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ON THE SIMPLICITY OF OPERATOR KNOTS

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The problem of the complete nonunitariness of contractions was studied in [1] and [2]. In this paper we obtain a general criterion for the complete nonunitariness of an arbitrary contraction. The obtained results are applied to the study of concrete model operators.

1. INTRODUCTION

Let H, E be Hilbert spaces, F, G bounded operators from E into H, T, S contractions in H, E, respectively. The totality

$$\alpha = \begin{pmatrix} H & T & H \\ F & & G \\ E & S & E \end{pmatrix}$$

is called the operator knot (cf. [5]) if

$$I - T^* T = G G^*, \; I - S^* S = G^* G, \; TG = FS,$$
$$I - T T^* = F F^*, \; I - S S^* = F^* F.$$

The knot α is called simple if the closed linear span H_0 of the vectors $T^n Fe$, $(T^*)^n Ge$ $(e \in E;\ n = 0, 1, \ldots)$ is the whole space H. H_0 is called the main subspace of the knot α and we have the simple following

LEMMA 1. *H_0 is the closed linear span of the vectors*

$$f_{\zeta e}\ (I - \zeta T)^{-1} Fe,\ g_{\zeta e} = (I - \zeta T^*)^{-1} Ge \ (e \in E,\ |\zeta| < 1).$$

LEMMA 2. *The following statements are equivalent:*

(i) α *is a simple knot,*

(ii) T *is a completely nonunitary contraction.*

The operator — function

$$\theta(\zeta) = S - \zeta E^* (I - \zeta T^*)^{-1} G$$

is called a characteristic function of α.

Let $\alpha_k = \begin{pmatrix} H_k & T_k & H_k \\ F_k & & G_k \\ E & S_k & E \end{pmatrix}$, (k = 1, 2) be knots with the common outer space E, then the knot

$$\alpha = \begin{pmatrix} H_1 \oplus H_2 & T_1P_1 + T_2P_2 - F_1G^*_2P_2 & H_1 \oplus H_1 \\ F_1S^*_1 + F_2 & & G_1 + G_2S_1 \\ E & S_2S_1 & E \end{pmatrix}$$

is called a product of α_1, α_2 and denoted by $\alpha = \alpha_2\alpha_1$.

In this work we will consider the product

$$\alpha = \begin{pmatrix} H & T & H \\ F & & G \\ E & S & E \end{pmatrix} = \alpha_n, \dots, \alpha_2\alpha_1$$

$$\text{of } n \text{ knots } \alpha_k = \begin{pmatrix} H_k & T_k & H_k \\ F_k & & G_k \\ E & S_k & E \end{pmatrix} \; (k = 1, 2, \dots n).$$

From the definition we have

$$H = H_1 \oplus H_2 \oplus \dots \oplus H_n,$$

$$(Tf)_k = T_k f_k - F_k \sum_{i=k+1}^{n} \Big(\overset{\curvearrowright}{\prod_{j=k+1}^{i-1}} S_j^* \Big) G_i^* f_i \tag{1. 1}$$

$$(Fe)_k = F_k \overset{\curvearrowright}{\prod_{j=k+1}^{n}} S_j e, \quad Ge = G_k \overset{\curvearrowleft}{\prod_{j=1}^{k-1}} S_j e, \quad S = \overset{\curvearrowleft}{\prod_{k=1}^{n}} S_k \tag{1. 2}$$

where $f = f_1 \oplus f_2 \oplus \dots \oplus f_n$ denotes an element of H.
And it is easy to see that

$$(f_{\zeta e})_k \ [(I-\zeta T)^{-1} Fe]_k = (I-\zeta T_k)^{-1} F_k R(k,\zeta)e \qquad (1.3),$$

$$(g_{\zeta e})_k = [(I-\zeta T)^{-1} Ge]_k = (I-\zeta T_k^*)^{-1} G_k \Phi(k,\zeta)e \qquad (1.4),$$

where

$$R(k,\zeta) = \overset{\curvearrowright}{\prod_{j=k+1}^{n}} \theta_j^*(\zeta), \quad \Phi(K,\zeta) = \overset{\curvearrowleft}{\prod_{j=1}^{k-1}} \theta_j(\zeta) \qquad (1.5).$$

Let $B_k = P_1 + P_2 + \dots + P_k$ i.e. B_k is the orthoprojector from $H = H_1 \oplus H_2 \oplus \dots \oplus H_n$ onto $H_1 \oplus H_2 \oplus \dots \oplus H_k$

