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NECESSARY AND SUFFICIENT CONDITIONS FOR SOME GENERAL OPTIMIZATION PROBLEMS

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1. INTRODUCTION

The theory of sufficient conditions for optimality has not been yet fully developed. Some sufficient conditions for optimization problems with equality-type constraints in Banach spaces have been given by Ioffe and Tikhomirov [2]. A few necessary and sufficient conditions for optimization problems with equality-type and inequality-type constraints have given by Levitin, Milyutin, Osmolovskii [5], Alepseev, Tikhomirov, Fomin [4]. Note, however, that in all the mentioned works, only the inequality-type constraint including a finite number of functionals is considered.

In the present paper, using the locally M-surjective mapping theorem in [1] we obtain a general sufficient condition of second-order for the problem with an inequality constraint relative to a closed convex cone and a form of local K-function of this problem. Also, some necessary conditions are derived in the form of certain local K-functions for the case where the cone M is the non-positive orthant. Finally, a necessary condition of second-order is established. From these results we obtain as special cases certain earlier results, including one of Ioffe and Tikhomirov in [2].

2. A GENERAL OPTIMALITY CONDITION FOR PROBLEM (I) AND ITS CONSEQUENCES

Let X, Y be real Banach spaces. Assume that Y is ordered by a closed convex cone M. Let D be an open subset of X, f_0 a real valued function on D, and F a mapping from D into Y. Let us consider the following problem:

$$\text{(I)} \qquad \begin{cases} f_0(x) \to inf \\ F(x) \in M \end{cases}$$

The Lagrangian of Problem (I) is defined as follows:

$$\mathcal{L}(x, \wedge) = f_0(x) + <\wedge, F(x)>,$$

where $\wedge \in Y^*$.

Assumptions.

(i) f_0, F are continuously Frechet differentiable in a neighbourhood of $\mathbf{x}_0 \in D$;

(ii) At $\mathbf{x}_0$ the Kuhn-Tucker necessary optimality condition holds. This means that there is continuous linear functional $\wedge \in (-M)^*$ such that

$$\mathcal{L}'(x_0, \wedge) = 0;$$

(iv) $F(x_0) = 0$.

We now recall some notations and results (see [1], [3]), to be used in the present paper.

Let f_1, f_2 be two mappings from an open subset U of X into Y. The mapping f_2 is said to be ([1]) in the (U, α) - Lipschitz proximity of f_1 if:

$$(\forall x, x' \in U) \; \| f_1(x) \; f_2(x) - f_1(x') - f_2(x') \| \leqslant \alpha \| x - x' \|$$

In the case that f_l is continuously Fréchet differentiable in U, the mapping $f_2(x) = f_l(x_0) + f'_l(x_0)(x - x_0)$ is in the (U, α) — Lipschitz proximity of f_l for some $\alpha > 0$.

The inverse of a convex process G (from X into Y) is the convex process $G^{-1}(Y \to X)$, whose norm is defined by

$$\| G^{-1} \| = \inf \left\{ \gamma \geqslant o ; \forall y \in \text{range } G, \exists x \in G^{-1}(y) : \| x \| \leqslant \gamma \| y \| \right\}$$

If G is a convex process, then by a lemma in [1] $\| G^{-1} \| < \infty$.

THEOREM 2.1 [1].

Let f be a mapping from an open subset U of X into Y and $X_0 \in U$. Assume there is in a (U, α) — Lipschitz proximity of f a M — surjective mapping $f(x_0) + g(x - x_0)$ such that $g : X \to Y$ is linear continuous, and $\alpha \, \| G^{-1} \| = \theta < 1$, where G denotes the convex process $G(x) = g(x) - M$.

Then for every $u \in U$, for every v such that $\| v - f(u) \| < \frac{1 - \theta}{\| G^{-1} \|} \rho(u, X \setminus U)$ and for every $\delta > 0$ the equation

$$f(x) \in v + M$$

has at least one solution x satisfying

$$\| x - u \| \leqslant C \, \| v - f(u) \| \quad \left(C = \frac{1 + \delta}{1 - \theta} \, \| G^{-1} \| \right),$$

where $\rho(u, X \setminus U)$ *denotes the distance from x to $X \setminus U$.*

Hence f is M — surjective at every point of U and the mapping $F_l(x) = f(x) - M$ carries every open subset of U onto an open subset of $F_l(U)$.

To derive the general sufficient condition we need the following.

LEMMA 2. 1. *Under the same hypotheses as in Theorem 2. 1., for every $\delta > 0$ there exists a neighbourhood U' of x_0 such that for every $x \in U'$ there is $\hat{x} \in U$ such that*

$$f(\hat{x}) \in f(x_0) + M \tag{2. 1}$$

$$\| \hat{x} - x \| \leqslant C \, \| f(x) - f(x_0) \| . \tag{2. 2}$$

This Lemma contains as a special case the generalized Ljusternik's theorem [2].

Proof. Taking a number $r > 0$ such that $B(x_0, 2r) \subset U$, where $B(x_0, 2r)$ denotes the open ball of radius $2r$ around x_0, by virtue of the continuity of f, we see that for every $\delta > 0$ there exists a neighbourhood U' of x_0 ($U' \subset B(x_0, r)$) such that

$$(\forall x \in U') \; \| f(x) - f(x_0) \| < \frac{(1 - \theta) r}{(1 + \delta) \| G^{-1} \|} . \tag{2. 3}$$

Observe that

$$(\forall x \in U') \qquad \rho(x, X \setminus U) > r. \tag{2. 4}$$

By (2. 3), (2. 4) we get

$$(\forall x \in U') \ \| f(x) - f(x_0) \| < \frac{(1-\theta)}{\| G^{-1} \|} \rho(x, X / U).$$

Applying Theorem 2. 1 to the point $v = f(x_0)$, for every $x \in U'$ there exists $\widehat{x} \in X$ such that (2.1), (2.2) hold.

It follows from (2.2), (2.3) that

$$\| \widehat{x} - x_0 \| < 2r,$$

which means $\widehat{x} \in U$. This completes the proof.

DEFINITION 2. 1. *We say that the mapping f is M-regular at $x_0 \in U$ if f is Fréchet differentiable at x_0 and $f'(x_0)$ is a M-surjection of X onto Y i. e. the multivalued mapping $f'(x_0) - M$ is surjective.*

When $M = \{0\}$ we obtain as a special case the usual concept of regular mapping.

