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STATE ESTIMATIONS FOR THE MARKOV PROCESS DRIVEN BY A POINT PROCESS

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INTRODUCTION

More a decade have passed since the appearance of the famous paper by H. Kunita [1] on nonlinear filtering of Markov process. During this time the nonlinear filtering theory has grown out of works of M. Fujisaki, G. Kallianpur, H. Kunita, T. Duncan, E. Wong, M. Zakai, P. Brémaud, Marc Yor, Van Schuppen, etc [3, 4, 5, 6, 7]. Most of these works concern the model of dynamical Wiener driven systems with white noise while a filtering theory for the case of point process observations is only at an early stage of development (See [2]).

It may be worthwhile to insist on the fact that there is a parallelism between systems driven by Ito differential equations and point process systems and almost the results on the former found counterparts in the latter. Such is also the case of filtering problems: Almost the essential results of Kunita in [1] can be translated into the case of dynamical systems of « point process noise ».

In this context the present paper aims at considering the problem of nonlinear filtering of Markov processes with point process observations, by reference probability method, i.e. by alternative to innovations method that is taken use in [1]. The paper is constructed as follows:

In Section 1, we recall some facts on point process martingales and state the problem in which we are concerning the so-called « quasi — filtering » $\overline{\pi_t}(f)$ instead of the filtering $\pi_t(f)$ that is defined by the conditional expectation

$$\mathbf{E}\,[f\ (X_t) \mid \mathcal{F}_t^Y]$$

In Section 2, after recalling a quasi-filtering equation (due to P. Brémaud) for Poisson driven Markov processes, we prove that this equation is, in some sense, equivalent to that of Kunita 's type with point process innovations.

Section 3 is devoted to a theorem on existence and uniqueness of the solution of a stochastic differential equation for quasi-filtering process.

1. PRELIMNARIES AND NOTATIONS

Suppose that the signal process X_t is a Feller Markov process on the probability space $(\Omega, \mathcal{F}, P)$. The state space U is a compact separable Hausdorff space and P_t, $t \geqslant 0$ is the Fellerian semigroup associated with the transition probabilities $P_t(X, E)$, that is

$$P_t(X) = \int P_t(X, dy)\, f(y) \tag{1.1}$$

maps $C(U)$ into itself for all $t \geqslant 0$ and satisfied

$$\lim_{t \to 0} P_t f(X) = f(X)$$

uniformly in U for all $f \in C(U)$, where $C(U)$ is the space of all real continuous functions over U.

Let Y_t be a point process adapted to some history $\mathcal{F}_t$ (that is to a non-decreasing sub σ — fields $\mathcal{F}_t \subset$, $t \geqslant 0$) and h_t is an $\mathcal{F}_t$ — progressively measurable such that for every $t \geqslant 0$,

$$\int_0^t h_s ds < \infty \quad P - a.s. \tag{1.2}$$

The point process Y_t is said to have the $(P, \mathcal{F}_t)$ — intensity h_t if the following relation holds

$$\mathbf{E}\left[\int_0^\infty \varphi_s \, dY_s\right] = \mathbf{E}\left[\int_0^\infty \varphi_s h_s \, ds\right] \tag{1.3}$$

for every $\mathcal{F}_t$ — predictable process φ_t.

It is well-known that if (2.) is satisfied and Y_t is a P — non-explosive point process then $M_t = Y_t - \int_0^t h_s ds$ is a F_t — lo al martingale [2]. This relation is also a martingale characterization of the intensity of a point process by an extension of Watanabe's Theorem [2]: Let Y_t be a non-explosive point process adapted to F_t and suppose that for some nonnegative F_t — progressive process $h_t Y_t - \int_0^t h_s ds$ is a local martingale, then h_t is the F_t — intensity of Y_t.

Thus, in our stochastic dynamical system, we consider a Feilerian system process X_t direct observasion of which is not possible, and data concerning X_t is observed by point process Y_t of intensity h_t, i. e. as an observation of the form

$$Y_t = \int_0^t h_s ds + M_t \tag{1.4}$$

where M_t is a $\mathcal{G}_t$ — martingale and called the « point process noise ».

Denote by $\mathcal{G}_t^Y$ the σ — field generated by the family of $(Y_s, 0 \leqslant s \leqslant t)$. The family $(\mathcal{G}_t^Y, t > 0)$ is called the internal history of the process Y_t

Let $\mathcal{B}(X_0)$ be a sub σ — field $\sigma(X_0)$ and lel $\mathcal{G}^t = \mathcal{F}_t^Y \vee \mathcal{B}(X_0)$.

