ACTA MATHEMATICA VIETNAMIGA
Vol 7. No I (1982)

SOME RESULTS ON THE DUAL SERIES EQUATIONS

NGUYEN VAN NGOC
Institute of Mathematics
Hanoi

1. INTRODUCTION

The purpose of the present paper is to discuss some mathematical problems of dual series equations with general kernels of the form

$$\begin{aligned} &\sum_{n=0}^{\infty} A_n \Upsilon(n)\psi_n(x) = f_k(x) \qquad (x \in I_k,\ k = \overline{1, N}) \\ &\sum_{n=0}^{\infty} A_n\ \psi_n(x) \quad = 0 \qquad (x \in I \setminus \bigcup_{k=1}^{N} I_k) \end{aligned} \tag{1.1}$$

where

$$I = (a, b) \subset R,\ \text{mes}\ I < \infty$$

$$I_k = (a_k, b_k) \subset I,\quad \overline{I_k} \cap \overline{I_j} = \phi \qquad (k \neq j),$$

$f_k(x)$, $\varphi_n(x)$ and $\Upsilon(n)$ are given functions, the coefficients A_n are to be determined. we assume that $\psi_n(x) \in L^2(I)$ and that $\{\psi_n^n(x)\}_{n=0}$ is a complete orthonormal set in $L^2(I)$

The dual equations (1.1) are generalization of some equations which are usually encountered in mixed boundary value problems of mathematical physics and in cotact problems of elasticity.

Based on generalized integral tranformations [1], we shall show that the dual equations (1.1) can be reduced to an equivalent system of integral equations. Then a solution of the latter system for some classes of π-kernels [cf.2] will be obtained by the method of successive approximations.

2. FUNCTIONAL CLASSES AND PRELIMINARY CONSIDERATIONS

We first recall some definitions and results from the theory of orthonormal series expansions for generalifed functions [1].

Denote by $\mathcal{A}(I) = \mathcal{A}(I, \mathcal{N}, \psi_n)$ the set of all test functions $\varphi(x)$ and by $\mathcal{A}'(I)$ the set of all distributions on $\mathcal{A}(I)$. (cf. [1]). In [1] it was shown that, the eigenfunctions $\psi_n(x)$ of the differential operator $\mathcal{N}$ ($\mathcal{N}\psi_n = \lambda_n\psi_n$, $|\lambda_n| \to \infty$, $n \to \infty$) belong to $\mathcal{A}(I)$ for all $n = 0, 1, 2, \ldots$

Throughout this paper we shall assume that $I = (a, b) \subset R$, mes $I < \infty$ and that the functions are real-valued. Let us denote by $\langle f, \varphi \rangle$ the values of the distributions $f \in \mathcal{A}'(I)$ on a test function $\varphi \in \mathcal{A}(I)$ and by (u, v)-a scalar product in $L^2(I)$ with norm $\| u \| = [(u, u)]^{1/2}$.

Let $\rho(x)$ be a non-negative function on I such that $\sqrt{\rho(x)} = L^r(I)$ $(r > 2)$. Wet denote by $L^2_\rho(I)$ the class

$$L^2_\rho (I) = \{ u(x) \mid \| u \|^2_\rho = \int_a^b \rho(x)\, u^2(x)\, dx < \infty \},$$

by $L^2\rho_1, \rho_2$ $(I \times I)$ the class of functions $f(x, y)$ such that

$$\| f \|^2_{\rho_1, \rho_2} = \int_a^b \rho_1(x)\, \rho_2(y)\, f^2(x, y)\, dx\, dy < \infty$$

and by M^2_ρ (I) the class

$$M^2_\rho (I) = \{ \rho(x)\, u(x) \mid u(x) \in L^2_\rho (I) \}.$$

It is easy to prove that if $g(x) \in M^2_\rho(I)$ then $g(x) \in L^p(I)$ $(p = 2r/r+2) \subset L^1(I)$.

