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THE TEST OF THE DISCRIMINATION FOR TWO POISSON PROCESSES

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In this paper we give first the test of the discrimination for two Poisson processes. Next we study the asymptotic normality of the statistics determinating the test and the convergence rate in this normal asymptotic process.

I – A RESTRICTED TEST PROBLEM

Let T be a separable metric space, $\mathfrak{B}$ be the σ-field of all Borel subsets of T. Standard notations and concepts in [1] (cf. Chapters II, III) will be used throughout the paper. Assume that a realization μ of the Poisson process Q is observed and we have to choose one of two posibilities: $Q = Q_{\rho_0}$ or $Q = Q_{\rho_1}$, where Q_{ρ_0}, Q_{ρ_1} are Poisson processes with densities ρ_0, ρ_1 (ρ_0, ρ_1 are Radon measures on T). Thus we have the following test problem:

(H) $\begin{cases} \text{Test of the hypothesis } Q = Q_{\rho_0} \\ \text{with the alternative hypothesis } Q = Q_{\rho_1} \end{cases}$

In general, this problem has no testable solution by a simple observation and we are led to the following restricted test problem:

(H^K) $\begin{cases} \text{Test of the hypothesis } Q^K = Q^K_{\rho_0}, \\ \text{with the alternative hypothesis } Q^K = Q^K_{\rho_1} \end{cases}$

where K is a compact subset of T and Q^K, $Q^K_{\rho_0}$, $Q^K_{\rho_1}$ are images of Q, Q_{ρ_0}, Q_{ρ_1} under the mapping $\mu \to \mu^K$ (μ^K is the restriction of μ on K).

Let ρ be a Radon measure on T such that

$$\rho i \ll \rho \qquad (i = 0,1)$$

Then for every compact subset K of T

$$\rho_i^K \ll \rho^K \qquad (i = 0,1).$$

Therefore, from [1] (Theorem 1, Chapter VI), we have:

$$Q^K_{\rho_i} \ll Q^K_{\rho} \qquad (i = 0,1)$$

with

$$\frac{dQ^K_{\rho_i}}{dQ^K_{\rho}}(\mu) = exp\{-(\rho i(K) - \rho(K)) + \mu(lnf_i)\},$$

$$fi(t) = \frac{d\rho_i}{d\rho}(t) \qquad (i = 0,1).$$

By the Neymann — Peason's basis Lemma (cf, [3], p. 95-96), it follows that the most powerful test with significance level α of the problem (H_K) have the critical domain of the form

$$S_K^{\alpha}\left\{\mu : \mu^K\left(ln\frac{f_1}{f_0}\right) > c(\alpha)\right\},$$

where $c(\alpha)$ is defined so that

$$Q^K_{\rho_0}\left\{S_K^{\alpha}\right\} = \alpha.$$

Hence, the most powerful test with significance level α of the problem (H_K) is completely determined by the statistics

$$Z_K(\mu) = \mu^K\left(ln\frac{f_1}{f_0}\right).$$

Now we investigate the distribution of Z_K.

Since

(1) $\{Z_K > -\infty\} \Leftrightarrow supp\ \mu \subset K \cap \{f_1 > 0\},$

and

$$(2) \qquad \{Z_K < +\infty\} \Leftrightarrow supp\, \mu \subset K \cap \{f_0 > 0\},$$

its follows that

$$Q_{\rho_0}\, \{Z_K = -\infty\} = 1 - exp\,\{-\rho_0(K \cap \{f_1 = 0\})\}\,; \quad Q_{\rho_0}\,\{Z_K < +\infty\} = 1,$$

$$Q_{\rho_1}\, \{Z_K = +\infty\} = 1 - exp\,\{-\rho_1(K \cap \{f_0 = 0\})\}\,; \quad Q_{\rho_1}\,\{Z_K > -\infty\} = 1.$$

Further, from (1) we infer that under the hypothesis $Q^K = Q^K_{\rho_0}$ the characteristic function of Z_K with $Z_K > -\infty$ is of the form

$$\varphi^0_K\,(u) = \exp\,\{\rho_0\,[I_{K \cap \{f_1 > 0\}}\,(\exp\left(iuln\,\frac{f_1}{f_0}\right) - 1)]\}.$$

Analogously, from (2) it follows that under the alternative hypothesis $Q^K = Q^K_{\rho_1}$, the characteristic function of Z_K with $Z_K < +\infty$, is of the form

$$\varphi^1_K\,(u) = \exp\,\{\rho_1\,[I_{K \cap \{f_0 > 0\}}\,(\exp\left(iuln\,\frac{f_1}{f_6}\right) - 1)]\}.$$

These results give us all informations to determine the critical point $c(\alpha)$ and the power of the test of the problem (H_K).

II — THE ASYMPTOTIC NORMALITY

Let K_1, K_2,... denote an increasing sequence of compact subsets of T such that $\lim_{n\to\infty} K_n = T$. Put

$$Z_n(\mu) = Z_{K_n}(\mu), \qquad n = 1, 2, \ldots.$$

Denote by $L_i\,(X)$ the distribution of the statistics X with respect to the probability measure Q_{ρ_i} $(i = 0,1)$.

