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MULTIDIMENSIONAL QUANTIZATION. II. THE COVARIANT DERIVATION

DO NGOC ZIEP
Institute of Mathematics, Hanoi

In some previous works the author has proposed a construction of unitary representations by K—orbits. In the present paper we give a physical illustration of this construction. The main result is the following fact: The partially invariant holomorphically induced represention of a connected and simply connected Lie group coincides with the representation arising in the procedure of multidimesional quantization of this group.

1. NOTATIONS AND STATEMENT OF THE MAIN RESULT

In this section we first recall some notations which have been introduced in the provious articles ([1], [2]), then we give the statement of the main result.

1.1. Partially invariant holomorphically induced representations by K—orbits

Let G be a connected Lie group, $\mathcal{G}$ its Lie algebra and $\mathcal{G}^*$ the dual space of $\mathcal{G}$. It is clear that the coadjoint representation (shortly, *K—representation*) of the group G in $\mathcal{G}^*$ divides $\mathcal{G}^*$ into *K—orbits*. We denote by $\mathcal{O}(G)$ the space of all K—orbits of the group G.

From now on we fix a K—orbit $\Omega \in \mathcal{O}(G)$ and a point F in it. Assume that G_F is the stabilizer of the point F, G_F is the Lie algebra of G_F. It is well known that in the category of homogeneous G—spaces we have $\Omega \approx G_F \setminus G$. It is not hard to verify that $2\pi iF$ is a representation of $\mathcal{G}_F$. The K—orbit is called *integral* iff there exists a unitary representation (character) χ_F of G_F, the differential of which is $2\pi iF$. Suppose that $(G_F)_o$ is the connected component of the identity of G_F and $(G_F)_o = S.R$ is its E. Cartan — Levi — Malcev's decomposition, $\widetilde{\sigma}$ is an irreducible unitary representation of G_F such that $\widetilde{\sigma}|_{R^o}=I$, and $\widetilde{\rho} = d\widetilde{\sigma}$ is the corresponding representation of the Lie algebra $\mathcal{G}_F$ and hence also of its complexification $(\mathcal{G}_F)_C = \mathcal{G}_F \otimes C$. Thus $\chi_F . \widetilde{\sigma}$ is an irreducible unitary representation of G_F such that its restriction to R (the solvable part of the connected component of the identity of the stabilizer G_F of the point F) is a multiple of χ_F, and its differential is

$$d(\chi_F . \widetilde{\sigma}) = 2\pi i F + \widetilde{\rho}.$$

If the representation $2\pi i F + \widetilde{\rho}$ can be extended in suitable sense [2, Def. 1.1] to a representation ρ of a complex Lie subalgebra $\mathcal{P}$ in $\mathcal{G}_C$ and $\chi_F . \widetilde{\sigma}$ to an irreducible unitary representation σ_0 of the subgroup H_0 which is the connected closed subgroup of G, the Lie algebre of which is $\mathcal{H} = \mathcal{G} \cap \mathcal{P}$, then the triple $(\mathcal{P}, \rho, \sigma_0)$ is called a $(\widetilde{\sigma}, F)$ — *polarization* of the K-orbit Ω. We also denote by M_0 the connected closed subgroup of G, the Lie algebra of which is $\mathcal{M} = (\mathcal{P} + \overline{\mathcal{P}}) \cap \mathcal{G}$, and $H = G_F . H_0$, $M = G_F . M_0$

In the works [1. 2] it is proved that:

1) There exists a structure of mixed manifold of type (k, l, m) on the G-space $\Omega = G_F \setminus G$, where $k = \dim G - \dim M$, $l = (\dim M - \dim H) / 2$, $m = \dim H - \dim G_F$,

2) On the (associated with the representation $\sigma|_{G_F}$) smooth G-bundle $\mathcal{E}_{\sigma}|_{G_F} = G \underset{G_F}{\times} V$ there exists a structure of a partially invariant partially holomorphic G-bundle $\mathcal{E}_{\sigma,\rho}$ such that the representation of the group G arising in the space of partially invariant partially holomorphic sections of $\mathcal{E}_{\sigma,\rho}$ is equivalent to the representation of this group by right translations in the space $C^\infty(G; \mathcal{P}, F, \rho, \sigma_0)$ of smooth functions f on G with values in V and satisfying the following system of equations:

$$f(hx) = \sigma(h) f(x) \quad ; \quad h \in H \quad , \quad x \in G,$$
$$L_X f + \rho(X) f = 0, \quad X \in \mathcal{P}$$

where L_X is the Lie derivation along the right invariant vector field ξ_X on G, corresponding to X.

To obtain an unitary representation we apply the usual *construction of unitary G-bundle* [1] (see also 1.2 of this paper). We denote the obtained unitary representation by $\mathrm{Ind}(G; \mathcal{P}, F, \rho, \sigma_0)$ and we call it the *partially invariant holomorphically induced* representation of G.