We make the following assumptions concerning $\psi_n(x)$ and λ_n:

$$\psi_n(x) \in L^2_\rho (I), \ \| \psi_n \|_\rho \leqslant \text{const } (\forall n),$$

$$\lambda_n = 0(n^\lambda)(\lambda > 0, \ n \to \infty).$$

Under these conditions instead of $\mathcal{A}(I)$ and $\mathcal{A}'(I)$ we shall write $\mathcal{A}_\rho(I)$ and $\mathcal{A}'_\rho(I)$ respectively.

The following facts are obvious:

LEMMA 2.1. $\mathcal{A}_\rho(I) \subset L^2_\rho(I)$.

COROLLARY 2.1. For every $f(x) \in L^2_\rho(I)$, $g(x) \in M^2_\rho(I)$ the scalar products $(f, \varphi)\rho$ (g, φ) where $\varphi(x) \in \mathcal{A}\rho(I)$, define continuous linear forms on $\mathcal{A}\rho(I)$. Therefore, $L^2(I)$, $L^2_\rho(I)$ and M^2_ρ are subsets of $\mathcal{A}'_\rho(I)$.

Now consider the series

$$R[A_n](x) = \sum_{n=0}^{\infty} A_n\, \psi_n(x). \tag{2.1}$$

From Theorem 9.6.1 of [2] and the above Corollary 2.1, we obtain:

LEMMA 2.2. Suppose that $A_n = 0(n^p)(n \to \infty)$. Then the series (2.1). is convergent in $\mathscr{A}'_\rho(I)$ and Its sum g belongs to $M^2_\rho(I)$ if and only if the coefficients A_n can be represented by the formulae

$$A_n = \int_a^b g(x)\, \varphi_n(x)\, dx \qquad (n=0, 1, 2,...) \tag{2.2}$$

Let us cntreduce the class

$$l^\gamma_p \left\{ \{A_n\}_{n=0}^{\infty} \Big| \sum_{n=0}^{\infty} \gamma(n) \mid A_n \mid^p < \infty \right\},$$

where

$$\gamma(n) > 0 \ (\forall n),\ \gamma(n) = 0(n^{-\lambda})(\lambda \geqslant 1,\ n \to \infty).$$

It is easy to prove the following lemmas.

LEMMA 2.3. For every $g(x) \in M^2_\rho(I)$, $\{(g, \psi_n)\}_{n=0}^{\infty} \in l^\gamma_2$.

LEMMA 2.4 If $\{A_n\}_{n=0}^{\infty} \in l^\gamma_1$ then the series $\sum_{n=0}^{\infty} \gamma(n) A_n\, \psi_n(x)$ is absolutely convergent in $L^2(I)$and in $\mathbf{L}^2_\rho(I)$. Let us define

$$H(x, y) = \sum_{n=0}^{\infty} \gamma(n)\ \psi_n(x)\ \psi_n(y). \tag{2.3}$$

Obviously. $H(x, y) \in L^2(I \times I)$. We sball assume that

$$H(x, y) \in L^2_{\rho, \rho}(I \times I) \cap L^2_{1, \rho}(I \times I). \tag{2.4}$$

THEOREM 2.1. *If $f(x) \in L^2_\rho(I)$, then*

$$\sum_{n=0}^{\infty} \gamma(n)\ (f, \psi_n)_\rho\ \psi_n(x) = (H(x, y), f(y))_\rho, \tag{2,5}$$

$$\sum_{n=0}^{\infty} \gamma(n) \mid (f, \psi_n)_\rho \mid^2 = (R[\gamma(n)(f, \psi_n)_\rho](x), f(x))_\rho \tag{2.6}$$

Proof. Observe that, under the assumption (2.4),

$$x \to (H(x, y), f(y)) \in \mathbf{L}^2(I) \cap \mathbf{L}^2(I)$$

for every $f(x) \in L^2(I)$. Using Lemmas 2.3 and 2.4, we have

$$\sum_{n=n}^{\infty} \gamma(n)(f, \psi)_\rho\ \psi_n(x) \in L^2(I) \cap L^2_\rho(I)$$

