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SOME FIXED POINT THEOREMS FOR NOWHERE NORMAL – OUTWARD SET VALUED MAPPINGS

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The aim of this paper is to present some fixed point theorems for nowhere-normal-outward set-valued mappings. In Section I we first recall some definitions and auxiliary facts concerning properties of projection onto closed convex sets in Banach spaces. Next, we present a fixed point theorem for compact nowhere-normal-outward set-valued mappings. A more precise version of this result will be given in Section II for the case of Hilbert space. Finally, in Section III we give two fixed point theorems for non-compact set-valued mappings, which are condensing or non-expansive. The readers are referred to [2], [3], [4], [5] for the fixed point results, concerning set-valued mappings, satisfying other boundary conditions.

I. A FIXED POINT THEOREM FOR NOWHERE-NORMAL-OUTWARD MAPPINGS IN BANACH SPACES.

First we recall some definitions and properties of the projection onto a non-empty closed convex set in a Banach space. A norm $\|\,.\,\|$ in a linear normed space X is said to be smooth if the unit ball in X has at each boundary point exactly one supporting hyperplane, it is said to be rotund (or strictly convex) if there is no open line segment in the unit ball meeting the unit sphere. We say that a linear normed space X has property (H) if:

$$x_k \underset{w}{\to} x \text{ and } \|x_k\| \to \|x\| \quad \text{imply} \quad \|x_k - x\| \to 0.$$

LEMMA 1: *(Lemma 3, [3]). Let C be a non-empty closed convex set in a reflexive Banach space X with a rotund norm. Then for every $x \in X$ there is exactly one $y = \pi_C(x) \in C$ satisfying*

$$\| y - x \| = \min_{u \in C} \| u - x \| .$$

The mapping $\pi_C : X \to C$ is weakly continuous in the sense that : $x_k \to x$ implies $\pi_C(x_k) \underset{w}{\to} \pi_C(x)$. If C is compact or X has property (H), then π_C is continuous.

Observe that $\pi_C(x) = x$ for all $x \in C$ and $\pi_C(x) \in \partial_a C$ for all $x \notin C$ (here $\partial_a C$ denotes the algebraic boundary of C). The mapping π_C is called the projection onto C.

DEFINITION 1: Let X be a Banach space and C a closed non-empty convex subset in X . Let x be a point of the algebraic boundary $\partial_a C$ of C and π_C the projection onto C . Then $\pi_C^{-1}(x) = \{y : \pi_C(y) = x\}$ is called the *normal outward set* at **x** . A mapping $F : C \to 2^X$ is called *nowhere-normal-outward* iff for al $d \in \partial_a C$ we have :

$$\pi_C^{-1}(x) \cap F(x) \subset \{x\}.$$

DEFINITION 2 : The set $2x - \pi_C^{-1}(x)$ is called normal-inward set at x. F is called nowhere-normal-inward iff for all $x \in \partial_a C$ we have :

$$(2x - \pi_C^{-1}(x)) \cap F(x) \subset \{x\}.$$

THEOREM 1 : *Let C be a non-empty closed convex subset in a reflexive Banach space X, $F : \to 2^X$ an upper semicontinuous set-valued mapping with non-empty closed convex values, such that :*

1) F is nowhere-normal-outward

2) The image F(C) is relatively compact.

If, in addition, either C is compact or X has property (H) then F has a fixed point.

Proof : Since the projection $\pi_C : X \to C$ is continuous (Lemma 1), the mapping F_1: clco $F(C) \to 2^X$ defined by $F_1(x) = F(\pi_C(x))$ is upper semi continuous with non-empty, closed convex values. Moreover clco F (C) is compact. Hence, according to the Tikhonov-Kakutani-Ky Theorem [1], $F_1(x)$ has a fixed point. Let x be a fixed point of F_1, i.e., $x^* \in F_1(x)$. We claim that $x^* \in C$. Suppose contrarily that $x^* \in C$. Then $\pi_C(x^*) \in \partial_a(C)$ and $F_1(x^*) = F(\pi_C(x^*)) \ni x^*$. Thus, we obtain

$$x^* \in \pi_C^{-1}(\pi_C(x^*)) \cap F(\pi_C(x^*)).$$

This contradicts the nowhere-normal-outward condition of F. We have now $\pi_C(x^*) = x^* \in F_1(x^*) = F(x^*)$. This completes the proof.