LEMMA 3.

$$\left(B_k f_{\zeta e}, f_{\mu a}\right) = \frac{1}{1-\overline{\mu}\zeta}\left([R^*(k,\mu)R(k,\zeta) - R^*(0,\mu)R(0,\zeta)]e, a\right) \qquad (1.6)$$

$$\left(B_k, g_{\zeta e}, g_{\mu a}\right) = \frac{1}{1-\overline{\mu}\zeta}\left([I - \Phi^*(k+1,\mu)\Phi(k+1,\zeta)]e, a\right) \qquad (1.7)$$

$$\left(B_k, g_{\zeta e}, f_{\mu a}\right) = \frac{1}{\zeta-\overline{\mu}}\left([R^*(o,\mu) - R^*(k,\mu)\Phi(k+1,\zeta)]e, a\right) \qquad (1.8).$$

Proof. From (1.1), (1.2) we have

$$(B_k f_{\zeta e}, f_{\mu a}) =$$

$$= \sum_{j=1}^{k}\left(R^*(j,\mu)F_j^*(I-\mu T_j^*)^{-1}(I-T_j)^{-1}F_j R(j,\zeta)e, a\right)$$

but, on the other hand, the relation

$$F_j^*\left(I-\overline{\mu}T_j^*\right)^{-1}\left(I-\zeta T_j\right)^{-1}F_j = \frac{I-\theta_j(\overline{\mu})\theta_j^*(\overline{\zeta})}{1-\overline{\mu}\zeta}$$

holds [5], therefore

$$\left(B_k, f_{\zeta e}, f_{\mu a}\right) = \frac{1}{1-\overline{\mu}\zeta}\sum_{j=1}^{k}\left([R^*(j,\mu)R(j,\zeta) - R^*(j-1,\mu)R(j-1,\zeta)]e, a\right)$$

$$= \frac{1}{1-\overline{\mu}\zeta} ([R^*(k,\mu) R(k,\zeta) - R^*(0,\mu) R(0,\zeta)] e, a).$$

The relations (1.7) — (1.8) are proved analogously.

2. THE FUNCTIONAL MODEL

We will use the following functional model [5]. Let α be a knot with the characteristic function $\theta(\zeta)$. Then $\theta(\zeta)$ is also a characteristic function of the following simple knot

$$\widehat{\alpha} = \begin{pmatrix} \widehat{H} & \widehat{T} & \widehat{H} \\ \widehat{F} & & \widehat{G} \\ E & \widehat{S} & \widehat{E} \end{pmatrix}$$

where

$$\widehat{H} = [H^2(E) \oplus \overline{\Delta L^2(E)}] \ominus \{\theta u \oplus \Delta u;\ u \in H^2(E)\},$$

$$\widehat{T}\{\varphi \oplus \psi\} = \{e^{it}\ \varphi(e^{it}) \ominus \theta\ (e^{it}) C_{\varphi\psi} \oplus e^{it}\psi(e^{it}) - \Delta(e^{it}) C_{\varphi\psi}\},$$

$$C_{\varphi\psi} = \frac{1}{2\pi}\int_0^{2\pi} e^{it}[\theta^*(e^{it})\varphi(e^{it}) + \Delta(e^{it})\ \psi(e^{it})]\, dt,$$

$$\widehat{F} = \{(\theta(e^{it})S^* - I)e \oplus \Delta(e^{it}) S^* e\},$$

$$\widehat{G} = \{e^{-it}(\theta\ (e^{it}) - S)e \oplus e^{it}\Delta(e^{it})e\}.$$

$$\Delta(e^{it}) = \left(I - \theta^*(e^{it})\theta(e^{it})\right)^{\frac{1}{2}},$$

$$\widehat{S} = \theta(0) = S,\ e \in E,\ \{\varphi \oplus \psi\} \in \widehat{H}.$$

Hence the simple part of the knot α is unitarily equivalent to $\widehat{\alpha}$ by the following unitary transformation U

$$Uf_{\zeta e} = U(I - \zeta T)^{-1} Fe = (I - \zeta\widehat{T})^{-1}\widehat{F}e \equiv \widehat{f}_{\zeta e} \qquad (2.1),$$
$$Ug_{\zeta e} = U(I - \zeta T^*)^{-1} = (I - \zeta\widehat{T^*})^{-1}\widehat{G}e \equiv \widehat{g}_{\zeta e}$$