LEMMA 2. 2. *Assume f is M-regular at $x_0 \in U$. Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $\zeta \in B(x_0, \delta)$ satisfying $f(\zeta) \in f(x_0) + M$ there is $\eta \in X$ such that $\| \eta \| < \varepsilon \| \zeta - x_0 \|$ and*

$$\zeta + \eta - x_0 \in \{\xi : f'(x_0) \xi \in M\}.$$

Proof. It has been established in [1] that if a mapping $g: X \to Y$ is M—convex, M — closed, M—surjective, then g is locally M — surjective. Thus the mapping $f'(x_0)$ is locally M—surjective i.e. the associated multivalued mapping $F_l(x) = u'(x_0)x - M$ is locally surjective. Hence there exists a number $r > o$ such that

$$F_l(B_x(0, l)) \supset B_y(0, r) \tag{2.5}$$

(here $B_x(0, l), B_y(0, r)$ denote the open balls of radius l, r around 0 in X, Y resp).

For $\varepsilon > 0$, by virtue of the differentiability of f at x. there exists a number $\delta > 0$ such that for every $\zeta \in B(x_0, \delta)$,

$$\| f(\zeta) - f(x_0) - f'(x_0)(\zeta - x_0) \| < \varepsilon r \| \zeta - x_0 \| \tag{2.6}$$

By (2.5), (2.6), there exists $\eta' \in X$ such that $\| \eta' \| < 1$ and

$$\frac{f(\zeta) - f(x_0)}{\varepsilon \| \zeta - x_0 \|} - f'(x_0) \left(\frac{\zeta - x_0}{\varepsilon \| \zeta - x_0 \|} \right) \in F_1(\eta'),$$

which implies that

$$f'(x_0)\eta' \in - f'(x_0) \left(\frac{\zeta - x_0}{\varepsilon \| \zeta - x_0 \|} \right) + \frac{f(\zeta) - f(x_0)}{\varepsilon \| \zeta - x_0 \|} + M \tag{2.7}$$

Obviously, $f(\zeta) \in f(\ _0) + M$ implies $\dfrac{f(\zeta) - f(x_0)}{\varepsilon \| \zeta - x_0 \|} \in M$ (2.8)

It follows from (2.7), (2.8) that

$$f'(x_0) \left(\eta' + \frac{\zeta - x_0}{\varepsilon \| \zeta - x_0 \|} \right) \in M \tag{2.9}$$

Setting $\eta = \varepsilon \| \zeta - x_0 \| \eta'$ we obtain $\| \eta \| < \varepsilon \| \zeta - x_0 \|$ and

$$f'(x_0) (\eta + \zeta - x_0) \in M,$$

which completes the proof.

We are now in a position to formulate a general sufficient optimality condition of second-order for Problem (I).

THEOREM 2.2. *Assume that Assumptions (i) — (iv) are satisfied for Problem (I), and that the mapping F is M-regular at x_0. Furthermore, assume that the mappings f_0, F are twice continuously Fréchet differentiable at x_0, and there is a number $\sigma < 0$ such that*

$$\mathcal{L}''_{xx}(x_0, \Lambda)(\xi, \xi) \geqslant \delta \| \xi \|^2, \tag{2.10}$$

$$(\forall \xi \in \{ \zeta : F'(x_0) \zeta \in M \})$$

Then x_0 is a local solution of Problem (I).

Proof. Ioffe and Tikhomirov have proved that if a local K function of Problem (I) at x_0 can be constructed, then x_0 is a local solution of this problem (see [2]). Hence it suffices to verify that the function φ defined as follows is a local K-function of Problem (I) at x_0:

$$\varphi(x) = f_0(x_0) - < \Lambda, F(x) > - \gamma \| F(x) \|. \tag{2.11}$$

where γ is the number in Assumption (iii).

Recall that the function $\varphi: X \to R$ is said to be a local K-function of Problem (I) at x_0 if there is a neighbourhood U of x_0 such that

a) $f_0(x_0) = \varphi(x_0)$,

b) for every admissible point $x \in U: \varphi(x) \geqslant \varphi(x_0)$,

c) $f_0(x) - \varphi(x) \geqslant 0 \qquad (\forall\, x \in U)$

We now prove that the function φ defined by (2,11) satisfies these conditions a)—c).

Condition a). In view of Assumption (i), it follows from (2.11) that

$$f_0(x_0) = \varphi(x)$$

Condition b) Let x be an arbitrary admissible point of Problem (I) i. e. $F(x) \in M$. According to Assumption (iii), we have

$$> \Lambda, -F(x) > \geqslant \gamma \, \| F(x) \|,$$

which implies

$$\varphi(x) = f_0(x_0) + < \Lambda, -F(x) > - \gamma \, \| F(x) \| \geqslant f_0(x_0) = \varphi(x_0).$$

Condition c setting $p(x) = \mathcal{L}(x, \Lambda) - f_0(x_0)$ we obtain

$$f_0(x) - \varphi(x) = p(x) + \gamma \, \| F(x) \|.$$

Because the mappings f_0, F are twice continuously Fréchet differentiable at x_0, for every $\varepsilon > o$ there exists a neighbourhood U_1 of x_0 $(U_1 \subset D)$ such that for every $x \in U_1$, $| p(x) - p(x_0) - < p'(x_0), x - x_0 >$

$$- \frac{1}{2} p''(x_0)(x - x_0, x - x_0) \mid < \frac{\varepsilon}{2} \| x - x_0 \|^2. \tag{2.12}$$

Putting $F_1(x) = F'(x_0)x - M$ we see that is a convex process and $\| F_1^{-1} \| < \infty$. Hence, for a positive number $\theta < 1$ there are a neighbourhood U_2 of x_0 $(U_2 \subset U_1)$ and a number $\alpha > 0$ such that $\alpha \, \| F_1^{-1} = \theta < 1$, and, the mapping $F(x_0) + F'(x - x_0)$ is in the (U_2, α) – Lipschitz proximity of $F(x)$.