COROLLARY : Theorem *1* is still valid if the condition « F is nowhere-normal-outward » is replaced by « F is nowhere-normal-outward »,

Proof : Obviously, the mapping $F_1 : C \to 2^X$ defined by $F_1(x) = 2x - F(x)$ satisfies the conditions of Theorem 1. So F_1 has a fixed point. Since, obviously, every fixed point of F_1 is also fixed point of F (and *vice versa*), the corollary follows.

II. A FIXED POINT THEOREM FOR NOWHERE—NORMAL—OUTWARD MAPPINGS IN HILBERT SPACES

In this section we show that the projection onto a non-empty closed convex set in Hilbert space H is non-expansive. i. e.

$$\| \pi_C(x) - \pi_C(y) \| \leqslant \| x - y \| \quad \text{for all } x, y \in H.$$

We also prove that the normal-outward set $\pi_C(x)$ coincides with the normal-outward cone, which is defined below.

Let H be a Hilbert space C a non-empty closed convex set in it and **x** a point of $\partial_a C$. The cone

$$N_C(x) = \{ y \in H : \text{Re} < y - x, u - x > \leqslant 0 \text{ for all } u \in C \}$$

is called the *normal-outward cone* of C at x.

Let x_0, e be two vectors in H with $e \neq 0$. Consider the straight line $L = \{x_0 + \lambda e : \lambda \in R\}$. It is easy to see that for each $x \in H$ its orthogonal projection, $\pi_L(x)$, is of the form

$$\pi_L(x) = x_0 + \frac{Re\langle e, x - x_0\rangle}{\| e \|^2} e \tag{1}$$

Hence, for any $x, x' \in H$, weget

$$\| \pi_L(x) - \pi_L(x') \|^2 = \frac{| Re\langle e, x - x'\rangle |^2}{\| e \|^4} \langle e, e\rangle = \frac{| Re\langle e, x - x'\rangle |^2}{\| e \|^2} \leqslant$$

$$\frac{| \langle e, x - x'\rangle |^2}{\| ee \|}$$

By means of the Cauchy-Schwartz inequality it follows that

$$\| \pi_L(x) - \pi_L(x') \| \leqslant \| x - x' \|$$

i. e.

the projection π_L is non-expansive,

LEMMA 2: *The projection π_C onto a non-empty closed convex set in Hilbert space is non-expansive.*

Proof: It is well known that π_C is well defined (Lemma 1). Let x,x' be two points with $\pi_C(x) \neq \pi_C(x')$. Consider the straight line passing through $\pi_C(x)$ and

$$\pi_C(x') : L = \{\pi_C(x) + \lambda (\pi_C(x') - (\pi_C(x)) : \lambda \in R\}$$

Observe that on the straight line L the points $\pi_C(x)$ and $\pi_C(x')$ correspond to the values $\lambda = O$ and $\lambda = l_1$, respectively. The interval $[\pi_C(x), \pi_C(x')]$ corresponds to [O, 1] and is entirely contained in C because of the convexity of C. Suppose $\pi_L(x) = \pi_C(x) + \lambda_o(\pi_C(x') - \pi_C(x))$. We claim that $\lambda_o \leqslant O$. First λ_0 can not belong to (0,1]. Suppose contrarily that $\lambda_0 \in (0,1]$. Since $\pi_L(x) \neq \pi_L(x')$ and $\pi_L(x) \in C$ we have $\| \pi_L(x) - x \| < \| \pi_C(x) - x \|$. This contradicts the definition of π_C, Further, λ_0 can not be greater than 1, Otherwise, we should have

$$\| \pi_C(x') - x \| > \| \pi_C(x) - x \|$$

which again contradicts the definition of π_C. Similarly

$\pi_C(x') = \pi_C(x) + \lambda_l(\pi_C(x)' - \pi_C(x))$ with $\lambda_l \geqslant 1$.

Then

$$\pi_L(x') - \pi_L(x) = (\lambda_1 - \lambda_o) \| \pi_C(x') - \pi_C(x) \| > \pi_C(x') - \pi_C(x)$$

It follows from the non-expansiveness of π_C that:

$$\| \pi_C(x') - \pi_C(x) \| \leqslant \| \pi_L(x') - \pi_L(x) \| \leqslant \| x' - x \|$$

This proves the non-expansiveness of π_C.