Let $\theta(\zeta) = \theta_2(\zeta)\theta_1(\zeta)$ be the regular factorization (cf. [6]) of the characteristic function $\theta(\zeta)$ of the knot α. Then we have the following invariant subspace for $\widehat{T}$

$$\widehat{H}_1 = \{\theta_2 u \oplus Z^{-1}(\Delta_2 u \oplus v) : u \in H^2(E),\ v \in \overline{\Delta_1 L^2(E)}\} \ominus$$

$$\ominus \{\theta w \oplus \Delta_w : w \in H^2(E)\},$$

where Z is an unitary operator mappings $\Delta L^2(E)$ onto $\overline{\Delta_2 L^2(E)} \oplus \overline{\Delta_1 L_2(E)}$ and defined (cJ. [6]) by

$$Z(\Delta v) = \Delta_2 \theta_1 v \,\omega\, \Delta_1 v$$

Let P_1 denote the orthoprojector from $\widehat{H}$ onto $\widehat{H}_1$. Then we have the following (cf. [4]):

LEMMA 4.

$$(\widehat{P}\,\widehat{F}_{\zeta e}, \widehat{F}_{\mu a}) = \frac{1}{1-\overline{\mu}\zeta}\Big(\theta_2(\overline{\mu})\,\theta_2^*(\overline{\zeta}) - \theta(\overline{\mu})\,\theta^*(\overline{\zeta})\Big]e,\ a\Big),$$

$$(\widehat{P}_1\,\widehat{g}_{\zeta e}, \widehat{g}_{\mu}) = \frac{1}{1-\overline{\mu}\zeta}\Big(\Big[I - \theta_1^*(\mu)\,\theta_1(\zeta)\Big]e,\ a\Big),$$

$$(\widehat{P}_1\,\widehat{g}_{\zeta e}, \widehat{F}_{\mu a}) = \frac{1}{\zeta-\overline{\mu}}\Big(\Big[\theta(\overline{\mu}) - \theta_2(\overline{\mu})\,\theta_1(\zeta)\Big]e,\ a\Big).$$

3. SIMPLICITY OF KNOTS

Let there be given n knots $\alpha_k = \begin{pmatrix} H_k & T_k & H_k \\ F_k & & G_k \\ F & S_k & E \end{pmatrix}$

and $\alpha = \alpha_n \ldots \alpha_1 \alpha_2$. Then for the characteristic functions we have

$$\theta(\zeta) = \theta_n(\zeta) \ldots \theta_2(\zeta)\,\theta_1(\zeta) = \overset{n}{\underset{k=1}{\overset{\curvearrowright}{\Pi}}} \theta_k(\zeta).$$

Using the notation (1.5) we will write

$$\theta(\zeta) = R^*(k, \overline{\zeta})\,\Phi(k+1, \zeta).$$

THEOREM 5. *The knot $\alpha = \alpha_n \ldots \alpha_2 \alpha_1$ is simple if and only if for $k = 1, 2, \ldots$ in the following conditions hold:*

(i) *the factorization (3. 1) are regular,*

(ii) *the vectors* $P_k f_{\zeta e}, P_k g_{\zeta e}$ $(e \in E, |\zeta| < 1)$ *are dense in* H_k

Proof. *Sufficiency.* By means of its characteristic function the functional mode $\widehat{\alpha}$ of the simple part of the knot α is built according to the scheme of 2. From the regularity of the factorization (3. 1) and Lemma 4 we have

$$(\widehat{P}_1 \widehat{F}_{\zeta e}, \widehat{F}_{\mu a}) = \frac{1}{1-\overline{\mu}\zeta} [R^*(k, \mu) R(k, \zeta) - \theta(\overline{\mu}) \theta(\overline{\rho})] e, a),$$

$$(\widehat{P}_1 \widehat{g}_{\zeta e}, \widehat{F}_{\mu a} \ \frac{1}{1-\overline{\mu}\zeta} [I - \varphi^*(k+1, \mu) \varphi(k+1, \zeta)] e, a)$$

$$(\widehat{P}_1 \widehat{g}_{\zeta e}, \widehat{g}_{\mu a}) = \frac{1}{\zeta - \overline{\mu}} ([\theta(\overline{\mu}) - R^*(k, \mu) \varphi(k+1, \zeta)] e, a)$$

