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# ON THE GENERALIZED COMPLEMENTARITY PROBLEMS IN LOCALLY CONVEX SPACES

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## 1. INTRODUCTION

A well known result of Browder (see e.g. [2]) states that if C is a nonempty compact and convex subset in $R^n$ and $f : C \times [0,1] \to C$ is a continuous mapping, then there is a closed and connected subset T of $C \times [0,1]$ which meets both $C \times \{0\}$ and $C \times \{1\}$ such that $x \in f(x, t)$ for every $(x, t)$ in T.

In the last decade there have been several attempts to generalize this important theorem. Saigal [8] extended it to multivalued mappings from $R^n$ to $R^n$ with compact convex values and Mas–Colell [5[ proved that Saigal's theorem remains true even for mappings with compact and contractible values.

One important application of Browder's theorem is in nonlinear complementarity problems. In [2], Eaves used it to prove the basic theorem of complementarity. Saigal [8] used Mas – Colell's extension of Browder's theorem to extend Eaves' result to multi–valued mappings with compact and contrac tible values.

In the first section of the present paper we shall extend Saigal's fixed point theorem to an arbitrary locally convex space. In the second section this result is used to extend Eaves' basic theorem of complementarity to barrel spaces. Finally, as an application we shall give an existence theorem for the generalized complementarity problem.

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## 2. A FIXED POINT THEOREM

**THEOREM 1.** *Let X be a seprated locally convex space and K be a compact convex subset of X. Assume that $f : K \times [0,1] \to 2^K$ is an u.s.c. multi – valued mapping with f(x,t) nonempty compact and convex for every (x.t) in $K \times [0,1]$. Let :*

$$C_f = \{(x,t) \in K \times [0,1] : x \in f(x,t)\}$$

*Then there is a closed connected subset T ot $C_f$ such that*

$$T \cap K \times \{0\} \neq \phi \text{ and } T \cap K \times \{1\} \neq \phi$$

**Proof.** Put

$$C_f^*: = \{(x, t, x) \in K \times [0,1] \times K : x \in f(x, t)\}$$

$$L : = \{(x, t, y) \in X \times [0,1] \times X : x = y\}$$

Clearly $C_f^* = G(f) \cap L$ where $G(f)$ is the graph of f. To prove the theorem, it suffices to show that $C_f$ has a component meeting both $K \times \{0\} \times K$ and $K \times \{1\} \times K$.

Suppose the contrary that there is no such component. Then there exist two compact subsets $C_1$ and $C_2$ of $C_f$ such that $C_1 \cup C_2 = C_f^*$ and $C_1 \cap C_2 = \phi$

$$C_1 \cap K \times \{0\} \times K \neq \phi \quad \text{but} \quad C_1 \cap K \times \{1\} \times K = \phi$$
$$C_2 \cap K \times \{1\} \times K \neq \phi \quad \text{but} \quad C_2 \cap K \times \{0\} \times K = \phi$$

So we can find an open convex (and symetric) neighborhood W of the origin in X and a positive number $\varepsilon$ such that

$$(C_1 + W \times (-\varepsilon, \varepsilon) \times W) \cap (C_2 + W \times (-\varepsilon, \varepsilon) \times W) = \phi$$
$$(C_1 + W \times (-\varepsilon, \varepsilon) \times W) \cap K \times \{1\} \times K = \phi$$
$$(C_2 + W \times (-\varepsilon, \varepsilon) \times W) \cap K \times \{0\} \times K = \phi$$

On the other hand, since $C_f^* = G(f) \cap L$, the compact set $G(f)$ is disjoint from $L \setminus (C_f^* + W \times (-\varepsilon, \varepsilon) \times W)$. Therefore, there exists an open and convex neighborhood W' of the origin and a positive number $\varepsilon'$ such that:

$$(G(f) + \overline{W'} \times (-\varepsilon', \varepsilon') \times \overline{W'}) \cap [L / (C_f^* + W \times (-\varepsilon, \varepsilon) \times W] = \phi$$

or equivalently :

$$(G(f) + \overline{W'} \times (-\varepsilon', \varepsilon') \times \overline{W'}) \cap L \subset C_f^* + W \times (-\varepsilon, \varepsilon) \times W$$

