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LINEAR SYSTEMS WITH STATE CONSTRAINTS IN BANACH SPACES

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§ I. INTRODUCTION

Let X, U be real Banach spaces. L(U, X) denotes the Banach space of all bounded linear operators from U in X. Throughout the paper we assume that U is separable.

An evolution operator on X is a function $(t, s) \longmapsto E(t, s)$ from $\{(t, s): t \geqslant s \geqslant t_0 > -\infty\}$ into L(X, X) with the following properties:

(i) $E(t, t) = I$ for all $t \geqslant t_0$ (I is the identity operator on X);

(ii) $(t, s) \longmapsto E(t, s)$ is strongly continuous;

(iii) $E(t, s)\, E(s, \tau) = E(t, \tau)$ for all $t \geqslant s \geqslant \tau \geqslant t_0$.

Suppose that $B(s): [t_0, \infty) \longmapsto L(U, X)$ is a given locally integrable function. Here, and in what follows, the integral is understood in the sense of Bochner. We consider the equation

$$x(t\,;\, t_0, x_0, u) = E(t, t_0)\, x + \int_{t_0}^{t} E(t, s)\, B(s)\, u(s)\, ds \qquad (1)$$

$$(t_0 \leqslant t \leqslant T)$$

where T is a given number, x_0 is a point of X and u(s) is a control, i.e. u(s) is a U – valued function, strongly Lebesgue measurable and essentially bounded on $[t_0, T]$.

A control u(.) is said to be admissible on a subinterval $[s, \tau] \subset [t_0, T]$ if it satisfies the constraint

$$u(t) \in \Omega(t) \qquad (2)$$

a.e. on $[s, \tau]$, where $\Omega(.)$ i a given measurable multi-function on $[t_0, T]$ taking nonempty closed values in some ball $S_r(0)$ of the Banach space U.

We recall that the multifunction $\Omega(.): [t_0, T] \longmapsto 2^U$ is said to be measurable if the set $\{t \in [t_0, T] : \Omega(t) \cap A \neq \phi\}$ is Lebesgue measurable for every open set $A \subset X$.

If $x(.)$ satisfies (1) for an admissible control u(.) on $[t_0, T]$ we say that $x(.)$ is a trajectory of the system (1)–(2) and $x(.)$ is generated by u(.).

It is well known that for every admissible control u(.), the function $x(.)$ defined by (1) is strongly continuous and can be considered as a mild solution of the system

$$\frac{dx}{dt} = A(t)x + B(t)u,$$

$$x(t_0) = x_0, \quad u(t) \in \Omega(t), \quad t \in [t_0, T],$$

where A(t) for all $t \in [t_0, T]$ are closed operators whose domain $D(A(t)) = D(A(t_0))$ is dense in X (see [4])

The qualitative study for infinite – dimensional control systems developed firstly by Fattorini H.O. has attracted a great deal of attention from many authors during the last decade [1 – 6]. Note, however, that investigations are devoted mainly to systems without any control and state constraints. The controllability of linear systems in Banach spaces with control constraints has been recently studied in [7–8].

In the present paper we shall be concerned with linear systems with constraints on both state and control variables. Namely, we consider the infinite – dimensional system described by the equation (1) with the control constraint (2) on $[t_0, T]$ and with the state constraints of the form.

$$x(t) \in N(t) \text{ for all } t \in [t_0, T], \tag{3}$$

where N(t) is a given lower semi — continuous multifunction from $[t_0, T]$ to closed convex subsets with nonempty interior in the Banach space X.

If x(.) is a trajectory of the system (1) – (2) satisfying the state constraint (3), we shall briefly say that it is a trajectory of (1) — (3), or (1)–(2)—(3) whenever the control constraint needs to be emphasized. We call x(.) interior if, in addition, x(t) belongs to the interior of N(t) for each $t \in [t_0, T]$.