Accordingly, for every $\varphi(x) \in \mathcal{A}_\rho(I)$, we get

$$\sum_{n=0}^{\infty} \gamma(n)(f, \psi_n)_\rho \psi_n(x), \varphi(x)) = \sum_{n=0}^{\infty} \gamma(n)(f, \psi_n)_\rho (\varphi, \psi_n) =$$

$$= (f, \sum_{n=0}^{\infty} \gamma(n)(\varphi, \psi_n) \psi_n(y))_\rho = (f, (\sum_{n=0}^{\infty} \gamma(n) \psi_n(x) \psi_n(y), \varphi(x)))_\rho$$

$$= (f(y), (H(x, y), \varphi(x)))_\rho.$$

Applying Fubini's theorem yields

$$(f(y), (H(x, y), \varphi(x)))_\rho = ((H(x, y), f(y))_\rho, \varphi(x)).$$

The equality (2.5) then follows in the sense of $\mathcal{A}'_\rho(I)$. According to Du Bois Reymond's theorem (cf. [3]), the equalily (2.5) is also true in the sense of $L^2(I)$ and $L^2_\rho(I)$.

In order to prove the equality (2.6), we have to use Lemmas 2.3–2.4 and the following proposition.

PROPOSITION 2.1. If $\mathcal{U}_n(x) \in L^2_\rho(I)$ and the series $\sum_{n=0}^{\infty} \mathcal{U}_n(x)$ is convergent in $L^2_\rho(I)$, then for every function $f(x) \in L^2_\rho(I)$, we have

$$(\sum_{n=0}^{\infty} \mathcal{U}_n(x), f(x))_\rho = \sum_{n=0}^{\infty} (\mathcal{U}_n, f)_\rho$$

Multiplying the series in (2.5) by $f(x)$, integrating the obtained equality with respect to x on I and using Proposition 2.1, we obtain the desired equality (2.6). Q.E.D.

3. DUAL SERIES EQUATIONS WITH GENERAL KERNELS

Let $I_k = (a_k, b_k)$ be subintervals of $I = (a,b)$, where $k = \overline{1, N}$ and $\overline{I}_k \cap \overline{I}_j = \phi$ $(k \neq j)$. Let $\rho_k(x)$ be non-negative functions defined in I_k such that $\sqrt{\rho_k(x)} \in L^{r_k}(I_k)$ $(r_k > 2)$.

We shall make the following assumptions

a) $\psi_n(x) \in L^2_{\rho k}(I_k)$, $\| \psi_n \|_{\rho k} \leqslant$ const $(\forall_n)$.

b) For $a_k < x, y < b_k$ $(k = \overline{1, N})$ the function $H(x,y)$ (cf. (2.3)) has the form

$$H(x,y) = \Pi_k(x,y) + \mathcal{D}_k(x,y), \tag{3.1}$$

where $\mathcal{D}_k(x,y) \in C(\overline{I}_k \times \overline{I}_k)$, the functions $\Pi_k(x,y)$ are π–kernels in the following sence (cf. [2] :

$$\int_{a_k}^{b_k} \Pi_k(x,y) \rho_k(y) \pi_{kn}(y) \, dy = \mu_{kn} \pi_{kn}(x), \tag{3.2}$$

$$\mu_{kn} \neq 0, \quad \{\mu_{kn}\}_{n=0}^{\infty} \in l_2,$$

where $\{\pi_{kn}(x)\}_{n=0}^{\infty}$ is a complete orthonormal system in $L^2_{\rho k}(I_k)$. Putting

$$\rho(x) = \sum_{k=1}^{N} \chi_k(x)\ \rho_k(x) \qquad (x \in I) \tag{3.3}$$

$$r = \min_{1 \leqslant k \leqslant N} \{ r_k \},$$

where $\chi_k(x)$ is the characteristic function of I_k, we see that all the results obtained in the previous section remain valid for this $\rho(x)$.