Now since K is compact, there exists a finite number of points $x_1, \ldots, x_m$ in K such that $K \subset \bigcup_{i=1}^{m} (x_i + W')$. Let S be the convex hull of $x_1, \ldots, x_m$ and define a multivalued mapping $F : S \times [0,1] \to 2^S$ by setting $F(x, t) : = (f(x, t) + \overline{W'}) \cap S$. By an easy argument one can verify that F is u.s.c. and $F(x, t)$ is a nonempty compact convex subset of S for every $(x, t)$ in $S \times [0,1]$. Therefore by Saigal's theorem [8], the set $C_F^* : = G(F) \cap L$ has a component meeting both $K \times \{0\} \times K$ and $K \times \{1\} \times K$.

On the other hand clearly $G(F) \subset G(f) + \overline{W'} \times (-\varepsilon', \varepsilon') \times \overline{W'}$ hence $C_F^* = G(F) \cap L \subset (G(f) + \overline{W'} \times (-\varepsilon', \varepsilon') \times \overline{W'}) \cap L \subset C_f^* + W \times (-\varepsilon, \varepsilon) \times W$

But by the choice of W and $\varepsilon$, $C_f + W \times (-\varepsilon, \varepsilon) \times W$ has no component meeting both $K \times \{0\} \times K$ and $K \times \{1\} \times K$. This contradiction completes the proof.

## 3. THE GENERALIZED COMPLEMENTARITY PROBLEM

Let X be a separated locally convex space whose dual X' is provided with the weak $-*$ topology. Let M be a closed convex cone in X and M' the polar of M:

$$M' = \{y \in X' : \langle x, y \rangle \geqslant 0 \quad \forall x \in M\}$$

Let $f : M \to 2^{X'}$ be a multi – valued mapping with $f(x)$ nonempty compact and convex for every x in M.

**DEFINITION 1**: A point $x \in M$ is called a solution of the *generalized complementarity problem* (GCP) with respect to the mapping f and the cone M if there is a point $y \in f(x)$ such that $y \in M'$ and $\langle x, y \rangle = 0$.

**DEFINITION 2**: Let C be a closed and convex subset of M. A point $x$ in C is called a *stationary point* of the pair (f, C) if there is a point y in $f(x)$ such that $\langle v-x, y\rangle \geqslant 0$ for every v in C.

The following propositions concerning the relatioship between the notions of stationary and solutions of GCP are simple and well – known.

**PROPOSITION 1**: A point $x$ is a solution to GCP if and only if $x$ is a stationary point of the pair (f, M).

**PROPOSINION 2**: Let C be a closed convex subset of M containing O. If $x$ is a stationary point of (f, C) such that $x \in \operatorname{int}_M C$ (the interior of C relative to M), then $x$ is a solution to GCP.

For our purpose we shall also need the following lemma:

**LEMMA**: Let X be a barrel space and f be u.s.c and C be a closed subset of M containg O. Assume that $D : C \to 2^{X'}$ is a continuous multi – valued mapping such that $D(O) = \{O\}$ and $D(x)$ is compact and contains x for every $x$ in C. Then the set $\Sigma = \{x \in C : x$ is a stationary point of $\{(f, D(x))\}$ is nonempty and closed.

**Proof.** Clearly $O \in \Sigma$ so $\Sigma$ is nonempty.

Let $(x_n)$ be any generalized sequence in $\Sigma$ which converges to $x$. We have to show that $x \in \Sigma$.

By definition, for every n there exists $y_n \in f(x_n)$ such that $\langle v - x_n, y_n\rangle \geqslant 0$ for every $v \in D(x_n)$. The set $\bigcup_n f(x_n) \cup f(x)$ being compact, [1. Ch.VI, Theorem 1.3] one can assume—by taking subsequence if necessary—that $(y_n)$ converges to a point y. Since f is u.s.c. we have $y \in f(x)$ and it remains to prove that $\langle v-x, y\rangle \geqslant 0$ for every $v \in D(x)$.