Since $p'(x_0) = 0$ and $p'(x)$ is continuous in a neighbourhood of x_0, there exists a neighbourhood U_3 of x_0 $(U_3 \subset U_2)$ such that for every $x \in U_3$,

$$p(x) - p(x_0) \leqslant \| x - x_0 \| \sup_{\xi \in U_3} \| p'(\xi) \| \tag{2.13}$$

Applying Lemma 2.2 to the mapping F we have a number $\delta_1 > 0$ such that $B(x_0, \delta_1) \subset U_3$ and for every $\zeta \in B(x_0, \delta_1)$ satisfying $F(\zeta) \in M$ there exists $\eta \in X$ such that $\| \eta \| < \varepsilon \| \zeta - x_0 \|$,

$$F'(x_0)(\zeta + \eta - x_0) \in M. \tag{2.14}$$

Applying Lemma 2.1 to the mapping F, yields a neighbourhood U of x_0 such that $U \subset B(x_0, \delta_1)$ and for every $x \in U$ there exists $\widehat{x} \in B(x_0, \delta_1)$ satisfying

$$F(\widehat{x}) \in M \tag{2.15}$$

$$\| \widehat{x} - x \| \leqslant C_l \, \| F(x) \| \tag{2.16}$$

From Lemma 2.2 applied to $\widehat{x}$ it then that there is $\eta \in X$ satisfying (2.14).

According to Assumption (ii), by (2.12), (2.13), (2.16), and the mean value theorem we have for every $x \in U$.

$$f_0(x) - \varphi(x) = p(x) - p(\widehat{x}) + p(\widehat{x}) + \gamma \| F(x) \| \geqslant$$

$$\geqslant - \sup \{ \| p'(\xi) \| : \xi \in [x, \widehat{x}], x \in U \} \| \widehat{x} - x \| + \gamma \| F(x) \| +$$

$$+ \frac{1}{2} p''(x_0)(\widehat{x} - x_0, \widehat{x} - x_0) - \frac{\varepsilon}{2} \| \widehat{x} - x_0 \|^2 \geqslant$$

$$\geqslant \frac{1}{2} \{ \mathcal{L}''_{xx}(x_0, \Lambda)(\widehat{x} - x_0, \widehat{x} - x_0) - \varepsilon \| \widehat{x} - x_0 \|^2 \}$$

In view of Lemma 2.2, and the fact that $\mathcal{L}''_{xx}(.)(\xi, \zeta)$ is a bilinear form, it follows from (2.10), (2.18) that for every $x \in U$,

$$\mathcal{L}''_{xx}(x_0, \Lambda)(\widehat{x} - x_0, \widehat{x} - x_0) - \varepsilon \| \widehat{x} - x_0 \|^2 \geqslant 0.$$

completing the proof. of Theorem 2.2.

Applications.

From Theorem 2.2 we obtain as special cases certain known results, including one of Ioffe and Tikhomirov in [2]. Let us mention some of these cases.

1) Consider the following problem

$$\text{(II)} \begin{cases} \text{minimize } f_o(x), \\ \text{subject to} \\ F(x) = o, \ x \in D, \end{cases}$$

where f_0, F, D are as in Problem (I).

COROLLARY 2.1 (Ioffe and Tikhomirov [2])

Assume that F is regular at x_0, $F(x_0) = 0$. Furthermore, assume there exist $\wedge \in Y^*$ such that $\mathcal{L}'_x(x_0, \wedge) = 0$, and a number $\delta > 0$ such that

$$\mathcal{L}''_{xx}(x_0, \wedge)(\xi, \zeta) \geqslant \|\sigma\| \|\xi\|^2 \ (\forall \xi \in Ker\ F'(x_0))$$

Then x_0 is a local solution of Problem (II).

Proof. It is easily seen that Assumption (iii) holds with $M = \{0\}$ and any number $\gamma > 0$. Hence the corollary follows from Theorem 2.2.

2) Consider the following

$$\text{(III)} \begin{cases} \text{minimize } f_o(x), \\ \text{subject to} \\ f_i(x) \leqslant 0, \ i = 1,\dots, k, \\ G(x) = 0, \ x \in D, \end{cases}$$

here f_o, D are as in Problem (I), and $f_i : D \to R$ $(i = l,\dots, k)$, $G: D \to Y$.

COROLLARY 2.2. *Suppose that G, f_i $(i = 0, l,\dots, k)$ are twice continuously Frechet differentiable at x_0, and $(G'(x_0), f'_1(x_0),\dots, f_k(x_0))$ is a M_l — surjection of X onto $Y \times R^k$ with $M_l = \{0\} \times R^k$, where R^k denotes the non-positive orthant of R^k. Furthermore, there are numbers $\lambda_i > 0$ $(= i\ l,\dots, k)$ and a functional $y^* \in Y^*$ such that $G(x_0) = 0$, $f_i(x_0) = 0$ $(i = l, \dots, k)$ and*

$$\mathcal{L}'_x(x_0, \lambda_1, \dots, \lambda_k, y^*) = 0,$$

where
$$\mathcal{L}(x, \lambda_1, \dots, \lambda_k, y^*) = f_0(x) + \sum_{i=1}^{k} \lambda_i f_i(x) + \langle y^*, G(x) \rangle,$$

$$\mathcal{L}''_{xx}(x_0, \lambda_1, \dots, \lambda_k, y^*)(\xi,\xi) \geqslant \sigma \|\xi\|^2, \ (\text{for some } \sigma > 0,$$

$$\forall \xi \in Ker\ G'(x) \wedge \{\xi : \langle f'_i(x_0), \xi \rangle \leqslant 0, \ i = 1, \dots, k\})$$

Then x_0 is a local solution of Problem (III).

Proof. Since $\lambda_i > 0$ (i = 1, ..., k), the functional (y^*, λ_1 ..., λ_n) is uniformly positive with respect to the cone M_1. Thus all the hypotheses of Theorem 2.2 are satisfied, and therefore the Corollary follows.

3) Consider the mathematical programming problem:

$$\text{(IV)}\quad \begin{cases} \text{minimize } f_0(x), \\ \text{subject to:} \\ f_i(x) = 0, \ (i = l, \ldots, k) \\ f_j(x) \leqslant 0, (j = k + l, \ldots, n), \ x \in D, \end{cases}$$

where f_0, D are as in Problem (I), $f_i : D \to R$ (i = 1, ..., n).