Using the notation (2.1), we can write the equations (1.1) in the form

$$\left.\begin{array}{ll} R\,[\Upsilon(n)\,A_n]\,(x) = f_k(x) & (x \in I_k,\ k = \overline{1,N}) \qquad (3.4) \\ R\,[A_n]\ \ (x) = 0 & (x \in I \setminus \bigcup_{k=1}^{N} I_k) \qquad (3.5) \end{array}\right\}$$

where the functions $\Upsilon(n)$ and $\psi_n(x)$ have the same properties as in the previous section.

Assuming $f_k(x) \in L^2(I_k) \cap L^2_{\rho k}(I_k)$, we shall determine the set $\{A_n\}_{n=0}^{\infty}$ in the classe $\mathcal{K}_1^{\Upsilon,N}\{\rho_k\}$, where

$$\mathcal{K}_p^{\Upsilon,N}\{\rho_k\} = \{ B_n \}_{n=0}^{\infty} \Big| \ |B_n| \leqslant \text{const}\ (\forall n),\ \{B_n\}_{n=0}^{\infty} \in l_p^{\Upsilon},$$

$$R\,[A_n]\,(x) \in M^2_{\rho k}(I_k),\ k = \overline{1,N} \}.$$

The convergence of series $R\,[A_n](x)$ and $R\,[\Upsilon(n)A_n](x)$ has been diseussed the previous section (Lemmas 2.2, 2.4).

Let $g \in \mathcal{A}_\rho'(I)$ ($\rho(x)$ is defined by (3.3)) be the sum of $R\,[A_n](x)$, where $\{A_n\}_{n=0}^{\infty} \in \mathcal{K}_1^{\Upsilon,N}\{\rho k\}$ and satisfies (3.5) According to Lemma 2.2, we get

$$A_n = \langle g, \psi_n \rangle = \int_a^b g(x)\psi_n(x)dx. \tag{3.6}$$

Since $\text{supp}\ g \subset \bigcup_{k=1}^{N} I_k$, for $x \in I_k$ we put $g_k(x) = g(x)$. Then we have

$$g_k(x) = R[A_n](x) \in M^2_\rho(I_k). \tag{3.7}$$

Hence $g(x) \in M^2_\rho\ (I)$. where $\rho(x)$ is defined by the formula (3.3) and we obtain from (3.6) (cf), (2.2))

$$A_n = \langle g,\ \psi_n \rangle = \sum_{k=1}^{N} \int_{ak}^{bk} g_k(x)\,\psi_n(x)dx. \tag{3.8}$$

THEOREM 3.1. *The dual equations (3.4) – (4.5) huve at most one solution in* $\mathcal{K}_1^{\Upsilon, N}\{\rho_k\}$.

Proof. To prove this theorem, we consider the homogeneous dual equati ons of (3.4–(3.5)

$$\left.\begin{array}{ll} R[\Upsilon(n)\, A_n](x) = 0 & (x \in I_k = \overline{1,N}) \\ R[A_n] \quad (x) = 0 & (x \in I \setminus \bigcup_{k=1}^{N} I_k) \end{array}\right\} \quad (3.9)$$

Now we multiply (3.9) by $g_k(x)$, and integrate with respect to x on I_k Summing the obtained equalities over k from 1 to N and using (2.6), (3,8) we conclude

$$\sum_{n=0}^{\infty} \Upsilon(n)|A_n|^2 = 0.$$

Since $\Upsilon(n) > 0$ ($\forall$ n), this implies $A_n = 0$ ($\forall$ n) and the proof is complete.