The conditional distributi on of X_t by the observation data $\mathcal{G}^t$,

$$\pi_t(f) = E_P [f(X_t) \mid \mathcal{G}^t], \; f \in C(U) \tag{1.5}$$

is called the filtering of X_t based on the data $\mathcal{F}_t^Y \vee \mathcal{B}(X_0)$

In the method of reference probabitity, the probability P actually governing the statistics of the observation Y_t is obtained from a probability Q by an absolutely continuous change $Q \rightarrow P$. We assume that Q is the reference probability such that Y is a $(Q, \mathcal{G}_t)$ — Poisson process of intensity 1, where

$$\mathcal{G}_t = \mathcal{G}_t^Y \vee \mathcal{G}_\infty^X$$

($\mathcal{G}_\infty^X$ is the σ — field $\sigma(X_t\ t \geqslant 0)$ which records all the events linked to the system process X_t)

Denoting for every $t \geqslant 0$ by P_t and Q_t the restrictions of P and Q respectively to $(\Omega, \mathcal{G}_t)$ we have $P_t \ll Q_t$, and the corresponding Radon — Nikodym derivative is given by

$$L_t = \frac{dP_t}{dQ_t} = \Big(\prod_{0 < s \leqslant t} h_s \Delta Y_s \Big) \exp\Big\{ \int_0^t (1 - h_s) ds \tag{1.6}$$

where h_s is a nonnegative bounded measurable and $\mathcal{G}_t$ — predictable process

The following assertions are known [2]

1. L_t is a $(Q, \mathcal{F}_t)$ — martingale and $M_t = Y_t - \int_0^t h_s ds$ is a $(P, \mathcal{F}_t)$ — martingale.

2. The restrictions Q_0 and P_0 of Q and P respectively to $(\Omega, \mathcal{F}_0) = (\Omega, \mathcal{F}_\infty^x)$ are such that $L_0 = dP_0/dQ_0 = 1$.

3. $\mathcal{F}_\infty^Y$ and $\mathcal{F}_\infty^X$ are independent

4. Let Z_t be a real — valued and bounded process adapted to $\mathcal{F}_t$ then for every history $\mathcal{G}_t$ such that $\mathcal{G}_t \subset \mathcal{F}_t$, $t \geqslant 0$, then

$$E_Q[L_t \mid \mathcal{G}_t]\, E_P[Z_t \mid \mathcal{G}_t] = E_Q[Z_t L_t \mid \mathcal{G}_t],\ Q-\text{a.s} \tag{1.7}$$

or equivalently,

$$E_P[Z_t \mid \mathcal{G}_t] = \frac{E_Q[Z_t L_t \mid \mathcal{G}_t]}{E_Q[L_t \mid \mathcal{G}_t]}, \quad P\text{—a.s} \tag{1.8}$$

This analogy of Bayes formula allows us to replace the estimation problem under P by an estimation problem under Q. Namely, by putting $\mathcal{G}_t = \mathcal{F}_t = \mathcal{F}_t^y \vee \mathcal{F}(X_0)$ we have to be concerning with $E_Q[L_t Z_t \mid F^t]$ instead of $E_P[Z_t \mid \mathcal{G}^t]$. Let us introduce

DEFINITION. *Under the above assumptions on state process* X_t *and observation* Y_t, *the quantity*

$$\pi_t(f) = E_Q[L_t f(X_t) \mid \mathfrak{F}^t],\ f \in C(U) \tag{1.9}$$

is called the quasi-filtering of X_t *based on data* $\mathcal{F}^t = \mathcal{F}_t^Y \cup \mathcal{B}(X_0)$.

It is obvious from this definition that

$$\pi_t(f) = \frac{\overline{\pi_t}(f)}{\overline{\pi_t}(1)}. \tag{1.10}$$

The problem under consideration in this paper is that of finding a connection between the filtering equation of Kunita's type for $\pi_t(f)$ and the quasi-filtering equation for $\overline{\pi_t}(f)$, and that of proving the existence and uniqueness of the solut on for the latter.

Before dealing with the quasi-filtering, let us «translate» some facts in the Fujisaki – Kallianpur-Kunita Theorem [1,4] to our case of point process observation. The proofs can be fonnd in [7] and [2].