COROLLARY 2.4. *Assume that f_i ($i = 0, 1, \ldots, n$) are twice continuously Fréchet differentiable at x_0, and the system $f'_l(x_0)$), ..., $f'_n(x_0)$ is linearly independent. Suppose in addition that there exist numbers $\lambda_1, \ldots, \lambda_n$ with $\lambda_i > 0$ ($i = k+l, \ldots, n$) such that $f_i(x_0) = 0$ ($i = 1, \ldots, n$),*

$$\mathcal{L}'_x(x_0, \lambda_1, \ldots, \lambda_n) = 0,$$

where

$$\mathcal{L}(x, \lambda_1, \ldots, \lambda_n) = f_0(x) + \sum_{i=1}^{n} \lambda_i f_i(x),$$

$$\mathcal{L}''_{xx}(x, \lambda_1, \ldots, \lambda_n)(\xi, \xi) \geqslant \sigma \ \|\xi\|^2 \ (\text{for some } \sigma > 0,$$

$$\forall \xi \in \{\zeta : < f'_i(x_0)\,\zeta > = 0, < f'_j(x_0), \zeta > \leqslant 0,$$

$$i = 1, \ldots, k; \ j = k + 1, \ldots, u\}$$

Then x_0 is a local solution of Problem (IV).

Proof. Since the sytem $f'_l(x_0)$, ..., $f'_n(x^0)$ is linearly independent, ($f'_l(x_0)$, ... $f'_n(x_0)$) is a surjection. Furthermore, the functional $\wedge = (\lambda_1, \ldots, \lambda_n)$ is uniformly positive with respect to the cone $M = \{0\} \times R_-^{n-k}$, where $\{0\} \subset R^k$, as $\lambda_j > 0$ (j = k + 1, ..., n). Thus all the hypotheses of Theorem 2.2 hold and therefore the Corollary follows.

In the theory on necessary and sufficient optimality conditions, a key point is to incorporate certain constraints onto the objective function. In this section we shall prove a theorem of this type, which implies a result of Ioffe and Tikhomirov [2].

Consider the following problem:

$$(\mathrm{I'}) \begin{cases} \mathcal{L}(x, \Lambda) \to \inf, \\ F(x) \leftarrow M, \end{cases}$$

where Λ is the in Assumption (ii).

It is of interest to note that if x_0 is a local solution of Problem (I') then x_0 is a local solution of Problem (I), but the converse does not hold in general (under Assumptions (ii), (iv)).

THEOREM 3.1. *a) Let there be given f_0, F satisfying Assump'ion (i), (ii) (iv) Assume that the mapping F is M — regular at x_0. Then, any local solution x_0 of Problem (I'), is a local solution of the following problem for every number 0 $\beta > 0$:*

$$(\mathrm{V}) \qquad f_0(x) + < \Lambda \cdot F(x) > + \beta \| F(x) \| - f_0(x_0) \to \inf$$

b) It the mapping F satisfies Assumption (iii), then any local solution x_0 of Problem (V), is a local solution of Problem (I) here $\beta \leftarrow (0, \gamma)$, γ is the number in Assumption (iii)).

Proof. a) First, suppose that x_0 is a local solution of Problem (I') i.e. there exists a neighbourhood U_1 of x_0 such that for every $x \in U_1$ satisfying. $F(x) \in M$,

$$\mathcal{L}(x, \Lambda) - \mathcal{L}(x_0, \Lambda) \geqslant 0. \tag{3.1}$$

Putting $p(x) = f_0(x) + \langle \Lambda, F(x) \rangle - f_0(x_0)$, we get

$$p(x) = \mathcal{L}(x, \Lambda) - f_0(x_0) \tag{3.2}$$

For any positive number $\theta < 1$, we can take a number α so that $\alpha \; \| F_1^{-1} \| = \theta <$ < 1, where $F_1(x) = F'(x_0)x - M$, such that the mapping $F(x_0) + F'(x_0)(x - x_0)$ is in the (U_2, α) — Lipschitz proximity of $F(x)$ for some neighbourhood U_2 of x_0 $(U_2 \subset U_1)$.

By Assumptions (ii), (iv) it follows from (3. 2) that $p(x_o) = 0$, $p'(x_o) = 0$. Because of Assumption (i), $p'(x)$ is continuous in a neighbourhood of x_o. Hence for an arbitrary number $\beta > 0$, there exists a neighbourhood U_3 of x_o ($U_3 \subset U_2$ such that for every $x, x' \in U_3$,

$$|p(x) - p(x')| \leqslant \frac{\beta}{c} \| x - x' \| \left(c = \frac{2 \| F_1^{-1} \|}{1 - \theta}\right) \tag{3. 3}$$

By Lemma 2. 1 applied to the mapping F there is a neighbourhood U of x_o ($U \subset U_3$) such that for every $x \in U$ there exists $\widehat{x} \in U$ satisfying $F(\widehat{x}) \in M$ and

$$\| \widehat{x} - x \| \leqslant C \| F(x) \| \tag{3. 4}$$

From (3. 1) it follows that $p(\widehat{x}) \geqslant 0$, hence by (3. 3), (3. 4) we have for every $x \in U$,

$$p(x) \geqslant p(\widehat{x}) - p(x) \geqslant -\frac{\beta}{C} \| \widehat{x} - x \| \geqslant -\beta \| F(x) \| ,$$

This implies

$$f_0(x) - f_0(x_0) + < \Lambda, F(x) > + \beta \| F(x) \| \geqslant 0 \tag{3. 5}$$

Therefore x_0 is a local solution of Problem (V).

b) Now assume that x_0 is a local solution of Problem (V), so there exists a neighbourhood U of x_0 such that for every $x \in U$ (3. 5) holds.

With $F(x) \in M$ it follows from Assumption (iii) that

$$- < \Lambda, F(x) > \geqslant \beta \| F(x) \| .$$

whenever $0 < \beta \leqslant \gamma$.

Hence, for every admissible point $x \in U$ of Problem (I),

$$f_0(x) \geqslant - < \Lambda, F(x) > - \beta \| F(x) \| + f_0(x_0) \geqslant f_0(x_0)$$

This means that x_0 is a local solution of problem(I)

Remark. In the case where the mapping F is regular at x_0, Problem (I') can be replaced in Theorem 3.1a by Problem (I). Thus if F—regular at x_0, then x_0 is a local solution of Problem (I) if and only if x_0 is a local solution of Problem (V) (whenever $o < \beta \leqslant \gamma$).