THEOREM 1: *Assume that the following conditions are satisfied*

1) $\rho_0 \ll \rho_1$

2) $\int_T (f^{1/2}(t) - 1)^2\, \rho_1\,(dt) = \infty$ *where* $f(t) = \frac{d\rho_0}{d\rho_1}(t)$.

3) There exists $M > 0$ such that

$$(\rho_0 + \rho_1)\{x : | l_n f(x) | > M\} < \infty$$

Then as $n \to \infty$

$$L_0 \left(\frac{Z_n - a_n}{b_n} \right) \to \mathcal{N}(0,1)$$

where

$$a_n = \int_{K_n^M} l_n \frac{1}{f(t)} \rho_0(dt), \quad b_n = \left(\int_{K_n^M} l_n \frac{4}{f(t)} \rho_0(dt) \right)^{1/2}$$

$$K_n^M K_n \cap K^M, \quad K^M = \{x : | \ln f(x) | \leqslant M\}$$

Proof. Note that Z_n can be decomposed as follows

$$Z_n = Z_n^{(1)} + Z_n^{(2)}$$

where

$$Z_n^{(1)} = \int_{K_n^M} \ln \frac{1}{f(t)} \mu(t), \qquad Z_n^{(2)} = Z_n - Z_n^{(1)}.$$

We shall prove that

$$(3) \qquad L_0 \left(\frac{Z_n^{(1)} - a_n}{b^n} \right) \to \mathcal{N}(0,1)$$

and

$$Q_{\rho_0} \left\{ \frac{Z^{(2)}}{b_n} \to 0 \right\} = 1.$$

Since for Q_{ρ_0} — almost all μ there exists a finite limit

$$Z^{(2)} = \lim_{n \to \infty} Z_n^{(2)}$$

it suffices to prove (3) and

$$(4) \qquad \lim_{n \to \infty} b_n = \infty$$

From the condition 2) of the Theorem we have

$$(5) \quad \infty = \int_{K^M} \left(1 - \frac{1}{f^{1/2}}\right)^2 \rho_0(dt) + \int_{\overline{K^M}} \left(1 - \frac{1}{f^{1/2}}\right)^2 \rho_0(dt)$$

where $\overline{K^M} = \left\{ x : | \ln f(x) | > M \right\}$.

We can write

$$\overline{K^M} = K^M_{>} + \overline{K^M_{<}}$$

where

$$\overline{K^M_{>}} = \left\{ x : \frac{1}{f(x)} > e^M \right\}, \qquad \overline{K^M_{<}} = \overline{K^M} \setminus \overline{K^M_{>}}.$$

Thus we have the estimation

(6) $\left(1 - \frac{1}{f^{1/2}}\right)^2 < \frac{1}{f}$ on $\overline{K^M_{>}}$

and

(7) $\left(1 - \frac{1}{f^{1/2}}\right)^2 < 1$ on $\overline{K^M_{<}}$.

From (5) $\div$ (7) and from the condition 3) of the Theorem it follows that

(8) $$\int_{K^M} \left(1 - \frac{1}{f^{1/2}}\right)^2 \rho_0 \, (dt) = \infty$$

Further, we see that (8) is satisfied if and only if one of the following conditions holds:

i)) $$\int_{K^M \cap \{ |\ln f| \leqslant c\}} \left(1 - \frac{1}{f^{1/2}}\right)^2 \rho_0 \, (dt) = \infty \text{ for every } c > 0$$

(ii) There exists $c > 0$ such that

$$\int_{K^M \cap \{ |\ln f| > c\}} \left(1 - \frac{1}{f^{1/2}}\right)^2 \rho_0 \, (dt) = \infty$$

From (i) it follows there exists $c > 0$ such that if $|\ln f| < c$ then

(9) $$\ln^2 \frac{1}{f} > \left(1 - \frac{1}{f^{1/2}}\right)$$

which together with (8) implies (4).
To prove (3) let us denote

$$T_n \, (t) = \frac{1}{b_n} \ln \frac{1}{f(t)}.$$

Hence, the logarithm $\psi_n(u)$ of the characteristic function of

$Z_n = \dfrac{Z_n - a_n}{b_n}$ is of the form

$$\Psi_n(u) \int_{k_n^M} exp \, (iu \, T_n \, (t)) - 1 - iu \, T_n \, (t)\} \, \rho_0 \, (dt).$$

Thus we have the estimation

$$(10) \quad \left| \psi_n(u) + \frac{u^2}{2} \right| \leq \int_{K_n^M} \left| exp\,(uiT_n(t) - 1 - uiT_n(t) + \frac{2u}{2}\, T_n(t) \right| \rho_0\ (dt) \leq$$

$$\leq \frac{M \mid u \mid^3}{b_n}$$

which together with (3) implies (3).

Theorem 1 is proved.

Analogously we can prove the following

THEOREM 2. *Assume that following conditions are satisfied*

1') $\rho_1 << \rho_0$

2') $\int_T (g^{v2}(t) - 1)^2\, \rho_0\,(dt)\ \infty$ *where* $g(t) = \frac{d\rho_1}{d\rho_0}(t)$.