In section 2 we show that the set $G(x_0)$ of all trajectories of (1) – (2) has a convex closure and, obtain thereby a generalized version of the « bang – bang » principle Jor trajectories.

In section 3 we use the results obtained in § 2 to derive some properties of the set $L(x_0)$ of all trajectories of (1) — (3). As a consequence, it follows, in particular, that the closure of the reachable set of the system (1) – (3) is a convex and continuous multifunction of initial points x_0 if the system possesses at least one interior trajectory from every x_0.

In section 4 the results of § 3 are applied to several optimization and controllability problems jor the systems (1) – (3).

NOTATIONS : Let X be a Banach space and X^* be the dual of X. Besides the notations introduced above we shall denote by

X_w, X^*_w : the spaces X and X^* endowd with the topologies $\sigma(X, X^*)$ and $\sigma(X^*, X)$, respectively;

$\langle f, x \rangle$: the value of a functional $f \in X^*$ at a point $x \in X$;

$S_\varepsilon(x)$, $S^*_\varepsilon(x^*)$: the open balls of radius ε around $x \in X$ and $x^* \in X^*$, respectively;

S_ε : the ball $S_\varepsilon(0)$;

coA, intA, $\overline{A}$, ∂A : the convex hull of a set $A \subset X$, its interior, closure and topological boundary, respectively;

$\rho(x, A)$: the distance of a point $x \in X$ from a set $A \subset X$;

$C([t_o, T], X)$, $C([t_o, T], X_w)$: the vector spaces of all continuous functions $x(.)$ from $[t_o, T]$ to X and X_w, respectively, endowed with the topologies o- uniform convergence;

$L^\infty([t_o, T], X)$: the Banach space of all (equivalent classes of) strongly measurable functions from $[t_o, T]$ to X such that $\| x(t) \|$ is essentially bounded on $[t_o, T]$. The norm in $L^\infty([t_o, T], X)$ is defined by

$$\| x(.) \| = \text{ess sup} \{ \| x(t) \| , \ t_o \leqslant t \leqslant T \} ;$$

$L^1([t_o, T], X)$: the Banach space of all (equivalent classes of) Bochner integrable functions from $[t_o, T]$ to X with the norm

$$\| x(.) \|_1 = \int_{t_o}^{T} \| x(t) \| \, dt ;$$

$L^\infty([t_o, T], X^*_w)$: the Banach space of all (equivalent classes of) scalarly measurable functions from $[t_o, T]$ to X^* which take values atmost everywhere in equicontinuous subsets of X^*. It is well known that $(L^1([t_o, T], X))^* = L^\infty(t_o, T]$, $X^*_w)$ (see, for example [14]);

$\chi_P(\tau)$: the characteristic function of a set P.

$\prod_{\alpha \in I} X_\alpha$: the cartesian product of the family of topological spaces $\{X_\alpha : \alpha \in I\}$ endowed with the product topology ([18]).

Finally, we recall that a multifunction on a Banach space X taking nonempty values in a topological vector space Y is called upper semicontinuous (lower semicontinuous) at $x_o \in X$ if for every open O-neighborhood S in Y there exists $\delta > 0$ such that $F(x) \subset F(x_o) + S$ ($F(x_o) \subset F(x) + S$, respectively) whenever $\| x - x_o \| < \delta$. The multifunction F is said to be continuous at x_o if it is simultaneously upper and lower semicontinuous at x_o.

§ 2. DENSE CONVEXITY AND GENERALIZED « BANG – BANG PRINCIPLE » FOR TRAJECTORIES

DEFINITION 1. A subset A of a Banach space si said to be *densely convex* if $\overline{A}$ is convex.

For many purposes in control and optimization theory the dense convexity is a quite convenient property, particularly in cases, where the separation technique for convex sets plays a crucial role. In fact, as will be shown later, in §4, even if the convexity is not guaranteed one can obtain useful results by proving and using the dense convexity of sets.

We first derive some properties of densely convex sets which will be frequently used in the paper.