THEOREM 3. 2. *Let there be given f_0, F satisfying Assumptions (i) — (iv). Suppose that the mapping F is M — regular at x_0 and x_0 is a local solution of Problem (I') (thus x_0 is also local solution of Problem (I)). Then for each $\beta \in (o, \gamma)$, the function φ_β below is a local* **K** *— function of Problem (I) at x_0:*

$$\varphi_\beta(x) = f_0(x^0) - < \wedge, F(x) > -\beta \parallel F(x) \parallel , \qquad (3.6)$$

where γ is the number in Assumption (iii).

Proof. We shall check the three conditions of a local K—function to $\varphi_\beta(x)$.

In view of Assumption (iv) it follows from (3.6) that for any β, $\varphi_\beta(x_0) = f_0(x_0)$.

For an arbitrary admissible point **x**, we obtain by virtue of Assumption (iii) that

$$- < \wedge, F(x) > \geqslant \beta \parallel F(x) \parallel ,$$

whenever $o < \beta \leqslant \gamma$,
from which it follows by (3.6) that

$$\varphi_\beta(x) \geqslant \varphi_\beta(x_0) = f_0(x_0).$$

Thus the condition a), b) of a local K—function hold for $\varphi_\beta(\mathbf{x})$.

To prove condition c) we observe that

$$f_0(x) - \varphi_\beta(x) = f_0(x) - f_0(x_0) + < \wedge, F(x) > + \beta \parallel F(x) \parallel$$

Since x_0 is a local solution of Problem (I'), it is also a local solution Problem (V) by Theorem 3.la. This means that there is a neighbourhood U of x_0 such that

$$f_0(x) - f_0(x) + < \wedge, F(x) > + \beta \parallel F(x) \parallel \geqslant 0,$$

whenever $x \in U$, $o < \beta \leqslant \gamma$
In other words,

$$f_0(x) - \varphi_\beta(x) \geqslant 0.$$

This completes the proof.

COROLLARY 3.1. (Ioffe and Tikhomirov [2])

Assume that the mapping F is regular at $x_0 \in D$, x_0 is an admissible point of Problem (II). Then x_0 is a local solution of Problem (II) if and only if there is $\wedge \in Y^*$ such that for every $\beta > o$ the function $\varphi_\beta(x)$ below is a local **K** *— function of Problem (II) at x_0:*

$$\varphi_\beta(u) = f_0(x_0) - < >, F(x) > - \beta \parallel F(x) \parallel$$

Proof. In this case we see that x_0 is a local solution of Problem (I) if and only if x_0 is a local solution of Problem (I'). According to the Lagrange multiplier principle, there exists $\wedge \in Y^*$ such that $\mathcal{L}'_x\ (x_0, \wedge) = 0$. Taking $M = \{0\}$, Assumption (iii) holds for $\wedge$ and any number $\gamma > 0$. The Corollary now follows from Theorem 3.2.

In what follows, we shall give an example in which a solution of the problem under consideration is easily derived from Theorem 3.1.

Example 3.1. Consider the problem (with $X = Y = R^3$)

$$
\text{(VI)}\quad \begin{cases}
f_0(x_1, x_2, x_3) = 3x_1 + 2x_2 + x_3 + x_1^2 + 2x_2^4 \to inf, \\
f_1(x_1, x_2, x_3) = -x_1 - x_2 - x_3 + x_1^2 \leqslant 0, \\
f_2(x_1, x_2, x_3) = -x_1 - x_2 + x_3^2 \leqslant 0, \\
f_3(x_1, x_2, x_3) = -x_1 - x_2^4 \leqslant 0.
\end{cases}
$$

We have

$$f_0'\ (0, 0, 0) = (3, 2, 1), \quad f_1'\ (0, 0, 0) = (-1, -1, -1),$$

$$f_2'(0, 0, 0) = (-1, -1, 0), f_2'\ (0, 0, 0) = (-1, 0, 0)$$

It is easily seen that the Kuhn Tucker necessary optimality condition holds with Lagrange multipliers $\lambda_1 = \lambda_2 = \lambda_3 = 1$ and Assumption (iii) is satisfied with $\gamma = 1$. Since the system f_1', f_2', f_3' is linearly independent, the regularity condition holds.

Then by the Remark to Theorem 3.1, Problem (VI) is equivalent to the following one (in a neighbourhood of the point 0):

$$f_0 + f_1 + f_2 + f_3 + \sqrt{f_2^1 + f_2^2 + f_3^2} - f_0(0) \to inf,$$

which means that

$$g(x_1, x_2, x_3) = 2x_1^2 + x_2^4 + x_3^2 +$$

$$+ \sqrt{\left(x_1 + x_2^4\right)^2 + \left(x_1 + x_2 - x_3^2\right)^2 + \left(x_1 + x_2 + x_3 - x_1^2\right)^2} \to inf.$$

It follows that the point $(0, 0, 0)$ is a local solution of Problem (VI) because

$$g(x_1, x_2, x_3) \geqslant 0 \qquad (\forall (x_1, x_2, x_3) \in R^3)$$

4. SOME NECESSARY AND SUFFICIENT CONDITION FOR PROBLEM (III)

Let us now consider Problem (III) mentioned in Section 2:

$$\text{(III)} \begin{cases} f_0(x) \to \inf, \\ f_i(x) \leqslant 0, \ i = 1,\dots, k, \\ G(x) = 0, \end{cases}$$

Here $f_i(x) = \psi_i[p_i(x)]$ $(i = 0, 1,\dots, k)$, p_i is a mapping from D into a Banach space Y_i, ψ_i a sublinear continuous functional on Y_i, and $G : D \to Y$.

Suppose $p_0, \dots, p_k$, G are continuously Fréchet differentiable in a neighbourhood of $x_0 \in D$ and that the mapping G is regular at x_0.