LEMMA 3: *Let C be a non-empty closed convex set in Hilbert space H. Then the normal — out ward cone $N_C(x)$ coincides with the normal — outward set $\pi_C^{-1}(x)$ for every $x \in C$.*

Proof: We first prove that $N_C(x) \supset \pi_C^{-1}(x)$. Let $y \in N_C(x)$. For every $v \in C$ we have:

$\| v y \|^2 = \| v - x + v - x \|^2 = \| v - x \|^2 + 2\text{Re} < v - x, x - y > +$
$+ \| x - y \|^2 \geqslant \| x - y \|^2$

Thus, x is the point in C nearest to y. Therefore $\pi_C(y) = x$, that is $y \in \pi_C^{-1}(x)$

Now we prove $\pi_C^{-1}(x) \subset N_C(x)$. Suppose $\pi_C(y) = x$. We have to show that $\text{Re} < y - x, v - x > \leqslant 0$ for all $v \in C$.

Suppose contrarily that there exists $v \in C$ such that $\text{Re} < y - x, v - x > 0$. Clearly, $v \neq x$. Consider the straight line $L = \{x + \lambda (v - x) : \lambda \in R\}$ We have

$$\|y - x - \lambda (v - x)\|^2 = < y - x - \lambda (v - x), y - x - \lambda (v - x) > =$$
$$= \lambda^2 \| v - x \|^2 + 2\lambda \text{ Re} < v - x, x - y > + \| x - y \|^2.$$

This expression attains its minimum only at

$$\lambda = \lambda_0 = \frac{\text{Re} < v - x, x - y >}{\| v - x \|^2} > 0$$

Because of the convexity of C, for $0 < \lambda < \min \{l, \lambda_0\}$ we have $x + \lambda(V - X) \in C$.

On the other hand, $\| \bar{X} + \lambda(v - x) - y \| \leqslant \| x - y \|$.

Hence, $\pi_C(y) \neq x$. This contradicts the assumption. The proof is complete.

COROLLARY : *Let C be a non-empty closed convex set in a Hilbert space H, $F : C \to 2^H$ an upper semi-continuous set-valued mapping with non-empty closed convex values satisfying:*

1) *F is nowhere-normal-outward.*
2) *F (C) is relatively compact.*

Then F *has a fixed point.*

Proof: Consider the set-valued mapping F_l: $clcoF(C) \rightarrow 2clcoF(C)$ defined by $F_l(X) = F(\pi_C(x))$. Clearly, F_l is upper semi-continuous. An argument anologuous to that used in the proof of Theorem 1 shows that it has a fixed point x^*. Clearly x^* is also a fixed point of F.

Remark: This result is stated with the assumption «C is compact and $\text{int}C \neq \phi$» in [2]. Therefore, the space considered in Theorem 20 of [2] is in fact of finite dimension.

III. A FIXED POINT THEOREM FOR NOWHERE-NORMAL-OUTWARD SET-VALUED MAPPINGS WITH NON-COMPACT IMAGE IN A HILBERT SPACE.

Let (Y, d) be a metric space and R a bounded subset of Y. We put $a(B) = \inf \{r > 0 : B$ can be covered by a finite number of sets of diameter less than or equal to r) (compare with [5]). Let S be a non-empty set in (Y, d), $F : S \rightarrow 2^Y$ a mapping.

DEFINITION 3: F is called condensing if for every bounded subset B of S with $a(A) > 0$, the set $F(B) = \bigcup_{b \in B} F(b)$ is bounded and $a(F(B)) < a(B)$.

THEOREM 2: *Let C be a non-empty closed convex set in a Hilbert space H, $F : C \rightarrow 2^H$ an upper-semicontinuous condensing set-valued mapping with compact convex values, satisfying;*

1) $clco\, F(C)$ is bounded

2) F is nowhere-normal-outward.

Then F has a fixed point.

