c \Big/ M_{n+1}^c < +\infty, \tag{2. 1}$$

where M_n is a convex regularization of the sequence M_n by means of logarithms [5]. Suppose

$\overset{0}{W}{}^\infty \mathcal{L}\{\varphi_n, (0, a)\}$ is nontrivial and consider the class;

$$\overset{0}{W}{}^\infty \mathcal{L}\{\varphi_n, (0, a)\} \equiv \{U(x) \in C^\infty (0, a), \rho^\infty(U) < +\infty\}.$$

Further let there be given two sequences of real numbers b_m, c_m, $m = 0,1,...$

We wish to find a function $U(x) \in W^{\infty} \mathcal{L}\{\varphi_n, (0, a)\}$ such that

$$D^m u(0) = b_m;\quad D^m u(a) = c_m,\ m = 0,1\ldots$$

Here we give simple sufficient conditions for the sequences b_m, c_m to be the trace of a function $u(x) \in W^{\infty} \mathcal{L}\{\varphi_n, (0, a)\}$. As will be seen below, it suffices in the connection to consider the case $c_m = 0$, i.e. to consider conditions on b_m under which a function $u(x) \in W^{\infty}\mathcal{L}\{\varphi_n, 0, a)\}$ can be found satisfying the conditions

$$D^m u(0) = b_m,\ D^m u(a) = 0,\ m = 0,1,\ldots \tag{2.2}$$

THEOREM 2. 1. *Let the numbers b_m be boundary values of a function $f(x)$ that is analytic in a neighborhood of zero, i. e. such that the series*

$$f(x) = \sum_{m=0}^{\infty} \frac{b_m}{m!} x^m$$

converges on some interval $[0, b] \subset [0, a]$. Then there exists a function $u(x) \in W^{\infty}\mathcal{L}\{\Phi_n, (0, a)\}$ satisfying condition (2.2).

Proof. For the construction of the desired function we use the following classical result [5], which is a corollary of Lemma 1.1 in [3].

LEMMA 2. 1. *Suppose M_n^C is a sequence of positive number a satisfying conditions* (2.1). *Then for any $b > 0$ and $q \in (0, 1)$ there exists a function $F(x) \in C^{\infty}(0, b)$ satisfying the conditions*

1) $F(0) = 1,\ D^k F(0) = 0,\ k = 0,1,\ldots$

2) $D^k F(b) = 0,\ k = 0,1,\ldots$

3) $\max\limits_{x \in (0, b)} | D^k F(x) | \leqslant q^k M_k^c$.

Let

$$u(x) = \begin{cases} F(x) f(x), & x \in [0, b], \\ 0, & x \in [b, a], \end{cases}$$

where $F(x)$ is defined as in Lemma 2.1 for $b < a$ and $q < 1/2$. Clearly, the function $u(x)$ satisfies the boundary conditions (2.2). We claim that $\rho^{\infty}(u) < +\infty$, i.e. that $u(x) \in W_1^{\infty} \mathcal{L}\{\varphi_n\}(0, a)\}$.

Indeed, using Leibniz's formula and the analyticity of the f(x), we get for any $x \in (0, b)$

$$|D^n u(x)| \leqslant (2L)^n \sum_{k=0}^{n} (n-k)!\, M_k^c\, (q/L)^k, \tag{2.3}$$

where L is some constant. Further, using the logarithmic convexity of the sequences M_n^c and $n!$, we get from (2.3) that

$$|D^n u(x)| \leqslant (2L)^n\, n! \sum_{k=0}^{n} (l_n)^k, \tag{2.4}$$

where $l_n = (M_n^c / n!)^{1/n} q/L$. We now show that

$l_n \to \infty$ if $n \to \infty$, i.e

$$\lim_{n\to\infty} \left(n! / M_n^c\right)^{1/n} = 0, \tag{2.5}$$

Indeed, for any N any $n > N$ and the followng inequality is valid:

$$\left(M_N^c\right)^{-1/n} = \left(M_N^c\right)^{-1/n} \left\{ \left[\frac{M_{n-1}^c}{M_n^c} \cdot \frac{M_{n-2}^c}{M_{n-1}^c} \cdots \frac{M_N^c}{M_{N+1}^c} \right]^{\frac{1}{n-N}} \right\}^{\frac{n}{n-N}} \leqslant$$

$$\leqslant \left(M_N^C\right)^{-1/n} \left[\sum_{k=N+1}^{n} \frac{M_{k-1}^c}{M_k^c} \cdot \frac{1}{n-N} \right]^{\frac{n-N}{n}} \leqslant$$

$$\leqslant \left(M_N^C\right)^{-1/n} \left(\varepsilon_N \frac{1}{(n-N)}\right)^{\frac{n-N}{n}} \tag{2.6}$$