3) There exists $M > 0$ such that

$$(\rho_0 + \rho_1)\,\{x : \mid lng(x) \mid > M\} < \infty$$

Then as $n \to \infty$

$$L_1\ \left(\frac{z_n - c_n}{d_n}\right) \to \mathcal{N}\,(0.1)$$

where

$$c_n = \int_{K_n^M} lng(t)\ \rho_1(dt), \qquad d_n = (\int_{K_n^M} ln^2 g(t)\ \rho_1(dt))^{1/2},$$

K_n^M *is the same as in the Theorem 1.*

From Theorem 1 and 2 we have

THEOREM 3. *Assume that the following conditons are satisfied*

1'') $\rho_1 \approx \rho_0$

2) $\int_T (f^{1/2}(t) - 1)_2\ \rho_1(dt) = \infty$. *where* $f(t) = \frac{d\rho_0}{d\rho_1}(t)$

3) There exists $M > 0$ such that

$$(\rho_0 + \rho_1)\,\{ x : \mid lnf\ (x) \mid\ > M\} < \infty$$

Then as $n \to \infty$

$$L_0\ \left(\frac{z_n - a_n}{b_n}\right) \to \mathcal{N}(0,1), \qquad L_1\left(\frac{z_n - c_n}{d_n}\right) \to \mathcal{N}(0,1),$$

where a_n , b_n are the same as in Theorem 1 and c_n , d_n as in Theorem 2

III. THE CONVERGEENCE RATE

THEOREM 4. *Assume that the conditions of the Theorem 1 are satisfied. Then there exists $A > 0$ such that for every integer n*

$$(11) \qquad \sup_{x} \left| F_n(x) - \Phi(x) \right| \leqslant \frac{A}{b_n^{1/2}}$$

where

$$F_n(x) = Q_{\rho_0} \left\{ \mu : \frac{Z_n(\mu) - a_n}{b_n} < x \right\}$$

and

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{t^2}{2}\right) dt$$

Proof : Let us denote

$$F_n^{(1)}(x) = Q_{\rho_0} \left\{ \mu : \frac{Z_n^{(1)}(\mu) - a_n}{b_n} < x \right\}$$

Then for every $\varepsilon > 0$ we have

$$(12) \; F_n(x) - \Phi(x) = [F_n(x) - F_n^{(1)}(x+\varepsilon)] + [F_n^{(1)}(x+\varepsilon) - \Phi(x+\varepsilon)] + [\Phi(x+\varepsilon) - \Phi(x)].$$

Moreover, since

$$(13) \qquad 0 < \Phi(x+\varepsilon) - \Phi(x) < \frac{1}{\sqrt{2\pi}} \varepsilon.$$

and

$$| F_n(x) - F_n^{(1)}(x+\ ) | \leqslant Q_{\rho_0} \left\{ \frac{|Z_n^{(2)}|}{b_n} \geqslant \varepsilon \right\},$$

by Chebyshev's inequality we get

$$(14) \qquad | F_n(x) - F_n^{(1)}(x+\varepsilon) | \leqslant \frac{A_1}{\varepsilon b_n}$$

where

$$A_1 = \rho_0 \{ | en_t | > M \}.$$

Now let $\varphi_n(u)$ and $\varphi(a)$ denote the characteristic functions of $F_n^{(1)}$ and Φ, respectively. Then we have

$$| \varphi_n(u) - \varphi(u) | = \varphi(u) \, . \, \left| \exp\left(\psi_n(u) + \frac{u^2}{2}\right) - 1 \right| .$$

Hence and by (10) $| u | \leqslant \left(\frac{b_n}{M}\right)^{1/3}$, it follows that for any $| u | < \left(\frac{b_n}{M}\right)^{1/3}$

$$|\varphi_n(u) - \varphi(u)| \leqslant 2\varphi(u)\left|\psi_n(u) + \frac{u^2}{2}\right| \leqslant \frac{2M}{b_n} \cdot |u|^3 \cdot \exp\left\{-\frac{u^2}{2}\right\}.$$

Therefore, by the Esseen's inequality (cf. [2], p. 207), we infer that for

$$\Gamma = \left(\frac{b_n}{M}\right)^{1/3}$$

$$(15)\quad \sup_x \left| F_n^{(1)}(x) - \Phi(x) \right| \leqslant \frac{4}{\pi} \int_0^{\Gamma} \frac{M}{b_n} u^2 e^{-\frac{u^2}{2}} du + \frac{24}{\sqrt{2\pi^3} \cdot \Gamma} \leqslant \frac{A_2}{b_n} + \frac{A_3}{b_n^{1/3}},$$

where

$$A_2 = 4M/\pi, \quad A_3 = 24\sqrt{M} \,/\, \sqrt{2\pi^3}.$$

From (12) + (15) the relation (11) holds with

$$A = \max\left(A_3, \sup_n (A_2/b_n^{2/3}), \sup_n (A_4/b_n^{1/6}\right)$$

where

$$A_4 = \sqrt{A_1} \,/\, 2\sqrt{2\pi}.$$

Theorem 4 is proved.

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