THEOREM 3.2. *The dual equations (3.4) – (3.5) are equivalent to the system of integral equations*

$$\sum_{i=1}^{N} \int_{a_i}^{b_i} H(x, y)\, g_i(y)\, dy = f_k(x) \; (x \in I_k, k = \overline{1, N}). \quad (3.10)$$

where $g_i(x) \in {}^{2}_{\rho_i}(I_i)$ *and are retated to* A_n *by the formula (3.8).*

Proof. If $\{A_n\}_{n=0}^{\infty} \in \mathcal{K}_1^{\Upsilon,N}\{\rho_k\}$ and satisfies (3.4) – (3.5) then the formula (3.8) holds. Let us rewrite the latter in the form

$$A_n = (\widehat{g}\,,\, \rho_n), \quad (3.11)$$

where

$$\widehat{g}(x) = \sum_{k=1}^{N} \chi_k(x)\, \rho_k^{-1}(x)\, g_k(x).$$

Subtitusting A_n from (3.11) in (3.4) and using the equality (2.5), we obtain the system (1.10)

Conversely, assume that $g_i(x)$ $(i = \overline{1, N})$ belong to $M^2_{\rho_i}(I_i)$ and satisfy the system (3.10). Then rom (3.10) we obtain (3.4), wherethe coefficients A_n are give by the formula (3.8) Berus supp $g_i \subset I_i$ the latter formula can be writlen in the form (3.6), where

$$\langle g, \varphi\rangle = \langle g_i, \varphi\rangle \; (\forall \varphi \in \mathcal{A}_\rho(I), \text{ supp } \varphi \subset I_i) \text{ supp } g \subset \bigcup_{i=1}^{N} I_i.$$

Using a known result ([1], theorem 9.6.1) we obtain from (3.6)

$$R[A_n](x) = g(x) \quad (x \in I),$$

$$R[A_n](x) = 0 \quad (x \in I \setminus \bigcup_{k=1}^{N} I_k).$$

Finally, using Lemma 2.3 and the formula (3.7) it is easy to check that $\{A_n\}_{n=0}^{\infty} \in \mathcal{K}_1^{\Upsilon, N} \{\rho_k\}$. The therem is proved.

4. EXISTENCE OF SOLUTIONS TO THE SYSTEM (3.10)

In (3.10) we now replace $g_i(x)$ by $\rho_i(x)\, \varphi_i(x)$ ($\varphi_i(x) \in L^2_{\rho_i}(I_i)$, $i = \overline{1, N}$) and for $x \in I_k$ ($k = \overline{1, N}$) we put

$$x = \xi_k(t) = \frac{1}{\beta - \alpha}[\beta(b_k - a_k)t + \beta a_k - \alpha b_k],$$

$$\widetilde{\varphi}_k(t) = \varphi_k(\xi_k(t)), \quad \widetilde{\rho}_k(t) = \rho_k(\xi_k(t)),$$

$$\widetilde{f}_k(t) = f_k(\xi_k(t)), \quad \widetilde{\pi}_{kn}(t) = \sqrt{\theta_k}\, \pi_{kn}(\xi_k(t)),$$

$$\widetilde{\Pi}_k(t,\tau) = \theta_k \Pi_k(\xi_k(t), \xi_k(\tau)),$$

$$\widetilde{\mathcal{D}}_{kk}(t,\tau) = \theta_k \mathcal{D}_k(\xi_k(t), \xi_k(\tau)),$$

$$\widetilde{\mathcal{D}}_{ki}(t,\tau) = \theta_i H(\xi_k(t), \xi_i(t)) \quad (k \neq i),$$

where

$$\theta_k = (b_k - a_k) / (\beta - \alpha), \quad (\alpha, \beta) \subset R.$$

If we introduce the vector notation

$$\vec{\varphi} = [\widetilde{\varphi}_k]_{k=1}^{N}, \quad \vec{\rho} = [\widetilde{\rho}_k]_{k=1}^{N}, \quad \vec{f} = [\widetilde{f}_k]_{k=1}^{N},$$

$$\vec{\pi} = [\widetilde{\pi}_{kn}]_{k=1}^{N}, \quad \vec{\mu}_n = [\mu_{kn}]_{k=1}^{N},$$