Comparing these relations with the (1. 6), (1. 7), (1. 8) we have

$$(\widehat{P}_1 \widehat{g}_{\zeta e}, \widehat{F}_{\mu a} B_k F_{\zeta e}, F_{\mu a}),$$

$$(\widehat{P}_1 \widehat{g}_{\zeta e}, \widehat{g}_{\mu a}) = (B_k g_{\zeta e}, g_{\mu a}),$$

$$(\widehat{P}_1 \widehat{g}_{\zeta e}, \widehat{F}_{\mu a}) = (B_k g_{\zeta e}, F_{\mu a}).$$

Taking account of Relations (2. 1) (2. 2) and the fact that the elements $\widehat{F}_{\zeta e}, \widehat{g}_{\zeta e} (e \in E, |\zeta| < 1)$ are dense in $\widehat{H}$ we get

$$\widehat{P}_1 = U B_k U^*.$$

As $\widehat{P}_l$, B_k are orthoprojectors and U is unitary, the subspace $U^* \widehat{H} = H_o$ is invariant under B_k. On the other hand, since $f_{\zeta e} = U^* \widehat{f}_{\zeta e}$, $g_{\zeta e} = U^* \widehat{g}_{\zeta e}$ it follows that $B_k f_{\zeta e} \in U^* \widehat{H}$, $B_k g_{\zeta e} \in U^* \widehat{H}$ $(k = 1, 2, \dots, n)$. Therefore,

$$P_k f_{\zeta e} = (B_k - B_{k-1}) f_{\zeta e} \in U^* \widehat{H} = H_o \qquad (3.2),$$

$$P_k g_{\zeta e} = (B_k - B_{k-1}) g_{\zeta e} \in U^* \widehat{H} = H_o \qquad (3.3)$$

We shall now show that the knot α is simple. For this, it suffices to prove that $U^* \widehat{H} = H$. Let $U^* \widehat{H} \neq H$ i.e. $\exists h \neq 0, h \perp U^* \widehat{H}$ $(h = h_1 \oplus h_2 \oplus \dots \oplus h_n \in H)$.

Then from (3.2) (3.3) it follows that $h \perp P_k f_{\zeta e}$, $h \perp P_k g_{\zeta e}$. The last equation means that $h_k \perp P_k f_{\xi_e}, \perp P_k g_{\xi e}$ $(e \in E, |\xi| < 1)$, which under the conditions of the theorem implies that $h_k = 0$ (k = 1,2,.., n), and consequently $h = 0$.

Necessity. Let α be a simple knot. We can write

$$\alpha = \alpha'\alpha''$$

where

$$\alpha' = \alpha_n \dots \alpha_{k+2}\alpha_{k+1}, \alpha'' = \alpha_k \dots \alpha_2 \alpha_1,$$

with characteristic functions

$$\theta'(\zeta) = R^*(k, \bar{\zeta}), \theta''(\zeta)\ \Phi(k+1, \zeta) \text{ respectively.}$$

Then by Theorem 2 [5] the factorization (3.1) is regular. Suppose that the condition (ii) of the theorem does not hold, i.e.

$\exists h_k \in H_k$, $h_k \neq 0$; $h_k \perp P_{\zeta e}$; $h_k \perp P_k f_{\zeta e}$ $(e \in E, |\zeta| < 1)$. This means that $h = (0, \dots 0, h_k, 0, \dots, 0) \perp f_{\xi e}, g_{\xi e}$. Thus $f_{\zeta e}, g_{\xi e} (e \in E, |\zeta| < 1)$ are not dense in H, which contradicts the simplicity of the knot α. The proof is complete.

Incidentally we have proved the following

THEOREM 6. *If the factorizations (3.1) are regular for $k = 1, 2, \dots, n$ then the main subspace H_0' of the knot $\alpha = \alpha_n \dots \alpha_2\alpha_1$ is invariant with respect to orthoprojectors P_k $(k = 1, 2, \dots n)$.*