Under these hypotheses, Dubovitsky – Milyutin [6] have shown that if x_0 is a local solution of Problem (I), then there exist numbers α_i, and linear continuous functionals $y_i^* \in Y_i^*$ $(i = 0, 1, \dots, k)$, $y^* \in Y^*$ such that.

$$\begin{cases} \alpha_i \geqslant 0, < y_i^*, y_i > \leqslant \psi_i(y_i) \ (\forall y_i \in Y_i ; i = 0, 1,\dots, k) & (4.1) \\ < y_i^*, p_i(x_0) > = \psi_i[p_i(x_0)] \ (i = 0, 1,\dots, k), & (4.2) \\ \alpha_i f_i(x_0) = 0 \quad (i = 1,\dots, k), & \\ \sum_{i=0}^{k} \alpha_i [p_i'(x_0)]^* y_i^* + [G'(x_0)]^* y^* = 0, & (4.4) \\ \sum_{i=0}^{k} \alpha_i = 1 & (4.5) \end{cases}$$

Denote by Ω_0 the set of all $\lambda = (\alpha_0,\dots, \alpha_k, y_0^*,\dots y_k^*, y^*)$ satisfying (4.1) – (4.5) and for each number $\eta > 0$, denote by Ω_η the set of all λ satisfying the following:

$$\begin{cases} \alpha_i \geq 0, \langle y_i^*, y_i \rangle \leqslant \psi_i (y_i) \quad (\forall\, y_i \in Y_i\,;\, i = 0, 1,\dots, k) & (4.6) \\ \langle y_i, p_i(x_0) \rangle - f_i(x_0) \geqslant -\eta \qquad (i = 1,\dots k) & (4.7) \\ \alpha_i f_i(x) = 0 \qquad (i = 1,\dots, k) & (4.8), \\ \| \sum_{i=0}^{k} \alpha_i [p_i'(x_0)]^* y_i^* + [G'(x_0)]^* y^* \| \leqslant \eta, & (4.9) \\ \sum_{i=0}^{k} \alpha_i = 1 & (4.10 \end{cases}$$

In [5] Levitin, Milyutin, Osmolovsky have proved that Ω_η is convex weakly* compact.

Note that since G is regular at x_0, by virtue of Lyusternik's theorem, there exists a neighbourhood U of x_0, a number $C > 0$ and a mapping $\chi : U \to X$ such that

$$\begin{cases} \| \chi(x) \| \leqslant C \| G(x) - G(x_0) \|, & (4.11) \\ G(x + \chi(x)) = G(x_0) & (4.12) \end{cases}$$

We now consider an admissible x_0 of Problem (III).

THEOREM 4. 1. *x_0 is a local solution of Problem (III) if and only if for each $\varepsilon > 0$, the function $\varphi(x)$ below is a local K-function of Problem (I) at x_0:*

$$\varphi(x) = f_0(x) - \alpha_0 [\langle y_0^*, p_0(x_0 + \chi(x)) \rangle - f_0(x)] -$$

$$- \sum_{i=1}^{k} \alpha_i \langle y_i^*, p_i(x_0 + \gamma(x)) \rangle - \langle y^*, G(x_0) + \chi(x) \rangle - \varepsilon \| \chi(x) \|, \text{ where}$$

$$\lambda = (\alpha_0,\dots, \alpha_k, y_0^*,\dots, y_k^*, y^*) \in \Lambda_0 \tag{4.13}$$

Proof. It suffices to prove the necessity because the sufficiency follows from a result of Ioffe and Tikhomirov [2].

By (4.11), it follows from (4.2), (4.3) that condition a) for a local K-function for φ holds.

Now let x be an admissible point of Problem (III). By (4.11) we get $\varphi(x) = f_0(x)$. Because x_0 is a local minimum of Problem (III), there exists a neighbourhood U of x_0 such that for every admissible point $x \in \mathring{U}$:

$$f_0(x) - f_0(x_0) \geqslant 0.$$

This implies

$$\varphi(x) - \varphi(x) \geqslant 0,$$

i. e. condition b) of a local K-function for φ holds.

Setting

$g(x) = \alpha_0 [\langle y_0^*, p_0(x)\rangle - f_0(x)] + \sum_{i=1}^{k} \alpha_i \langle y_i^*, p_i(x)\rangle + \langle y^*, G(x)\rangle$, by virtue of the differentiability of, p_i, G, for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any x satisfying $\| x - x_0 \| < \delta$,

$$| g(x) - g(x_0) - \langle g'(x_0), x - x_0 \rangle | \langle \varepsilon \| x - x_0 \| \tag{4.14}$$

Since G is continuous at x_0, for $\delta > 0$ there is $\delta_1 > 0$ such that $\| x - x_0 \| < \delta_1$ implies $\| G(x) \| < \dfrac{\delta}{C}$ (we can choose δ_1 so that $B(x_0, \delta_1) \subset B(x_0, \delta) \subset U$).

It follows from (4.11) that $x + \chi(x) \in B(x_0, \delta)$, Hence by (4.14),

$| g(x_o + \chi(x)) - g(x_o) - < g'(x_o), \chi(x) > | \leqslant \varepsilon \| \chi(x) \|$, which implies by (4.2) — (4.4) that

$$g(x_o + \chi(x)) + \varepsilon \cdot \| \chi(x) \| \geqslant g(x_o) + < g'(x_o), \chi(x) > =$$

$$= \alpha_o [< y_o^*, P_o(x_o) > - f_o(x_o)] + \sum_{i=1}^{k} \alpha_i < y_i^*, P_i(x) > + < y^*, G(x_o) > +$$

$$+ < \alpha_o [P_o'(x_o)]^* y_o^*, \chi(x) > + \sum_{i=1}^{k} \alpha_i < [P_i'(x_o)]^* y_i^*, \chi(x) > +$$

$$+ < [G'(x_o)]^* y^*, \chi(x) > + < \sum_{i=o}^{k} \alpha_i [P_i'(x_o)]^* y_i^* + [G'(x_o)]^* y^*, \chi(x) > = 0$$

Thus

$$f_o(x) - \varphi(x) = g(x_o + \chi(x) + \varepsilon \| \chi(x)) \| \geqslant 0 \ (\forall x \in B(x_o, \delta_1)).$$

Therefore φ is a local K-function of Problem (III) at x_o.