1.2. Quantization operator

In general, quantization means a procedure of construction of quantum systems from given classical systems. A majority of the existing methods of quantization are subsumed under the following scheme [3, §15]. Consider the *physical quantities* associated with the system. Among these we single out a certain set of *primary quantities* forming a Lie algebra under the *Poisson brackets*. We suppose that when we go over to quantum mechanics, the *commutation relations* among primary quantities are preserved in the following sense. Let h be the *Planck's constant*, $\hbar = h/2\pi$ and $\widehat{f}$ the *quantum mechanical operator* corresponding to the primary quantity f. Then the following relation must be satisfied

$$\{f_1, f_2\}^{\wedge} = \frac{i}{\hbar} [\widehat{f_1}, \widehat{f_2}]$$

This means that the correspondence $f \longmapsto \frac{i}{\hbar}\widehat{f}$ is an operator representation of the Lie algebra of primary quantities. Ordinarily, constants are included among the primary quantities, and one requires that the relation

$$\widehat{1} = I(\text{identity operator})$$

holds.

Now we consider a fixed Hamiltonian system (Ω, B_Ω) where $\Omega \in O(G)$ and B_Ω is the Kirillov's form on Ω. Suppose that G_F if the stabilizer of F and the K–orbit Ω is integral, $\widetilde{\chi_{F}.\sigma}$ is the irreducible representation of G_F, which is described above, $(\mathcal{P}, \rho, \widetilde{\sigma_0})$ is a (σ, F) –polarization of the K–orbit Ω.

Suppose that Δ_G (resp., Δ_H) is the modular function of the group G (resp., H), $\delta^2(h) = \Delta_H(h)/\Delta_G(h)$, $h \in G_F \subset H$, is the non – unitary character of G_F. We consider the G–bundle $\mathcal{M} = G \times_H C$, associated with the non–unitary character δ^2 of the group G_F.

We denote by $\mathcal{M}^{1/2}$ the bundle associated with the character $\delta \mid G_F = (\Delta_H/\Delta_G)^{1/2} \mid G_F$. Thus the bundle $\widetilde{\mathcal{E}}_{\sigma,\rho} = \mathcal{E}_{\sigma,\rho} \otimes \mathcal{M}^{1/2}$ is an *unitary G – bundle* over $\Omega = G_F \backslash G$ If s is a section of $\widetilde{\mathcal{E}}_{\sigma,\rho}$ then $\| s \|^2_V$ is a section of the bundle $\mathcal{M}$, and we can take integral of it $\int_{H\backslash G} \| s \|^2_V(x) d\mu_{H\backslash G}(x)$.

To obtain a model of quantum system we choose the Hilbert space which is the completion of the space of all partially invariant partially holomorphic square–integrable sections of the unitary G bundle $\widetilde{\mathcal{E}}_{\sigma,\rho}$.

Let Γ be a connection on the principal bundle $H \to G \to H\backslash G$, α is the 1–form of the associated affine connection on the induced Hilbert bundle $V \longrightarrow \widetilde{\mathcal{E}}_{\sigma,\rho} \longrightarrow \Omega$ associated with the representation $\delta . \sigma$ of H and the natural morphism of the principal bundles

$$\begin{array}{ccc} G_F \longrightarrow G & \text{and} & H \longrightarrow G \\ \downarrow & & \downarrow \\ G_F\backslash G & & H\backslash G \end{array}$$

Now to each smooth function $f \in C^\infty(\Omega)$ let us correspond the operator $\widehat{f}$ acting by:

$$\widehat{f} = L\xi_f + \alpha(\xi_f) + f$$

It was shown in [1, 2] that the correspondence $f \to \widehat{f}$ defines a procedure of quantization iff the differential 1-form α satisfies the relation

$$-(I_V \otimes B_\Omega)(\xi, \eta) = \xi\alpha(\eta) - \eta\alpha(\xi) - \alpha([\xi, \eta]) + \frac{i}{\hbar}[\alpha(\xi), \alpha(\eta)].$$

If $X \in \mathcal{G}$ then X can be considered as a function on $\mathcal{G}^*$, and in particular, on Ω. It is easy to see that the function X on Ω is the generating function of the field ξ_X which corresponds to $X \in \mathcal{G}$.

The representation of the group G is defined by the formula

$$T(\exp X) = \exp(i/\hbar . \widehat{X}).$$

The relation

$$\widehat{\{f_1, f_2\}} = i/\hbar\, [\widehat{f_1}, \widehat{f_2}]$$

and self-adijointness of operator X guarantee that the condition $T(g_1 . g_2) = T(g_1) . T(g_2)$ holds and that the operators T(g) are unitary in certain neighborhood of the identity. This «local» representation admits an unique extension of a many valued representation of G, which will be single-valued on the simply connected covering $\widetilde{G}$ of the group G.