Let us consider anfarbitrary v in $D(x)$. By the continuity of D there is a sequence $v_n \to v$ such that $v_n \in D(x_n)$ for every n, Then $\langle v_n - x_n, y_n\rangle \geqslant 0$ for every n. But $y_n$ is relatively compact, hence weakly bounded. Sine X is a barrel space, $\{y_n\}$ is equicontinuous [7, Ch. IV, Proposition 1] Therefore:

$$\langle v_n - x_n, y\rangle \to \langle v - x, y\rangle$$

and hence $\langle x-x, y\rangle \geqslant 0$, which completes the proof.

We can now state our main result:

**THEOREM 2:** *Let X be a separated barrel space whose dual X' is provided with the weak $-*$ topotogy. Let M be a closed convex cone in X and $D : R - 2^M$ be a continuous multivalued mapping with compact convex values which satisfies the following conditions:*

*(a) $D(O) = \{O\}$ and $D(t) \subset D(s)$ for $t < s$*

*(b) the set $C = D(R_+)$ is closed.*

*Assume furthermore that : $M \to 2^{X'}$ is an u.s.c. multivalued mapping with $f(x)$ nonempty compact and convex for every $x$ in M. Then there is a closed and connected subset T of M such that:*

*(i) $\forall_x \in T\ \exists_t \in R_+$ : $x$ is a stationary point of $(f, D(t))$*

*(ii) $\forall_s \in T\ \exists_x \in T$ : $x$ is a stationary point of $(f. D(s))$.*

**Proof.** Consider the function $h: C \to R$ defined by

$$h(x) = \{\inf t \in R : x \in D(t)\}$$

By [1, Ch. VI., Theorems 3.1., and 3.2], h is continuous. Furthermore $h(O) = 0$ and $x \in D(t)$ if and only if $t \geqslant h(x)$.

Put $\Sigma = \{x \in C$ : x is a stationary point of $(f, D(h(x)))\}$ and $\Sigma' = \{x \in C$ : x is a stationary point of $(f, D(t))$ fro some t in $R_+\}$.

From the assumtions (a) and (b) and the definition of h, it follows that $O \in \Sigma = \Sigma'$ Applying Lemma with D being Doh we obtain that $\Sigma$ is closed. Let T be the component of $\Sigma$ containing O. Then clearly T is closed connected and satisfies (i). It remains to prove that T also satisfies (ii).

For $s = 0$, (ii) is obvious, so let $s \neq 0$. Since D(s) is compact, f(D(s)) is compact. But X being a barrel space, the closed convex hull E(s) of f(D(s)) is compact and convex in X' [7, Ch. VI., Proposition 1]. Let us define a mapping $F: D(s) \times E(s) \times [0, s] \to 2^{D(s) \times E(s)}$ by taking $F(x,y,t) = \arg\min_{v \in D(t)} \langle v, y\rangle \times f(x)$.

Clearly $F(x,y,t)$ is nonempty compact and convex for every $(x,y,t)$. To see F is u.s,c. (or equivalently, F is a closed mapping), it suffices to verify the closedness of the first component of F. that is of the mapping $(y, t) \mapsto \arg\min_{v \in D(t)} \langle v, y\rangle$.

For this purpose let $(t_n)$, $(y_n)$ and $(u_n)$ be arbitrary generalized sequences such that

$$u_n \in \arg\min_{v \in D(t_n)} \langle v, y_n\rangle \text{ and } t_n \to t,\ y_n \to y,\text{ and } u_n \to u.$$

We have to show that $u \in \arg\min_{v \in D(t)} \langle v, y\rangle$.

Using the continuity of D one can find for every $v \in D(t)$ a generalized sequence $(v_n)$ such that $v_n \to v$ and $v_n \in D(t_n)$ for every n. Then $\langle v_n, y_n\rangle \geqslant \langle u_n, y_n\rangle$. Again since X is a barrel space $(y_n)$ is equicentinuous so

$$\langle v_n, y_n\rangle \to \langle v, y\rangle$$

and

$$\langle u_n, y_n\rangle \to \langle u, y\rangle$$

Hence $\langle v, y \rangle \geqslant \langle u, y \rangle$, which means that $u \in \arg\min_{v\, D(t)} \langle v, y \rangle$ proving the upper semicontinuity of F.