where, using condition (2.1),

$$\varepsilon_N = \sum_{N=k+1} M_{k-1}^c \quad M_k^c \to 0,\ N \to \infty.$$

Hence by the Sterling's formula we obtain that

$$\left(\frac{n!}{M_N^c}\right)^{1/n} \leqslant \left[\frac{K}{M_N^c} \left(\frac{\varepsilon_N}{n-N}\right)^{n-N} \sqrt{2\Pi n} \right]^{1/n} \frac{n}{e} \leqslant$$

$$\leqslant \left(\frac{K.\sqrt{2\pi n}}{M_N^c}\right)^{1/n} \frac{n}{n-N} \left(\frac{n-N}{\varepsilon_N}\right)^{N/n} \frac{\varepsilon_N}{e}; \tag{2.7}$$

where K is a positive constant. It is obvious that for any fixed N with $n \to \infty$ we have:

$$\left(\frac{K\sqrt{2\pi n}}{K_N^c}\right)^{1/n} \frac{n}{n-N}\left(\frac{n-N}{\varepsilon_N}\right)^{\frac{N}{n}} \to 1$$

This means that for sufficently large n we get from (2.7) the inequality

$$\left(\frac{n!}{M_N^c}\right)^{1/n} \leqslant \frac{2\varepsilon_N}{e}. \tag{2.8}$$

By (2.5) the last inequality means

$$\left(\frac{n!}{M_n^c}\right)^{1/n} \to 0, \qquad n \to \infty$$

So the formula (2.5) is proved.

We now return to the inequality (2.4). Since $l_n \to \infty$ as $n \to \infty$, we have for large n and any $x \in (0,b)$

$$|D^n u(x)| \leqslant (2L)^n n! \sum_{k=0} (l_n)^k \leqslant (2L)^n n! \frac{l_n^{n+1}-1}{l_n - 1} \leqslant 2(2L)^n n!\ l_n^n \leqslant$$

$2(2q)_n M_n^c \leqslant q_0^n M_n^c$, where $q_0 < 1$ since $q < 1/2$. Hence

$$\rho^\infty(u) = \sum_{n=0}^\infty \int_0^a \Phi_n(D^n u(x))dx \leqslant K \sum_{n=0}^\infty \varphi_n(q_0^n M_n^c) \leqslant K \sum_{n=0}^\infty q_0^n\ 1/a < +\infty.$$

We have thus proved that $u(x) \in W^\infty \mathcal{L}\{\varphi_n, (0,a)\}$. Theorem 2.1 is proved.

We now consider the more general case. In order to state the theorem, we introduce the sequence of numbers

$$S_m = \sum_{k=0}^\infty M_{m+k}^c \Big/ M_{m+k+1}^c \tag{2.9}$$

Note that $S_m \to 0$ as $m \to \infty$ by virtue of (2.1).

THEOREM 2.2. *Suppose the sequence b_m satisfies the following condition:*

There exists a numbers $r < 0$ such that

$$\sum_{m=0}^\infty |b_m| max\left\{S_m \frac{r^m}{m!}, (M_m^c)^{-1}\right\} < +\infty. \tag{2.10}$$

Then there exists a function $u(x)$ in $W^\infty \mathcal{L}\{\Phi_n, (0,a)\}$ satisfying (2.2).

Proof. The proof consists in constructing the desired function $u(x) \in W^\infty \mathcal{L}\{\Phi_n, (0,a)\}$ and will be carried out in several steps:

1) We construct the basis functions $v_m(x)$ such that $v_m(x) \in C^{\infty}(0,b)$, where $b \leqslant a$, and

$$D^n v_m(0) = \delta_{nm}, n = 0,1,\ldots$$

where δ_{nm} is the Cronecker delta.

2) We put

$$v(x) = \sum_{m=0}^{\infty} b_m v_m(x)$$

(clearly $v(x)$ satisfies the boundary conditions $D^m v(0) = b_m$, $m = 0,1,\ldots$) and we establish that the boundary values $D^m v(b)$ satisfy conditions

$$| D^m v(b) | K^m m!$$

where K is a positive constant.

3) We construct the function $w(x)$ in $W^{\infty}\mathcal{L}\{\varphi_n, (b, a)\}$ such that (with the use of Theorem 2.1)

$$D^m w(b) = D^m v(b), \ D^m w(a) = 0, \ m = 0, 1,\ldots$$

4) We put

$$u(x) = \begin{cases} v(x), & x \in (0, b) \\ w(x), & x \in (b, a) \end{cases}$$

and show that $u(x)$ is the desired function.