PROPOSITION 1. Let A be a subset of a Banach space X and N be a convex set with nonempty interior in X. Then

(i) A is densely convex if and only if $\text{co}A \subset \overline{A}$;

(ii) If A is densely convex, then $A \cap \text{int}N$ is densely convex;

(iii) If A is densely convex and, in addition, $A \cap \text{int}N \neq \phi$, then $A \cap N$ is also densely convex. Furthermore, in this case, $\overline{A \cap N} = \overline{A \cap \text{int}N}$.

Proof. The assertion (i) is obvious. To prove (ii) assume $A \cap \text{int}N \neq \phi$ noting that the assertion is trivial in the converse case. Let $\varepsilon > 0$ be given and z be an arbitrary elemen of $\text{co}A \cap \text{int}N$. Since $z \in \text{int}N$ there exists $\delta > 0$ such that $S_\delta(z) \subset \text{int}N$. On the other hand, since $z \in \text{co}A$ and, in view of (i), $\text{co}A \subset \overline{A}$, one can find an element $a \in A$ such that $\| z - a \| < \min \{\delta, \varepsilon\}$. Thus, given $\varepsilon > 0$, for every $z \in \text{co}A \cap \text{int}N$ there exists $a \in A \cap \text{int}N$ such that $\| z - a \| < \varepsilon$. This means that $A \cap \text{int}N$ is dense in $\text{co}A \cap \text{int}N$ and, hece, in $\text{co}(A \cap \text{int}N)$ since $\text{co}(A \cap \text{int}N) \subset \text{co}A \cap \text{co}(\text{int}N) = \text{co}A \cap \text{int}N$. The assertion follows now from (i).

In order to prove (iii), it clearly suffices to show that $A \cap N \subset \overline{A \cap \text{int}N}$. Let $z_0 \in A \cap \text{int}N$ be given. Then, for each $z \in A \cap N$ and $\lambda \in [0, 1]$, setting $y(\lambda) = \lambda z + (1 - \lambda) z_0$, we observe that $y(\lambda) \in \text{co}A$ and $y(\lambda) \to z$ as $\lambda \to 1$. Besides, by convexity of N, $y(\lambda) \in \text{int}N$ for all $\lambda \in [0,1)$. Hence, for every $\varepsilon > 0$, there exists $\lambda_0 > 0$ such that $\| y(\lambda_0) - z \| < \varepsilon/2$ and $y(\lambda_0) \in \text{co}A \cap \text{int}N$. Then, as has been shown in the proof of (ii), there exists $a \in A \cap \text{int}N$ such that $\| y(\lambda_0) - a \| < \varepsilon/2$ and, consequently, $\| z - a \| < \varepsilon$. This implies the required inclusion.

Remark that in the assertion (iii) the condition $A \cap \text{int}N \neq \phi$ is essential. To see this let consider in the space $X = R_2$ subsets A, N defined as

$$A = \{(x, y) \in R_2 : 0 < x < 1,\ 0 < y < 1\} \cup \{(1,1), (1,0)\},$$
$$N = \{(x, y) \in R_2 : 1 \leqslant x \leqslant 2,\ 0 \leqslant y \leqslant 1\}.$$

Clearly, A is densely convex and N is convex with nonempty interior. Nethertheless, $A \cap N = \{(1,1), (1,0)\}$ and so is not densely convex. Here $A \cap \text{int}N = \phi$.

Now, consider the system (1) — (2) and assume that all hypotheses stated above with respect to the system are satisfied. Parallelly with the control constraint (2) we also consider the constraint of the form.

$$u(t) \in \overline{\text{co}}\,\Omega(t) \quad \text{a.e. on } [t_0, T] \tag{2)'}$$

and in this connection, we shall speak of the system (1) — (2)'.