THEOREM 1. 1. *Let* $\pi_t = E_P [f(X_t) \mid \mathcal{G}^t]$ *be the filtering of* X_t *based on* $\mathcal{G}^t = \mathcal{G}^Y_t \vee \mathcal{B}(X_0)$. *Then the process*

$$\widehat{M}_t = Y_t - \int_0^t \pi_s(h)\, ds \tag{1.11}$$

is an $\mathcal{F}^t$ — *martingale. Farthermore,* $\mathcal{G}^t$ *and* $\sigma(M_v - M_u;\ t \leqslant u \leqslant v)$ *are independent for all* $t \geqslant 0$

THEOREM 1.2. *If* m_t *is a separable square-integrable* $\mathcal{G}^t$ — *martingale, it is represented as*

$$m_t - m_0 \int_0^t H_s(dY_s - a_s ds) \tag{1.12}$$

where H_t *is a* $\mathcal{G}^t$ — *predictable process such that*

$$\int_0^t \mid H_s \mid h_s\, ds < \infty \quad P\text{-a.s.,} \quad 0 < t < \infty \tag{1.13}$$

A modification of this theorem can be found in [2] where the considered point process is a multivariate one of intensity $h_s = (h_s(i))$ $\quad 1 \leqslant i \leqslant N$.

THEOREM 1.3. *If* A *is the infinitesimal generator of the semigroup* P_t *of the signal process then* π_t *satisfies the following two type of stochastic differential equations:*

$$\pi_t(f) = \pi_0(f) + \int_0^t \pi_s(Af)ds + \int_0^t (\pi_s(fh) - \pi_s(f)\pi_s(h))\ d\widehat{M}_s$$

$$\forall f \in \mathcal{D}(A) \tag{1.14}$$

$$\pi_t(f) = \pi_0(P_t f) + \int_0^t (\pi_s((P_{t-s}f)h) - \pi_s(P_{t-s}f)\,\pi_s(h))\ d\widehat{M}_s$$

$$\forall f \in C(U) \tag{1.15}$$

J. Szpirglas (See [3] for instance) has proved that the two equations (1.9) and (1.10) in [1] are equivalent. Of course, the same thing can be done for our case to conclude that (1.14) and (1.15) are equivalent.

Now we turn back to the quasi — filtering.

2. QUASI-FILTERING EQUATION

The assumptions are the same as in the last Section. In particular, X_t is a U — valued Feller process with semigroup $(P_t, t \geq 0)$ and $Z = f(X_t)$, $f : U \to R$ is bounded and continuous. The following result is due to P. Brémaud:

The quasi-filtering $\overline{\pi}_t(f) = E_Q [L_t f(X_t) \mid \mathcal{G}^t]$ satisfied the following equation:

$$\overline{\pi}_t(f) = \overline{\pi}_0(f) + \int_0^t \overline{\pi}_{s-} ((h-1) P_{t-s} f) \, d\gamma_s, \tag{2.1}$$

where $\overline{\pi}_0(f) = E_Q[f(X_t)]$ and $\gamma_t = Y_t - t$ (which is a $(Q, \mathcal{G}^t)$ — martingale).

Now we will show that the quasi-filtering $\overline{\pi}_t$ (f) defines uniquely the filtering $\pi_t(f)$ satisfying the Kunita's equation (1.15).

THEOREM 2.1. *Let $\overline{\pi}_t(f)$ be a solution of the equation (2.1). Then $\pi_t(f) = \overline{\pi}_t(f) / \overline{\pi}_t(1)$ satisfied the equation* (1.15).

Proof. According to Ito's formula, we have

$$\frac{\overline{\pi}_t(f)}{\overline{\pi}_t(1)} = \frac{\overline{\pi}(f)}{1} + \int_0^t \frac{d\,\overline{\pi}_s(f)}{\overline{\pi}_s(1)} - \int_0^t \frac{\overline{\pi}_s(f)\, d\overline{\pi}_s(1)}{\overline{\pi}_s^2(1)} +$$

$$+ \int_0^t \frac{\overline{\pi}_s(f)}{\overline{\pi}_s^3(1)} \, d < \overline{\pi}_s(1), \overline{\pi}_s(1) > - \int_0^t \frac{d\langle \overline{\pi}_s(f), \overline{\pi}_s(1)\rangle}{\overline{\pi}_s^2(1)} \tag{2.2}$$

Denote by (1), (2), (3), and (4) respectively the integrands in the right hand side of (2.2).