THEOREM 4. 2. *x_o is a local solution of Problem (III) if and only if for each number $\eta > o$, the function φ_η below is a local K-function of Problem (III) at x_o:*

$$\varphi_\eta(x) = f_o(x) - \max_{\Omega_\eta} \{ \alpha_o[< y_o^*, P_o(x) > - f_o(x_o)] + \sum_{i=1}^{k} \alpha_i < y_i^*, P_i(x) + $$

$$+ < y^*, G(x) > \}$$

Proof. As with Theorem 4.1, it suffices to prove the necessity. In [5] it has been shown that

$$\max_{\Omega_\eta} \{ \alpha_o[< y_o^*, P_o(x_o) > - f_o(x_o)] + \sum_{i=1}^{k} \alpha_i < y_i^*, P_i(x_o) > +$$

$$+ < y^*, G(x_o) > \} = 0$$

Hence condition a) of a local K-function for φ_η holds.

By virtue of the compactness of Ω_η, there exists

$$\bar{\lambda} = (\bar{\alpha}_o, \dots \bar{\alpha}_R, \bar{y}_o^*, \dots, \bar{y}_R^*, \bar{y}^*) \in \Omega_\eta \text{ such that}$$

$$\varphi_\eta(x) = f_o(x) - \bar{\alpha}_o [< \bar{y}_o^*, P_o(x) > - f_o(x_o)] - \sum_{i=1}^{k} \bar{\alpha}_i < \bar{y}_i^*, P_i(x) > -$$

$$- \bar{y}^*, G(x) >$$

Since $< \bar{y}_i^*, y_i > \leqslant \psi_i(y_i)$ $(\forall y_i \in y_i)$, for any admissible point x, one has

$$-\bar{\alpha}_o [< \bar{y}_o^*, P_o(x) > - f_o(x_o)] \geqslant -\bar{\alpha}_o [\psi_o(P_o(x_o)) - f_o(x_o)] =$$

$$= -\bar{\alpha}_o [f_o(x) - f_o(x_o)] - \bar{\alpha}_i < \bar{y}_i^*, P_i(x) > \geqslant -\bar{\alpha}_i \psi_i [P_i(x)] = -\bar{\alpha}_i f_i(x) \geqslant$$

$\geqslant 0$ $(i = 1, \dots, k)$, which implies that

$$\varphi_\eta(x) - \varphi_\eta(x_o) \geqslant (1 - \bar{\alpha}_o) [f_o(x) - f_o(x_o)].$$

It follows from (4.10) that $1 - \bar{\alpha}_o \geqslant 0$. Therefore, for every admissible x belonging to some neighbourhood U of x_0,

$$\varphi_\eta(x) - \varphi_\eta(x_o) \geqslant 0,$$

as x_o is a local minimum of Problem (III). Thus Condition b) of a local K-function for φ_η holds.

In [5] it has been proved that for every x belonging to some neighbourhood V of x_o (which can be chosen so that $V \subset U$),

$$\max_{\Omega_\eta} \{ \alpha_o[<y_o^*, P_o(x)> - f_o(x)] + \sum_{i=1}^{k} \alpha_i < y_i^*, P_i(x)> + <y^*, G(x)>\} \geqslant 0.$$

Hence $fo(x) - \varphi_\eta(x) \geqslant 0$ ($\forall\, x \in V$)

Therefore φ_η is a local K-function of Problem (III) at x_o.

By a proof similar to that of Theorem 4.2, using results in [5] we obtain the following theorem.

THEOREM 4.3. *x_o is a local solution of Problem (III) if and only if there exists a number $\eta_o \geqslant o$ such that for every number η satisfying $o \leqslant \eta \leqslant \eta_o$, the function below is a local K-function of Problem (III) at x_o*

$$\varphi_\eta(x) = f_o(x) - \max_{\Omega_o} \{ \alpha_o[< y_o^*, P_o(x) > - f_o(x)] + \sum_{i=1}^{k} \alpha_i < y_i^*, P_i(x) > +$$

$$+ < y^*, G(x) > \} - \eta\, \sigma(x)$$

where $\sigma(x) = [f_o(x) - f_o(x)]^+ - \sum_{i=1}^{k} f_i(x) + \| G(x) \|$, $f^+ = \max \{ f, 0 \}$.

Finally we derive a second-order necessary condition for Problem (III) which contains as a special case a result in [4].

THEOREM 4. 4. *Suppose that f_0, ... f_k, G are twice continuously Fréchet differentiable in a neighbourhood of x_0, G—regular at x_0, x_0 is a local solution of Problem (III). Then for every ξ belonging to the set*

$$T = Ker\, G'(x_o) \cap \{ \zeta : < f_i'(x_o), \zeta > \leqslant 0, i = 1, ..., k\},$$

there exist numbers $\overline{\alpha_0}(\xi) \geqslant 0$, ..., $\overline{\alpha}_k(\xi) \geqslant 0$, and $\overline{y}^*(\xi) \in Y^*$ such that

$$L''_{xx}(x_0, \overline{\alpha_0}(\xi), ..., \overline{\alpha}_k(\xi), \overline{y}^*(\xi))(\xi, \xi) \geqslant 0 \;(\forall\, \xi \in T)$$

where $\mathcal{L}(x, \alpha_0, \alpha_k, y^*) = \sum_{i=0}^{k} \alpha_i f_i(x) + \langle y^*, G(x) \rangle$

Proof. For any $\xi \in T$, by virtue of the compactness of Ω_0 there exists $\overline{\lambda}(\xi) = (\overline{\alpha}_0(\xi), \ldots, \overline{\alpha}_k(\xi), \overline{y}^*_i(\xi), \ldots, \overline{y}^*_k(\xi), \overline{y}^*(\xi)) \in \Omega_0$ such that

$$Q_0(\xi) = \max_{\Omega_0} \{ \alpha_0 [\langle y_0^*, p_0(x_0+\xi)\rangle - f_0(x_0)] + \sum_{i=1}^{k} \alpha_i \langle y_i^*, p_i(x_0+\xi)\rangle + \langle y^*, G(x_0+\xi)\rangle \} = \overline{\alpha}_0(\xi) [\langle \overline{y}^*_0(\xi), p_0(x_0+\xi)\rangle - f_0(x_0)] + \sum_{i=1}^{k} \overline{\alpha}_i(\xi) \langle \overline{y}^*(\xi), p_i(x_0+\xi)\rangle + \langle \overline{y}^*(\xi), G(x_0+\xi)\rangle \quad (4.15)$$