Given an initial state $x_o \in X$, let $G(x_o)$ and $\widetilde{G}(x)$ denote the sets of all trajectories of the systems (1) — (2) and (1) — (2)', respectively. It is clear that $\widetilde{G}(x_o)$ is a convex subset of the Banach space $C([t_o, T], X]$. With respect to $G(x_o)$ we have the following

THEOREM 1. *$G(x_o)$ is a densely convex set in the Banach space $C([t_o, T], X)$ and, moreover, $\overline{G(x_o)} = \widetilde{G}(x_o)$.*

We first derive some auxiliary lemmas which will be used in the proof of Theorem I.

For every $t', t'', t_o \leqslant t' < t'' \leqslant T$, let us denote

$U(t', t'') = \{u(.) \in L^\infty([t', t''], U) : u(t) \in \Omega(t) \text{ on} [t', t'']\}$;

$\widetilde{U}(t' t'') = \{u(.) \in L^\infty([t', t''], U) : u(t) \in \overline{co}\, \Omega(t) \text{ on} [t', t'']\}$;

$$V(t', t'') = \{\int_{t'}^{t''} E(t'', s)\, B(s)\, u(s) ds : u(.) \in U(t', t'')\};$$

$$\widetilde{V}(t', t'') = \{\int_{t'}^{t''} E(t'', s)\, B(s)\, u(s) ds : u(.) \in \widetilde{U}(t', t'')\}.$$

LEMMA 1. Under the stated hypotheses, $V(t', t'')$ is a densely convex set in the Banach space X and, moreover, $\overline{V(t', t'')} = \widetilde{V}(t', t'')$.

Proof. This is an immediate consequence of Theorem 3 of [9].

Some more general result of this kind can be found in [12].

LEMMA 2. Suppose $\widetilde{x}(.)$ is a trajectory of system (1) — (2)'. Then, for every finite colection of points $t_i \in [t_o, T]$, $i = \overline{1, k}$, there exists a trajectory x_ε of the system (1) — (2) such that

$$\| x_\varepsilon (t_i) - x(t_i) \| < \varepsilon \quad \text{for } i = 0, 1, \dots k. \tag{4}$$

Proof. We can assume without loss of generality that $t_o < t_1 < \dots < t_k = T$. Consider the restriction of $\widetilde{x}(.)$ to subinterval $[t_o, t_1]$. From the definition we get $x(t_1) = E(t_o, t_1)\, x_o + \widetilde{v}_1$ with $\widetilde{v}_1 \in \widetilde{V}(t_o, t_1)$. By Lemma 1, for every $\delta > 0$ there exists $u_1^\delta(.) \in U(t_o, t_1)$ such that

$$\| \widetilde{v}_1 - \int_{t_o}^{t_1} E(t_1, s)\, B(s)\, u_1^\delta(s)\, ds \| < \delta.$$

Then, for the corresponding trajectory $x_1^\delta(.) = x(.; t_o, x_o, u_1^\delta)$ we obtain

$$\| x_1^\delta (t_1) - \widetilde{x}(t_1) \| < \delta. \tag{5}$$

Now consider $\widetilde{x}(.)$ on the subinterval $[t_1, t_2]$. Using property (iii) of the evolution operator E(t, s) we can write

$$\widetilde{x}(t_2) = E(t_2, t_1)\widetilde{x}(t_1) + \widetilde{v}_2 \text{ where } \widetilde{v}_2 \in \widetilde{v}(t_1\ t_2).$$

As above, it follows from Lemma 1 that there exists

$$u_2^{\delta}(.) \in U(t_1, t_2) \text{ such that } \| \widetilde{v}_2 - \int_{t_1}^{t_2} E(t_2, s)\, B(s)\, u_2^{\delta}(s)ds \| < \delta \tag{6}$$