1.3. THEOREM. *The partially invariant holomorphically induced representation* Ind (G; $\mathcal{P}$, ρ, F, σ_0) *of a connected and simply connected Lie group* G *coinsides with the representation* T *of this group, arising in the procedure of multidimensional quantization with the corresponding affine connection.*

An equivalent statement of this theorem is the following: The covariant derivation of the representation Ind (G,: $\mathcal{P}$, ρ, F, σ_0) is the quantization operators in 1.2, multiplied by constant $2\pi i/h$. To do this, firstly we must justify the notion of covariant derivation for our case of bundles with Hilbert fibres. Then we must show that this is a connection and compute it. We shall do this in the following section.

2. PROOF OF THEOREM

The proof of our theorem is long and requires a detailed analysis of the notion of affine connection. Thus we divide it into several steps.

2.1. Justification of the infinite dimensional bundle.

Firstly we recall the construction of the associated bundle $\mathcal{E}_{\sigma,\rho}$. We know that $\widetilde{\mathcal{E}}_{\sigma,\rho} = G \underset{G_F}{\times} V = G \times V/\sim$, where $\sim$ is the following equivalent relation: $(g, v) \sim (g', v')$ iff there exists $k \in G_F$ such that $g' = kg$ and $v' = \sigma(k(\sigma(k) v$. We remark that V is a Hilbert space, $\sigma(k) = (\Delta_H(k)/\Delta_G(k))^{1/2}$ and $\delta(k)$ is an unitary operator. Thus $\widetilde{\mathcal{E}}_{\sigma,\rho}$ is a vector bundle with (in)finite dimensional fibres which are Hilbert spaces. The structural group of $\widetilde{\mathscr{E}}_{\sigma,\rho}$ is a subgroup of the «projective» unitary group $\mathbf{C} \times U(V)$ of V. But our situation is also good, because the structural group really is a finite-dimensional Lie subgroup of $\mathbf{C} \times U(V)$, as a complete image of the Lie group G_F in the representation $\delta . \sigma$: Thus we can apply for this structural group the theorem of Stone, and we can speak about the finite dimensional Lie algebra of the structural group of $\mathscr{E}_{\sigma,\rho}$.

2.2. Covariant derivation of homomorphisms.

Now we modify the results concerning covariant derivation of homomorphisms of vector fibre bundles for the case of infinite dimensonal Hilbert bundle of type $\widetilde{\mathscr{E}}_{\sigma,\rho}$. The case of finie dimensional bundles is well-known, see for example the work of Kosmann — Schwarzbach [4].

Let us denote by E and E' two bundles of this kind over a homogeneous space $G_1 \backslash G$, which are associated with the unitary representations σ and σ' of the subgroup G_1 of G. Thus G acts on E and E' by the natural actions. We define the action of G on Hom (E, E') as follows. Assume that $u \in$ Hom (E, E') and u is projected onto a morphism u_M of the base $M = G_1 \backslash G$. Then we define $g . u = g . u . g^{-1}$, for $g \in G$. Thus $g . u \in$ Hom (E, E').

Assume that X is an elemet of the Lie algebra $\mathcal{G}$ of G and that $g_t = \exp(tX)$. For any small value t, $g_t . u = g_t \, u \, g_t^{-1}$ is well defined and are morphisms from E to E' over the morphisms $g_t \, u_M \, g_t^{-1}$ of the base $M = G_1 \backslash G$. The set Hom (E, E') can be considered as the subset of vector space of linear mappings from $\Gamma(E)$ in the into $\Gamma(E')$. And thus we can define $X . u = \frac{d}{dt} (g_t . u) \Big|_{t=0}$. As in the finite dimensional case (cf. [4]), in the infinite one the following results hold:

1) X.u is a linear differential operator of first ordre from E into $u_M^* E'$ and $X . u = X_E . u - u . X_{E'}$, where X_E and $X_{E'}$ are considered as linear differential operators of first order of sections of E and E' respectively (see [4, Prop.1]).

2) If u is projected onto identity and the actions of G on E and E' are projected onto the identity of M, then X.u is a homomorphism from E into E'. This is the Lie derivation oj u respective to X, considered as the sections of $E^* \otimes E'$, where E^* is the dual fiber bundle of the bundle E (see [4, Prop.2]).

3) If now E = M, a morphism u is a section of the bundle E'. And we obtain the results on connection.