Since $\overline{\pi}_t$ (f) is a solution of (2.1) one can see that

$$d\,\overline{\pi}_s(f) = \overline{\pi}_{s-} ((h-1)P_{t-s} f) d\gamma_s$$

hence

$$(1)\quad \frac{d\,\overline{\pi}_s(f)}{\overline{\pi}_s(1)} \; \frac{\overline{\pi}_{s-}[(h-1)\, P_{t-s} f]}{\overline{\pi}_s(1)} d\gamma_s = \pi_s \, [(h-1)\, P_{t-s} f] \, d\gamma_s,$$

Noticing that $L_t = 1 + \int_0^t L_{s-}\ (h_s - 1)\ d\gamma_s$ (see [2]) and therefore

$\overline{\pi_s}(1) = 1 + \int_0^t \overline{\pi_s}(h-1)\ d\gamma_t$, we have

$$(2)\quad \frac{\overline{\pi_s}(f)d\pi_r(1)}{\overline{\pi_s}^2(1)} = \frac{\overline{\pi_s}(f)}{\overline{\pi_s}(1)} \cdot \frac{\overline{\pi_s}(h-1)}{\overline{\pi_s}(1)}\ d\gamma_s = \pi_s(f)\ \pi_r\ (h-1)d\gamma_s.$$

A computation for the integrand (3) yields:

$$(3)\quad \frac{\overline{\pi_s}(f)}{\overline{\pi_s}^3(1)}\ d < \overline{\pi_s}(1), \overline{\pi_s}(1) > = \left[\frac{\overline{\pi_s}(h-1)}{\overline{\pi_s}(1)}\right] \frac{\overline{\pi_s}(P_{t-s}f)}{\overline{\pi_s}(1)}\ ds =$$

$$= \pi_s(P_{t-s}f)\ \pi_s^2(h-1)ds$$

And finally we have

$$(4)\quad \frac{d\langle \overline{\pi_s}(f), \overline{\pi_s}(1)\rangle}{\overline{\pi_s}^2(1)} = \frac{\overline{\pi_s}[(h-1)P_{t-s}f]}{\overline{\pi_s}^3(1)} \cdot \frac{\overline{\pi_s}(h-1)}{\overline{\pi_s}(1)}\ ds =$$

$$= \pi_s[(h-1)P_{t-s}f]\pi_s(h-1)ds.$$

So, summing up (1), (2), (3) and (4) we have:

$$\pi_s[h-1)P_{t-s}f]\ d(Y_s - s) - \pi_s(P_{t-s}f)\ \pi_s(h-1)\ d(Y_s - s) +$$
$$+ \pi_s(P_{t-s}f)\pi_s^2(h-1)ds - \pi_s(P_{t-s}f)\ \pi_s(h-1)\ (dY_s - ds - \pi_s(h-1)ds) =$$
$$= \pi_s(hP_{t-s}f)\ (dY_s - \pi_s(h)ds) - \pi_s(P_{t-s}f)\pi_s(h)\ (dY_s - \pi_s(h)ds) =$$
$$= [\pi_s(hP_{t-s}f) - \pi_s(P_{t-s}f)\pi_s(h)]d\ \hat{M}_s.$$

This completes the proof.

3. EXISTENCE AND UNIQUENESS OF THE SOLUTION OF A STOCHASTIC DIFFERENTIAL EQUATION

As in [1], the set of all probability measures on U is denoted by $\mathcal{M}(U)$.

Let γ_t be a point process $\mathcal{G}^t$ — martingale and $\overline{\pi_0}$ be an $\mathcal{M}(U)$ — valued variable independent of (γ), defined on a probability space $(\overline{\Omega}, \overline{\mathfrak{B}}, \overline{P})$.

Consider the following equation

$$\overline{\pi_t}(f) = \overline{\pi_0}(f) + \int_0^t \overline{\pi_{s-}}((h-1)P_{t-s}f)d\gamma_s \qquad (3.1)$$

A $\mathcal{M}(U)$ — valued stochastic $\overline{\pi_t}$ is called a solution of (3.1) if $\overline{\pi_t}$ is independent of $\sigma(\gamma_v - \gamma_u, s \leqslant u \leqslant v)$. The quasi-filtering $\overline{\pi_t}$ defined in Section 2 where $\gamma_t = Y_t - t$ is a solution of (3.1) in the above sense. The main result of this Sect on is an analogy of Theorem 2.1 in [1] The method of Kunita is applied but the proof is simpler than of [1] because the equation (3.1) here is simpher than (2.1) in [1]

THEOREM 3.1.. *There is a unique solution of (3.1) for arbitrary initial condition* $\overline{\pi_0}$.