4 THE COMPLETELY NONUNITARY CONTRACTION

We consider an application of the above results to the concrete triangular model (cf. [7]). Let E be a Hilbert space, $P(t)$ an operator function in E such that

$$\int_{t_1}^{t_2} \| P(t) \|_E^2 \, dt < \infty$$

and T — the operator in $L_E^2(t_1, t_2)$ of the form

$$(Tf)(x) = e^{i\varphi(x)} f(x) - 2e^{i\varphi(x)} P(x)\, \Pi^{-1}(x) \int_x^{t_2} \Pi(t)P(t)f(t)dt \qquad (4.1),$$

where

$$\Pi(x)=\int_{t_1}^{x} exp\left\{P(t)\,P^{*}(t)\right\} dt \qquad (4.2).$$

We shall find a criterion for the complete nonunitariness of T. Accordingly, let us introduce the operators $F: E \to L^2_E(t_1, t_2)$, $G: E \to L^2_E(t_1, t_2)$, $S: E \to E$ of the forms

$$Fe=\sqrt{2}\ e^{i\varphi(x)}\,P^*(x)\ \Pi^{-1}(x)\ \Pi(t_2)\ e \qquad (4.3),$$

$$Ge=\sqrt{2}\ P^*(x)\ \Pi^*(x)\ e \qquad (4.4),$$

$$Se=\Pi^*(t_2)\ e \qquad (4.5).$$

It is not difficult to prove the following

THEOREM 7. *The totality*

$$\alpha(t_1, t_2) := \begin{pmatrix} L^2_E(t_1, t_2) & T & L^2_E(t_1, t_2) \\ F & & G \\ E & S & E \end{pmatrix}$$

of the forms (4.1)—(4.5) is a knot. Moreover the biparametric family of the knots $\alpha(t_1, t_2)$ *it multiplicative*, i. e.

$$\alpha(t_1, t_2) = \alpha(t, t_2)\ \alpha(t_1, t)\ (t_1 < t < t_2) \qquad (4.6).$$

$$f_{se} = (I - ST)^{-1}\,Fe = \frac{\sqrt{2}\ e^{i\varphi(x)}}{1-\xi\, e^{i\varphi(x)}}\,P^*(x)\,[\theta^*(t_1, x; \xi)]^{-1}\theta^*(t_1, t_2; \bar{\xi}\, e) \quad (4.7),$$

$$g_{se} = (I - ST)^{-1}\,Ge = \frac{\sqrt{2}}{1-\xi\, e^{i\varphi(x)}}\,\theta(t_1, x; \xi)e \qquad (4.8),$$

where

and the characteristic function of the knot $\alpha\ (t_1, t2)$ is $\theta\ (t_1, t_2; \xi)$.

THEOREM 8 [4]. *The operator* T *is completely nonunitary if and only if the following conditions hold:*

(i) *the factorization*

$$\theta(t_1, t_2; \xi) = \theta(t, t_2; \xi)\ \theta(t_1, t; \xi) \qquad (4.10)$$

is regular for $\forall\ t \in (t_1, t_2)$,

(ii) *the equation* $P(x)\ f(x) = 0$ *has a unique solution* $f(x) = 0$ *for a. e.* $x \in (t_1, t_2)$.

Proof. *Necessity.* The complete nonunitariness of T is equivalent to the simplicity of the knot $\alpha(t_1, t_2) = \alpha(t, t_2)\ \alpha(t_1, t)$.

Then by Theorem 5 the factorization (4.10) is regular, the necessity of the condition(ii) follows from Lemma 1 and the relations (4.7) — (4.8).

Sufficiency. For $t_1 < t' < t'' < t_2$ we have

$$\alpha(t_1, t_2) = \alpha(t'', t_2)\ \alpha(t', t'')\ \alpha(t_1, t') \tag{4.11}$$

and from the condition (i) and Theorem 6 it follows that

$$P(t', t'')\ f_{se} \in U^* \widehat{H},\ P\ (t'\ t'')\ g_{se} \in U^* \widehat{H} \tag{4.12}$$

Let $h \in H = L^2_E\ (t_1, t_2)$ and $h \perp U^* H$. From (4.12), (4.8).
it follows that

$$\int_{t'}^{t''} \left(\frac{P^*(x)\,\theta\ (t_1, x; \xi)}{1 - \xi\, e^{-i\varphi(x)}}\, e,\ h(x) \right)_E dx = 0.$$

Because of the arbitrariness of $t', t'' \in (t_1, t_2)$, $e \in E$ and the reversibility of $\theta(t_1, x; \xi)$, the last relation implies $P(x)\ h(x) = 0$ (a. e), hence $h(x) = 0$. Therefore, α is simple and the operator T is completely nonunitary.

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