Let $x_2^{\delta}(.) = x(.; t, x_1^{\delta}(t_1), u_2^{\delta})$ be the corresponding trajectory. Then, setting

$$M = \max \{ \| E(t, s) \|, t_o \leqslant s \leqslant t \leqslant T \} \tag{7}$$

we get, on account of (5), (6), that

$$\| x_2^{\delta}(t_2) - \widetilde{x}(t_2) \| \leqslant (M + 1)\, \delta.$$

In general, an analogous argument shows that for every Sunbinveatrl $[t_{i-1}, t_i]$ $(i = \overline{1, k})$ there exists a control $u_i^{\delta}(.) \in U(t_{i-1}, t_i)$ such that the corresponding trajectory

$$x_i(.) = x(.; t_{i-1}, x_{i-1}(t_{i-1}), u_i^{\delta})$$

satisfies $\| x_i^{\delta}(t_i) - \widetilde{x}(t_i) \| \leqslant (M^{i-1} + ... + M + 1))\delta$

Setting $u^{\delta}(t) \equiv u_i^{\delta}(t)$ for $t_{i-1} \leqslant t < t_i$ $(i = \overline{1, k})$

we thus obtain an admissible control on $[t_o, T]$. Let $x_\varepsilon(.) = x(.; t_o, x_o, u^{\delta})$ be the trajectory of (1) — (2) generated by $u^{\delta}(.)$ with

$$\delta = \varepsilon/(M^{K-1} + ... + M + 1).$$

Then, obviously. $x_\varepsilon(.)$ satisfies (4), as was to be proved.

Proof of Theorem 1. Clearly, it suffices to show that $G(x_o)$ is dense in $\widetilde{G}(x_o)$, that is, given $\varepsilon > 0$ and $\widetilde{x}(.) \in \widetilde{G}(x_o)$ there exists $x_\varepsilon(.) \in G(x_o)$ such that $\| \widetilde{x}(.) - x_\varepsilon(.) \| < \varepsilon$. To this end we choose $\delta < 0$ so that

$$\int_E \| B(t) \| dt < \varepsilon / (4Mr) \tag{8}$$

for any measurable subset $E \subset [t_o, T]$ satisfying $\mu E < \delta$. Taking a finite colection of points $t_i \in [t_o, T]$. $i = \overline{1, k}$ with $\max \{ | t_i - t_{i-1} |, i = \overline{1, k} \} < \delta$ we find, in view of Lemma 2, a trajectory $x_\varepsilon(.) \in G(x_o)$ such that

$$\| x_\varepsilon(t_i) - \widetilde{x}(t_i) \| \leqslant \min \{ \varepsilon / 2. \varepsilon / (2M) \} \tag{9}$$

for $i = 0, 1. .... k$.

Denote by $u_\varepsilon(.)$ and $\tilde{u}(.)$ the control generating trajectories $x_\varepsilon(.)$ and $\tilde{x}(.)$, respectively. Let t be an arbitrary point of $[t_o, T]$ and let $(t_{i-1}, t_i]$ be a subinterval containing t. Then from (8), (9) it implies

$$\| x_\varepsilon(t) - \tilde{x}(t) \| \leqslant \| E(t, t_{i-1}) \| \; \| x_\varepsilon(t_{i-1}) - \tilde{x}(t_{i-1}) \| +$$

$$\int_{t_{i-1}}^{t} \| E(t, s) \| \; \| B(s) \| \; \| u_\varepsilon(s) - \tilde{u}(s) \| \, ds < \varepsilon/2 + 2Mr \int_{t_{i-1}}^{t_i} \| B(s) \| \, ds < \varepsilon$$

which completes the proof of Theorem.

Remark 1: If the control space U is reflexive, then as will be seen from the proof of Theorem 3 below, the set $\tilde{G}(x_o)$ of all trajectories of the system (1) – (2)' is actually closed. Hence, in this case, Theorem 1 asserts that $\overline{G(x_o)} = \tilde{G}(x_o)$.

Consider the evaluation operator $H_T : C([t_o, T], X) \to X$ defined by $H_T x(.) = x(T)$. Clearly, H_T is linear and continuous. We call $R(x_o) = H_T G(x_o)$ the reachable set from x_o of the system (1)–(2). Analogously, $\tilde{R}(x_o) = H_T \tilde{G}(x_o)$ is called the reachable set of the system (1)–(2)'.