Assume now that there exists $\xi_0 \in T$ such that $\|\xi_0\| = 1$, but

$$\mathcal{L}''_{xx}(x_0, \overline{\alpha}_0(\xi_0), \ldots, \overline{\alpha}_k(\xi_0), \overline{y}^*(\xi_0))(\xi_0, \xi_0) = -\gamma < 0$$

Choose $\eta > 0$ so that: $\eta \| G''(x_0)(\xi_0)\| < \frac{\gamma}{8}$,

$$\eta \| f_0''(x_0)(\xi_0, \xi_0)\| < \frac{\gamma}{8}, \quad \eta \| \sum_{i=1}^{k} f_i''(x_0)(\xi_0, \xi_0)\| < \frac{\gamma}{8}. \quad (4.16)$$

By virtue of the differentiability of f_i and G, there exists a number $\delta > 0$ such that for every x satisfying $\|x\| < \delta$,

$$\| G(x_0+x) - G(x_0) - G'(x_0)x - \frac{1}{2} G''(x_0)(x, x) \| < \frac{\gamma \|x\|^2}{16}, \quad (4.17)$$

$$| f_0(x_0+x) - f_0(x_0) - \langle f_0'(x_0), x\rangle - \frac{1}{2} f_0''(x_0)(x,x)| < \frac{\gamma \|x\|^2}{16}, \quad (4.18)$$

$$| g(x_0+x) - g(x_0) - \langle g'(x_0), x\rangle - \frac{1}{2} g''(x_0)(x, x)| < \frac{\gamma \|x\|^2}{16}, \quad (4.19)$$

$$(\text{where } g = \sum_{i=1}^{k} f_i),$$

$$(\mathcal{L}(x_0+x, .) - \mathcal{L}(x_0, .) - \langle \mathcal{L}'_x(x_0, .), x\rangle - \frac{1}{2} \mathcal{L}''_{xx}(x_0, .)(x, x)| < \frac{\gamma \|x\|^2}{16} \quad (4.20)$$

Setting $\overline{\mathcal{L}}(x,.) = \mathcal{L}(x,.) - \overline{\alpha_0}(\xi_0) f_0(x_0)$ and noting that $\overline{\Lambda}(\xi_0) \in \Omega_0$,

one has $\overline{\mathcal{L}}(x_0, \overline{\alpha_0}(\xi_0), ..., \overline{\alpha_k}(\xi_0), \overline{y^*}(\xi_0)) = 0, \overline{\mathcal{L}}'_x(x_0, \overline{\alpha_0}(\zeta_0), ..., \overline{y^*}(\xi_0)) =$

$=$ and

$$\mathcal{L}'_x(x_0, \overline{\alpha_0}(\xi_0), ..., \overline{y^*}(\xi_i)), \ \overline{\mathcal{L}}''_{xx}(x_0,.) = \mathcal{L}''_{xx}(x_0,.)$$

$$|\overline{\mathcal{L}}(x_0+x,.) - \overline{\mathcal{L}}(x_0,.) - \langle \mathcal{L}'_x(x_0,.), x\rangle - \frac{1}{2}\mathcal{L}''_{xx}(x_0,.)| <$$

$$< \frac{\gamma \| x \|^2}{16} \tag{4.21}$$

Hence for t satisfying $0 < t < \delta$, $\| t\, \xi_0 \| < \delta$ and for the function

$\sigma(x) = [f_0(x_0+x) - f_0(x)]^+ + \sum_{i=1}^{k} f_i(x_0+x) + \| G(x_0+x) \|$ (where

$f^+ = \max\{f, 0\}$), it follows from (4.16) — (4.19) that

$$\eta\sigma(t\,\xi_0) = \eta[f_0(x_0+t\xi_0) - f_0(x_0)]^+ + \eta \sum_{i=1}^{k} fi(x_0+t\xi_0) + \eta \| G(x_0+t\xi_0) \| < \frac{3\gamma t^2}{8} \tag{4.22}$$

From this and (4.21) we get

$$\overline{\mathcal{L}}(x_0+t\xi_0,.) + \eta\sigma(t\,\xi_0) < -\frac{\gamma t^2}{2} + \frac{\gamma t^2}{16} + \frac{3\gamma t^2}{8} = -\frac{\gamma t2}{16} < 0 \tag{4.23}$$

It follows from (4.1), (4.2), (4.23) that

$$Q_0(t\xi_0) + \eta\sigma(t\xi_0) \leqslant \overline{\alpha}_0(\xi_0)[\psi_0\big(P_0(x_0+t\xi_0)\big) - f_0(x_0)] +$$

$$+ \sum_{i=1}^{k} \overline{\alpha_i}(\xi_0)\, \psi_i\,[P_i(x_0+t\xi_0)] + \langle \overline{y^*}(\xi_0) G(x_0+t\xi_0)\rangle + \eta\sigma(t\xi_0) =$$

$$= \overline{\mathcal{L}}(x_0+t\xi_0, \overline{\alpha_0}(\xi_0), ..., \overline{y^*}(\xi_0)) + \eta\sigma(t\xi_0) < 0 \tag{4.24}$$

On the other hand, it has been proved in [5] that for every **x** belonging to some neighbourhood of 0:

$$Q_0(x) + \eta\sigma(x) \geqslant 0.$$

This implies that for t small enough,

$$Q_0(t\xi_0) + \eta\sigma(t\xi_0) \geqslant 0.$$

contradicting (4.24). The proof is complete.

COROLLARY 4.1 see [4].

Under the hypotheses of Theorem 4. 4, if x_0 *is a local solution of Problem (III), then for every* ξ *belonging to the set*

$$T_1 = \{ \xi : G'(x_0)\xi = 0, < f'_i(x_0), \xi > = 0, i = 0, 1, ..., k \},$$

there exist numbers $\alpha_i(\xi) \geqslant 0$, $i = 0, 1, ..., k$, $y^*(\xi) \in Y^*$ *such that*

$$\mathcal{L}''_{xx}(x_0, \alpha_0(\xi), ..., y^*(\xi))(\xi,\xi) \geqslant 0 \ (\forall \xi \in T_1)$$

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