It is not hard to verify that in our sitution the commutative diagram takes place

$$\begin{array}{ccccc}
\mathrm{Ant}(E) & \longrightarrow & \mathrm{Aut}_{\text{semi-linear}}(\Gamma(E)) & & \\
\downarrow & & \downarrow & & \\
\mathscr{A}(E) & \xrightarrow[\cong]{} & \mathrm{End}_{\mathrm{Der}}(E) & \longrightarrow & \mathrm{Diff}^1_{\mathrm{scal}}(E) .
\end{array}$$

In our infinte — dimensional case it is also easy to prove that: The differential of a C^∞ —homomhrphism of a Lie group on Aut (E) is a homomorphism of the Lie algebra into the Lie algebra $\mathrm{Diff}^1_{\mathrm{scal}}$ (see [4, Prop. 5]).

It also yields the following corollary: The Infinite — dimensional generator of a one parameter group class C^∞ of automorphisms of the linear bundle E is a differential linear operator with scalar symbol.

2.3. Connection

In general a connection is a fashion of identification of fibres of the bundle. This is given by a differential 1—form α with values in the Lie algebra of the structural group (see 2.1) of the G—bundle. Then with this form o-connection we can write the explicit formula for the covariant derivation

$$\nabla_{\xi_X} = \xi_X + + 2\pi i/h \, . \, \alpha_1(\xi_X)$$

This is comparable to the results from 2.2.

2.4. Identification and computation of the form of connection

We recall that $\widetilde{\mathscr{E}}_{\sigma,\rho}$ is identified with the set of pairs $(g, v) \in G \times V$, factorized by the equivalence relation $(g, v) \sim (g', v')$ iff there exists $k \in G_F$ such that $g' = kg$, $v' = \delta(k)\,\sigma(k)\,v$.

The sections of the bundle $\widetilde{\mathscr{E}}_{\sigma,\rho}$ are identified with functions on G which are G_F—equivariant, i.e.

$$f(kx) = \delta(k)\,\sigma(k)\,f(x) \text{ (see [3]).}$$

The action of $g \in G$ on a section s is identified withthe action by right translations of the function $f = f_s$, see [3].

With this identification it follows the exact formula for the covariant derivation (see also [3, § 15.4])

$$\nabla_{\xi_X} = \xi_X + 2\pi\, i/h. \ \rho_1(X) = \xi_X + \rho(X) + d\delta(X)$$

where ξ_X is the Lie derivation, ρ is the known representation $d\delta$ is the differential of non—unitary character $\delta = (\Delta_H/\Delta_G)^{1/2}_{G_F}$. Thus the form of connection α_1 is associated with the representation ρ_1 in the above formula.

2.5. The differential form β

We recall that each point F in the K—orbit Ω is at the same time a linear function on the Lie algebra $\mathcal{G}$. Thus we can consider the expression $\langle P,X \rangle$, for $X \in \mathcal{G}$ in two ways. On the one hand, the function f_X $\langle .., X \rangle$ on Ω is the generating function for the Hamiltonian field ξ_X (see [3]). On the other hand, we can consider $\langle F, . \rangle$ as a differential 1—form β on G by the following formula

$$\langle \beta, \xi \rangle (F) = \langle F, \xi(e) \rangle$$

where ξ is an arbitrary vector field on G, f(e) is its value at the identity of the group G. As F is linear functional in $\mathcal{G} = T_eG$, β is a real 1 — form.

As a corollary from this we have

$$f_X(F) = \langle F, X \rangle = \langle \beta, \xi_X \rangle (F)$$

where $X \in \mathcal{G}$, ξ_X is the corresponding left — invariant vector field, f_X is the generating function of ξ_X.

2.6. End of the proof of the theorem

We have from the above consideration

$$\nabla_{\xi X} = \xi_X + \frac{2\pi i}{h} \alpha_1 (\xi_X)$$
$$= \xi_X + \frac{2\pi i}{h} f_X + \frac{2\pi i}{\hbar} (\alpha_1(\xi_X) - \beta(\xi_X))$$

We denote by α the differential 1—form $\alpha_1 - \beta$. Thus we have

$$\nabla_{\xi X} = \xi_X + \frac{2\pi i}{h} (f_X + \alpha(\xi_X)) = \frac{2\pi i}{h} \left(\frac{h}{2\pi i} \xi_X + f_X + \alpha(\xi_X) \right) = \frac{i}{\hbar} \widehat{X}$$

and the theorem is proved.

3. SOME REMARKS

1. From the proof of the theorem follows the explicit formula for values of the form α, occuring in the formula for the quantization operator

$$\widehat{f} = \xi_f + f + \alpha (\xi_f) ; \; \alpha = \alpha_1 - \beta$$

where α_1 is the differential 1—form associated with the representation $\rho_1 = \frac{h}{2\pi i} (\rho + d\delta)$.

2. The partially invariant partially holomorphic sections of $\mathcal{E}_{\sigma,\rho}$ form a subset of H—equivariant sections which satisfy also a system of equations for the complex subalgebra $\mathcal{P}$.

3. The theorem was not proved in the case of 1—dimen-sional quantization (Kirillov's quantization) [2, § 15]. Thus our proof is the first one.

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