Now applying Theorem 1 to the mapping F, there exists a closed component A of the set

$$C_F = \{ (x, y, t) \in D(s) \times E(s) \times [0, s] : (x, y) \in F(x, y, t) \}$$

such that
$$A \cap D(s) \times E(s) \times \{0\} \neq \phi \quad (1)$$

and
$$A \cap D(s) \times E(s) \times \{1\} \neq \phi \quad (2)$$

For every $(x, y, t) \in A$ we have $(x, y) \in F(x, y, t)$ whichmeans that $x \in \arg\min_{v \in D(t)} \langle v\, y \rangle$ and $y \in f(x)$.

Consequently $x$ is a stationary point of $(f, D(t))$, i.e. $x \in \Sigma' = \Sigma$. Hence the set $B = \{x \in C : (x, y, t) \in A$ for some $y$ and $t\}$ is contained in $\Sigma$ and is connected. But (i) implies $O \in B$, therefore $B \subset T$. Furthermore (2) implies that there is a stationary point $x$ of $(f, D(s))$. Hence T satisfies (ii) as was to be proved.

In the case where $X = R^n$ this theorem reduces to Theorem 2.1. in [4] which is a generalization of the so called basic theorem of complementery of Eaves [2].

A corollary of this theorem is the following existence result for GCP, generalizing coresponding results of Eaves [2] and Saigal [8].

**COROLLARY.** Let X, M, D, f as in Theorem 2 with the further assumption that D maps $R_+$ onto M. Moreover suppose there is a bounded set U in $M \setminus \{0\}$ which separates 0 from $\infty$ such that

$$\forall x \in U\, \exists w \in \mathrm{int}\,_M D(h(x))\ \forall y \in f(x) : \langle w - x, y \rangle \leqslant 0 \}$$

Then GCP has a solution.

(We say that U separates O from $\infty$ if any connected and unbounded set in M containing O meets U).

**Proof.** Consider the connected set T whose existence has been proved in Theorem 2. We can distinguish two case : (a) T is bounded (b) T is unbounded.

In the case (a) we claim that T is compact. Indeed $h^{-1}[0,1]$ is open in M and contains O. Since T is bounded and contained in the cone M, there is a natural number n such that $T \subset n \,.\, h^{-1}[0,1] \subset n \,.\, h^{-1}[0,1] = n \,.\, D(1)$. D(t) being compact. T is compac. On the other hand $T \subset M = \cup \{\mathrm{int}_M D(t) : t \in R_+ \}$ Hence there is $t > O$ such that $T \subset \mathrm{int}_M D(t)$ and Proposition 2 then implies that GCP has a solution.

In the case (b), since T is connected and contains O, there is a point $x$ in $T \cap U$. But $x \in T$ implies that

$$\exists y \in f(x)\ \forall v \in D(h(x)) : \langle v - x, y \rangle \geqslant 0 \quad (3)$$

while $x \in U$ implies that

$$\exists w \in \mathrm{int}_M D(h(x))\ \ \forall z \in f(x) : \langle w - x, z \rangle \leqslant 0 \quad (4)$$

Therefore $\langle w - x, y \rangle = 0$.

Since $w \in \text{int}_M D(h(x))$, there exists for every $u \in M$ an $\varepsilon > 0$ such that $w + \varepsilon u \in D(h(x))$. By (3), $0 \leqslant \langle w + \varepsilon u - x, y \rangle = \varepsilon \langle u, y \rangle$. Hence $\langle u, y \rangle \geqslant 0$, i.e., $y \in M'$.

Finally $O \in D(h(x))$ implies $0 \leqslant \langle 0 - x, y \rangle$ and consequently $\langle x, y \rangle = 0$ since $x \in M$ and $y \in M'$. Thus $x$ is a solution to GCP.

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