$$\vec{\Pi} = [\widetilde{\Pi}_k]_{k=1}^{N}, \quad \mathcal{D} = [\widetilde{\mathcal{D}}_{ki}]_{k,i=1}^{N}$$

and the multiplication of vectors

$$\vec{u} \square \vec{v} = [u_k v_k]_{k=1}^{N}$$

then the system (3.10) (under the assumption (3.1)) and the formula (3.2) can be written respectively in the forms

$$\int_{\alpha}^{\beta} \vec{\Pi}(t,\tau) \square \vec{\rho}(\tau) \square \varphi(\tau)\, d\tau + \int_{\alpha}^{\beta} [\mathcal{D}(t,\tau) \cdot (\vec{\rho}(\tau) \square \vec{\varphi}(\tau))] d\tau = \vec{f}(t), \tag{4.1}$$

$$\int_{\alpha}^{\beta} \vec{\Pi}(t,\tau) \square \vec{\rho}(\tau) \square \vec{\pi}_n(\tau)\, d\tau = \vec{\mu}_n \square \vec{\pi}_n(t) \quad (n = 0, 1, 2, \ldots). \tag{4.2}$$

Let us introduce the space

$$L^{2,N}_{\vec{\rho}}(\alpha,\beta) = \{\vec{u}(t) \mid u_k(t) \in L^2_{\tilde{\rho}_K}(\alpha,\beta),\ k = \overline{1,N}\}.$$

Clearly this is Banach space with the norm

$$\| \vec{u} \|_{\vec{\rho}} = (\sum_{k=1}^{N} \| u_k \|^2_{\tilde{\rho}_K})^{1/2}$$

Denote

$$(\vec{u}, \vec{v})_{\vec{\rho}} = \int_{\alpha}^{\beta} \vec{\rho}(t) \square \vec{u}(t) \square \vec{v}(t)\, dt,$$

$$|\vec{\xi}|_N = (\sum_{k=1}^{N} |\xi_k|^2)^{1/2} \quad (\vec{\xi} \in R^N),$$

$$\vec{\mu}^{-1} = [\mu_k^{-1}]_{k=1}^{N} \quad (\mu_k \neq 0,\ k = \overline{1,N}).$$

Then the following lemmas can easily be verified.

LEMMA 4.1. Let M(t) be a functional matrix, whose elements are $M_{ki}(t \in \in L^2_{\tilde{\rho}_i}(\alpha,\beta)$ $(k, i = \overline{1,N})$. For every $\vec{\xi}, \vec{\eta} \in R^N$ and $\vec{u}(t) \in L^{2,N}_{\vec{\rho}}(\alpha,\beta)$ we have

$$|\vec{\xi} \square \vec{\eta}|_N \leqslant |\vec{\xi}|_N\ |\vec{\eta}|_N,$$

$$\left| \int_{\alpha}^{\beta} [M(t) . (\vec{\rho}(t) \square \vec{u}(t)]\, dt \right|_N \leqslant \| \vec{u} \|_{\vec{\rho}} \left(\sum_{k,i=1}^{N} \| M_{ki} \|^2_{\tilde{\rho}_i} \right)^{1/2}.$$

LEMMA 4.2. For every $\vec{u}(t) \in L^{2,N}_{\vec{\rho}}(\alpha,\beta)$ we have the expansion

$$\vec{u}(t) = \sum_{n=0}^{\infty} \vec{\xi}_n \square \vec{\pi}_n(t), \tag{4.3}$$

$$(\vec{\xi}_n = (\vec{u}, \vec{\pi}_n)_{\vec{\rho}},\ n = 0,1,2,...),$$

where the series (4.3) is convergent in $L^{2,N}(\alpha, \beta)$. Moreover we have

$$\| \vec{u} \|^2_{\vec{\rho}} = \sum_{n=0}^{\infty} |\vec{\xi}_n|^2_N \tag{4.4}$$