1. Construction of the basis functions. We denote by $C > 0$ the sum of the series (2.9). Let m be a non-negative integer and let $b > \Gamma/2$. We choose the numerical sequence

$$b_{km} = \frac{M_m^C}{d_m^k M_{m+1}^k},$$

where $d_m = 4 S_m / b$ and S_m is defined by (2.8). It is easily seen that

$$\sum_{k=0}^{\infty} \frac{b_{km}}{b_{k-1,\,m}} = \sum_{k=1}^{\infty} \frac{1}{d_m} \frac{M_{m+k-1}^C}{M_{m+k}^C} < b/3.$$

Then by virtue of Lemma 2.1 there exists a function $F_m(x) \in C^{\infty}(0, b)$ such that

a) $F_m(0) = 2C$, $D^n F_m(0) = 0$, $n = 1, 2,\ldots$

b) $D^n F_m(b) = 0$, $n = 0,1\ldots$

c) $\max_{x \in (0,b)} | D^k F_m (x) | \leqslant \frac{q^k d_m^k M_{m+k}^c}{M_m^c}$, (2.11)

where $q < 1$ is a positive number. We now put for $x \in (0, b)$

$$v_0(x) = \frac{1}{2C} F_0(x),$$

$$v_1(x) = \frac{d_1}{2C} \int_0^{x/d_1} F_1(\eta) d\eta, \ldots$$

$$v_m(x) = \frac{d_m}{2C(m-2)!} \int_0^x (x-\xi)^{m-2} \int_0^{\xi/d_m} F_m(\xi_1)\, d\xi_1\, d\xi, \; m \geqslant 2.$$

This is the desired family of basis functions.

Cleary, the functions $v_m(x)$ satisfy the conditions $D^n v_m(0) = \delta_{nm}\; n = 0,1,\ldots$ In addition, by the costruction of $v_m(x)$ and (2.11) we have that for any $x \in (0, b)$

$$| D^n v_m (x) | \leqslant \frac{d_m b^{m-1-n}}{2C(m-1-n)!}, \quad n \leqslant m-1, \tag{2.12}$$

$$| D^n v_m (x) | \leqslant \frac{q^{n-m} M_n^c}{2C \cdot M_m^c}, \quad n \geqslant m. \tag{2.13}$$

2. **Construction of the function $v(x)$.** We put

$$v(x) = \sum_{m=0}^{\infty} b_m v_m(x) \tag{2.14}$$

It is evident that $D^m v(x) = b\; m$, $m = 0, 1\ldots$ We consider the values $D^m v(b)$, $m = 0, 1, \ldots$ and we wish to show that

$$\left| D^m v(b) \right| \leqslant K^m m!.$$

Indeed

$$D_n v(b) = \sum_{m=n}^{\infty} b_m D^n v_m(b),$$

since $D^m(b) = 0$ as $m < n$. Hence, by virtue of (2. 12) we have

$$\left| D^n v(b) \right| \leqslant \frac{1}{2C} \sum_{m=n}^{\infty} |b_m| \frac{d_m b^{m-1-n}}{(m-1-n)!} \leqslant$$

$$\leqslant \frac{1}{2C} . n! b^{-n} \sum_{m=n}^{\infty} |b_m| d_m \frac{(2b)^{m-1}}{(m-1)!} \leqslant K . b^{-n} . n!$$

where $K > 0$ is a constant (Here we use the condition $b < r/2$ and the inequality (2. 10)).

3. Construction of the function $W(x)$. On the basis of Theorem 2. 1 we now choose a function $W(x) \in W^{\infty} \mathcal{L} \{\varphi_n, (b, a)\}$ satisfying the conditions

$$D w(b) = D^n w(b) = D^n v(b), \quad D^n w(a) = 0, \quad n = 0, 1, \ldots$$

4. Construction of the desired function. Finally, we put

$$u(x) = \begin{cases} v(x), & x \in (0, b) \\ w(x), & x \in (b, a) \end{cases}$$

and show that $u(x)$ is the desired function. Clearly, $u(x)$ satisfies conditions (2. 2). Further, it is enough to show that