Proof: Consider the set-valued mapping $F_l : clcoF(C) \rightarrow 2clcoF(C)$ defined by $F_l(x) = F(\pi_C(x))$. Clearly, F_l is upper semi-continuous and takes non-empty closed convex values. Moreover, F_l is condensing. Indeed, let B be a bounded set in $clcoF(C)$ with $a(B) = O$. We have to show that $a(F_l(B)$. Since π_C is non-expansive, we have $a(\pi_C(B) \leqslant a(B)$. If $a(\pi_C(B)) > O$ the inequalities

$$a(F(\pi_C(B)) < (a(\pi_C(B) \leqslant a(B)$$

follows immediately from the assumptions on F. Consider the case where

$$a(\pi_C(B)) = O.$$

Since $\pi_C(B)$ is totally boundéd, cl $\pi_C(B)$ is compact in C, $F(\text{cl}\, \pi_C(B))$ is compact by the upper semi-continuity of F. This implies that $E(\pi_C(B))$ is totally bounded, i.e. $a(F(\pi_C(B))) \doteq O$ and we have $a(F_l(B)) < a(B)$. Thus, the mapping F_l satisfies the conditions of Theorem (4.1) in [4] and hence, admits a fixed point $x \in F_l(X)$. This point x can not lie outside C. Indeed, if $x^* \in C$, then $\pi_C(x^*) \in \delta_a C$ and hence $x^* \in N_C(\pi_C(C))$. This contradicts the assumption 2) of Theorem 3. Thus, $x^* \in C$ and we have $x^* \in F(\pi_C(X)) = F(X^*)$.

This completes the proof.

THEOREM 3: *Let C be a non-empty closed convex subset in a Hilbert space H. $F: C \to 2^H$ a non-expansive set-valued mapping non empty compact convex values satisfying:*

1) clco F (C) is weakly compact.

2) $F(x) \cap N_C(x) \subset \{x\}$ for all $x \in \delta_a c$.

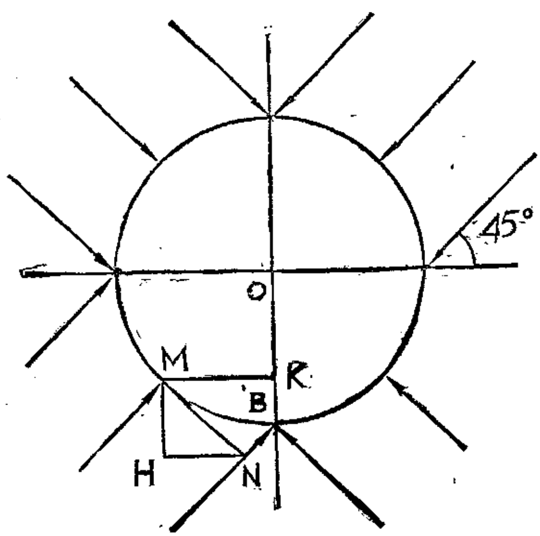
Then F has a fixed point.

Proof: Consider F_1: clco $F(C) \to 2^{clco\, F(C)}$ defined by $F_l(x) = F(\pi_C(x))$. Since π_C is non-expansive and single-valued, F_l is non-expansive. By Theorem 5.4 in [5] F_1 has a fixed point $x^* \in F_l(x^*)$. From the assumption 2), it follows that $x^* \in C$ and x^* is a fixed point of F.

It should be noted that in the proof of Theorems 2 and 3 we used the non-expansiveness of the projection π_C. In Banach spaces the projection π_C is not always non-expansive and it can really vary in different equivalent norms. This is illustrated in the following example.

Example: Consider the unit ball.

$S = \{(x,y) \in R^2 : x^2 + y^2 \leqslant 1\}$ in R^2. S is non-empty closed and convex set in every equivalent norm. Consider an another norm defined by $\|(x,y)\|' = \max\{\|x\|, \|y\|\}$. Then the projection π_S in $(R^2, \|.\|')$ is described as in the following figure:



Take M near B and N such that $MN \perp NB$. Denote by d and d' the usual distance in R^2 and the distance in $(R^2, \| \cdot \|')$ respectively. We have $d'(M,N) = d(M,H) = d(H,N)$ and $d'(M,B) = d(M,K)$. It is clear that $\pi_S(N) = B$, $\pi_S(M) = M$, $\frac{d'(M,B)}{d(M,N)}$ tends to 2 as M tends to B. Thus the projection π_S in $(R^2, \| \cdot \|)$ is not non-expansive.

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