Before studying the system (4.1), let us introduce the following notations

$$(S_{\vec{f}})(t) = \sum_{n=0}^{\infty} \vec{\mu}_n^{-1} \square (\vec{f}, \vec{\pi}_n)_{\vec{\rho}} \square \vec{\pi}_n(t), \tag{4.5}$$

$$R_n^{ki}(\tau) = \int_{\alpha}^{\beta} \widetilde{\mathfrak{D}}_{ki}(t,\tau) \widetilde{\rho_k}(t) \widetilde{\pi}_{kn}(t)\, dt,$$

$$\sigma_N = \Big(\sum_{n=0}^{\infty} \left| \vec{\mu}_n^{-1} \right|_N^2 \sum_{k,i=1}^{N} \left\| R_n^{ki} \right\|_{\rho_i}^2 \Big)^{1/2},$$

$$R_n(t) = [R_n^{ki}(t)]_{k,i=1}^{N}.$$

The following théorem holds.

THEOREM 4.1. *Suppose that* $\sigma_N < 1$ *and*

$$\left\{ \left| \vec{\mu}_n^{-1} \square (\vec{f}, \vec{\pi}_n)_{\vec{\rho}} \right|_N \right\}_{n=0}^{\infty} \in l_2$$

Then the system (4.1) *has a unique solution in* $L_{\vec{\rho}}^{2,N}(\alpha,\beta)$ *which can be obtained by the method of successive approximations from the system*

$$\vec{\varphi}(t) = (S_{\vec{f}})(t) - \sum_{n=0}^{\infty} \vec{\pi}_n(t) \square \vec{\mu}_n^{-1} \square \int_{\alpha}^{\beta} [R_n(\tau) \cdot (\vec{\rho}(t) \square \vec{\varphi}(\tau))]\, d\tau. \tag{4.6}$$

Proof. Using (4.2), (4.3) and the method of [2] (for N = 1), after some transformations, we obtain the system (4.6), where $(S_{\vec{f}})(t)$ and the matrix $R_n(\tau)$ are given by (4.5).

Denoting by $T_{\vec{\varphi}}$ the expression en the right–hand side ef. (4.6), we see that T maps $L_{\vec{\rho}}^{2,N}(\alpha,\beta)$ into $L_{\vec{\rho}}^{1,N}(\alpha,\beta)$.

Now we show that T is a contraction operator. Indeed, using the equality (4.4) from (4.6) we obtain

$$\left\| T_{\vec{\varphi}} - T_{\vec{\psi}} \right\|_{\vec{\rho}}^2 = \sum_{n=0}^{\infty} \left| \vec{\mu}_n^{-1} \square \int_{\alpha}^{\beta} [R_n(\tau) \cdot (\vec{\rho}(t) \square (\vec{\varphi}(\tau) - \vec{\psi}(\tau)))] d\tau \right|_N^2. \tag{4.7}$$

Applying the inequalities in Lemma (4.1), we get from (4.7)

$$\| T_{\vec{\varphi}} - T_{\vec{\psi}} \|_{\vec{\rho}} \leqslant \sigma_N \| \vec{\varphi} - \vec{\psi} \|_{\vec{\rho}}.$$

Since under the conditions of this theorem $\sigma_N < 1$, it follows that T is a contraction operator. Therefore the system (4.1) has a unique solution in $L^{2,N}_{\overrightarrow{\rho}}(\alpha,\beta)$ which can be obtained from (4.6) by the method of successive approximations. The theorem is proved.

Il follows from all the above that

THEOREM 4.2. *Under the conditions of the Theorem 4.1, the dual series equations (3.4)–(3.5) in* $\mathcal{K}^{\gamma,N}_{l}\{\rho_k\}$ *have a unique solution.*

Received December 15, 1981

REFERENCES

1. A.H. Zemanian. *Generalized integral trausformations.* Moscow 1974 (in Russian).

2. G.J. Popov. *On the method of orthogonal polynomials in contact problems of elasticity.* MM. T. 33. N° 3, 1969 (in Russian).

3. V.S. Vladimirov. *Generalized functions in mathematical physics.* Moscow 1976. (in Russian).