$v(x) \in W^{\infty} \mathcal{L} \{ \varphi_n, (0, a) \}$ since

$$\operatorname{supp} v(x) \cap \operatorname{supp} w(x) = \varphi.$$

By virtue of (2. 12) and (2. 13) for any $x \in (0, b)$ we have

$$\left| D^n v(x) \right| \leqslant \sum_{m=0}^{\infty} |b_m| \left| D^n v_m(x) \right| \leqslant$$

$$\leqslant \sum_{m=0}^{n} \frac{1}{2C} |b_m| q^{n-m} \frac{M_n^c}{M_m^c} + \sum_{m=n+1}^{\infty} \frac{1}{2C} |b_m| \frac{d_m b^{m-1-n}}{(m-1-n)!}$$

whence

$$\varphi_n\left(D^n v(x)\right) \leqslant \frac{1}{2} \varphi_n \left(\left(\sum_{m=0}^{n} \frac{|b_m| q^{n-m}}{C . M_m^c} \right) M_n^c \right) +$$

$$+ \frac{1}{2} \varphi_n \left(\sum_{m=n+1}^{\infty} \frac{|b_m| d_m b^{m-1-n}}{C(m-1-n)!} \right) \equiv$$

$$\equiv I_1 + I_2 \tag{2. 15}$$

By virtue of (2. 10) we can write

$$I_1 \leqslant \frac{1}{2C} \sum_{m=0}^{n} \frac{|b_m| q^{n-m}}{M^c_m} \varphi_n(M^c_m) \leqslant$$

$$\leqslant \frac{1}{2C} \sum_{m=0}^{n} \frac{|b_m| q^{n-m}}{M^c_m} \frac{1}{a}. \tag{2.16}$$

Further, using (2.5) and (2.10) the following inequality is valid for sufficiently large N.

$$I_2 \leqslant \frac{1}{2} \varphi_n \left(\sum_{m=n+1}^{\infty} \frac{|b_m| d_m b^{m-1-n}}{C(m-1-n)!} \right) \leqslant$$

$$\leqslant \frac{1}{2} \varphi_n \left(b^n n! \sum_{m=n+1}^{\infty c} \frac{|b_m| d_m (2b)^{m-1}}{C(m-1)!} \right) \leqslant$$

$$\leqslant \frac{1}{2C} \varphi_n (b^{-n} n!) \sum_{m=n+1}^{\infty} \frac{|b_m| d_m (2b)^{m-1}}{(m-1)!}. \tag{2.17}$$

From (2.15), (2.16) and (2.17) we deduce.

$$\varphi_n (D^n v(x)) \leqslant \frac{1}{2C.a} \sum_{m=0}^{n} \frac{|b_m| q^{n-m}}{M_m} +$$

$$+ \frac{1}{2C} \varphi_n (b^{-n} n!). \sum_{m=n+1}^{\infty} \frac{|b_m| d_m (2b)^{m-1}}{(m-1)!}.$$

Consequently,

$$\rho^{\infty}(u) \equiv \sum_{n=0}^{\infty} \int_0^b \varphi_n (D^n v(x)) dx \leqslant$$

$$< \frac{b}{2Ca} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{|b_m| q^{n-m}}{M^c_m} +$$

$$+ \frac{b}{2C} \sum_{n=0}^{\infty} \varphi_n (b^{-n} n!) \sum_{m=n+1}^{\infty} \frac{|b_m| d_m (2b)^{-}}{(m-1)!} \tag{2.18}$$

We show that series in the right hand side of (2.18) converge when $b < r/2$. Indeed, by virtue of (2.10).

$$\sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{|b_m| q^{n-m}}{M_m^c} \leqslant$$

$$\leqslant \sum_{n=m}^{\infty} |b_m| (M_m^c)^{-1} \sum_{k=0}^{\infty} q^k < +\infty.$$

Further, from (2. 10), (2. 8) and (2. 1) we get that

$$\sum_{n=0}^{\infty} \varphi_n (b^{-n} n!) \sum_{m=n+1}^{\infty} \frac{|b_m| d_m (2b)^{m-1}}{(m-1)!} \leqslant$$

$$\leqslant c \sum_{m=0}^{\infty} \varphi_n (b^{-n} n!) < +\infty.$$

These concludes our proof that $v(a) \in W^{\infty} \mathcal{L} \{ \varphi_n, (0, a) \}$. The theorem is proved.

Example. Suppose $\overset{0}{W}{}^{\infty} \mathcal{L} \{ \varphi_n (0, a) \}$ is a nontrivial class [1, 3] and $\varphi_n^{-1}(a)$, $n = 0,1,\ldots$ is a logarithmically convex sequence, then

$$M_m^c = \varphi_n^{-1}(a),$$

$$Sm = \frac{\varphi_{m+1}^{-1}(a)}{\varphi_m^{-1}(a)} + \frac{\varphi_{m+2}^{-1}(a)}{\varphi_{m+1}^{-1}(a)} + \ldots \leqslant K. (M_m^c)^{-1},$$

where $K >$ is a constant. Consequently, for large m

$$max \left(Sm \frac{r^{m-1}}{(m-1)'}, (M_m^c) \right)^{-1} = 1/\varphi_m^{-1}(a)$$

then the condition (2. 10) takes the form

$$\sum_{m=0}^{\infty} \frac{|b_m|}{\varphi_m^{-1}} < +\infty$$

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