Proof. First we show the uniqueness. Assume $\overline{\pi_t}$ and $\overline{\pi_t'}$ are two solutions of (3.1) corresponding to the same condition $\overline{\pi_0}$. Put

$$\rho_t(f) = \overline{E}(\mid \overline{\pi_t}(f) - \overline{\pi_t'}(f) \mid^2), \tag{3.2}$$

where $\overline{E}$ is the mathematical escpetation under $\overline{P}$, then

$$\rho_t(f) \leqslant 2\overline{E}(\mid \overline{\pi_t}(f) \mid + \overline{\pi_t'}(f) \mid^2) \leqslant 4\| f \|^2, \tag{3.3}$$

where

$$\| f \| = \sup_{X \in U} \mid f(X) \mid$$

We have also

$$\rho_t(f) \leqslant \int_0^t \rho_{s-}((h-1))\, P_{t-s}f)\, ds \tag{3.4}$$

Substituting (3.3) into the integrand of (3.4) we get

$$\rho_t(f) \leqslant 4 \| h - 1 \|^2 \| f \|^2 t \tag{3.5}$$

Applying repeatly this estimation n times to the right hand side of (3.4) we see that

$$\rho_t(f) \leqslant \| h - 1 \|^2 \| f \|^2 \frac{t^n}{n!} \tag{3.6}$$

Letting n tend to infinity we have $\rho_t(f) = 0$ for all $t > 0$ and $f \in C(U)$ and this fact shows the uniqueness of the solution of (3.1).

To prove the existence we notice at first that in above Section where $\gamma_t = \gamma_t - t$, the quasi — filtering $\overline{\pi_t}$ based on $\mathcal{G}_t^Y \vee \sigma(\pi_0)$ is a solution of (3.1). This quasi — filtering $\overline{\pi_t}$ can be expressed as a functional of $(\pi_0, \gamma_s - \gamma_0, 0 \leqslant s \leqslant t)$ which is denoted by

$$\Phi(\overline{\pi_0}, \gamma_s - \gamma_0, 0 \leqslant s \leqslant t) \tag{3.7}$$

Now in our situation, by replacing this $(\overline{\pi}_t, \gamma_t)$ by $(\overline{\pi}'_t, \gamma'_t)$ where π'_0 is anew initial condition and γ'_t is a point process $\mathcal{G}^t$ martingale we see that

$$\overline{\pi}'_t = \Phi(\pi'_0, \gamma'_s, 0 \leqslant s \leqslant t)$$

is a solution of (4.1). The proof of Theorem 3.1 is complete.

Remarks.

1. We can see that the solution $\overline{\pi}_t$ of (3.1) is $\sigma(\gamma_s - \gamma_0; 0 \leqslant s \leqslant t)$ — mesurable by applying the successive approximation method of Kunita [1] to solve the equation (3.1). Once again, the proof is simpler than that of [1] because of the simple form of (3.1).

2. In the case where $\gamma_t = Y_t - t$, it is not difficult to see that $\sigma(\gamma_s - \gamma_0;$ $0 \leqslant s \leqslant t)$ coincides with $\sigma(\widehat{M}_s, 0 \leqslant s \leqslant t)$ where $\widehat{M}_t = Y_t - \int_0^t \pi_s(h)\, ds$ is the point innovation process mentioned in Theorem 1.1.

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REFERENCES

[1] Kunita, H. (1971)
Asymptotic Behavior of the Nonlinear Filtering Errors of Markov Processes, Journal of Multivariate Analysis, Vol 1, No 4, pp 365 – 393.

[2] Brémaud P. (1980)
Point Processes and Queues. Martingale Dynamics, Springer Series in Statistics, Springer-Verlag.

[3] Marc Yor.
Sur la théorie du filtrage, Cours, Preprint, Paris.

[4] Fujisaki, M., Kallianpur. G., and Kunita, H. (1972)
Stochastic differential equations for non-linear filtering problem, Osaka J. Math. 9, pp 19 — 40.

[5] Kallianpur, G. (1980).
Stochastic Filtering Theory, Springer Series in Applications of Mathematics. Vol. 13, Springer — Verlag.

[6] Kunita, H. (1974 – 1975)
Théorie du filtrage, Cours de 3ème cycle, Université Paris VI.

[7] Van Schuppen. J. (1977)
Filtering, Prediction and Smoothing for coun-ing process observations, a martingale approach SIAM J. Appl. Math. 32. pp 552 – 570.