By the continuity of H_T, Theorem 1 implies the following well–known result (see, for exemple, [7], [10]).

COROLLARY 1. The reachable set $R(x_o)$ of the sytem (1) – (2) is a densely convex set in the Banach spare X and, moreover, $\overline{R(x_o)} = \tilde{R}(x_o)$

Remark 2: If the control space U is reflexive, then, in the preceding corollary, $\overline{R(x_o)} = \tilde{R}(x_o)$.

Let, besides the stated hypotheses, the restraint control set $\Omega(t)$ be compact, convex for everery $t \in [t_o, T]$. Denote by $\ddot{\Omega}(t)$ the set of extreme points of $\Omega(t)$. Since, by the Krein–Milman theorem, $\overline{co}\, \ddot{\Omega}(t) = \Omega(t)$, from Theorem 1 we immediately obtain the following.

COROLLARY 2. Assume that $\Omega(t)$ is compact and convex for every $t \in [t_o, T]$. The set of all trajectories of the system (1) generated by extreme controls $u(t) \in \ddot{\Omega}(t)$ is dense in $G(x_o)$ for the norm of uniform convergence.

Theorem 1, therefore, can be considered as a generalized version of the well–known « bang–bang principle » applied to trajectories.

§3. THE SET OF TRAJECTORIES OF A LINEAR SYSTEM WITH BOTH STATE AND CONTROL CONSTRAINTS

Consider now the system (1) with the control contraint (2): $u(t) \in \Omega(t)$ a. e. on $[t_o, T]$, and the state contraint (3): $x(t) \in N(t)$ for every $t \in [t_o, T]$.

Recall that $N(t)$ is assumed to be a lower semi continuous multifunction from $[t_o, T]$ to closed convex subsets with nonempty interior in X. Let us define

$$L(x_o) = \{x(.) \in G(x_o) : x(t) \in N(t) \ (\forall t \in [t_o, T])\} \tag{10}$$

(the set of all trajectories of system (1) – (3)) ; and

$$L_o(x_o) = \{x(.) \in G(x_o) : x(t) \in \text{int } N(t) \ (\forall t \in [t_o, T])\} \tag{11}$$

(the set of all interior trajectories of system (1) — (3)).

In a similar way, but replacing $G(x_o)$ by $\widetilde{G}(x_o)$ we define $\widetilde{L}(x_o)$ and $\widetilde{L}_o(x_o)$.

THEOREM 2. *$L_o(x_o)$ is a densely convex set in $C([t_o, T], X)$. If, in addition, $L_o(x_o)$ is nonemply, then $L(x_o)$ is alson densely convex and, moreover, $\overline{L(x_o)} = \overline{L_o(x_o)}$.*

To prove Theorem 2 we need the following

LEMMA 3. Let there be given a lower semicontinuous multifunction $N(t)$ from $[t_o, T]$ to closed convex subsets with nonempty interior in a Banach space X. Let $\mathcal{N}_o$ be the set of all continuous interior sections of $N(t)$ (i.e. $\mathcal{N}_o = \{z(.) : z(t) \in \text{int}N(t)$ for all $t \in [t_o, T]\}$) and let $\mathcal{N}$ be the set of all continuous sections of $N(t)$. If $\mathcal{N}_o$ is nonempty then $\mathcal{N}_o = \text{int } \mathcal{N}$.

Proof. The inclusion $\text{int } \mathcal{N} \subset \mathcal{N}_o$ is obvious. To prove the inverse inclusion, let $z(.)$ be an arbitrary element of $\mathcal{N}_o$. Setting $\delta(t) = \rho(z(t), \partial N(t))$ we observe that $\delta(t) > 0$ and $S_{\delta(t)}(z(t)) \subset N(t)$ for every $t \in [t_o, T]$. Moreover, as will be seen below, $\delta(t)$ is a lower semicontinuous function on $[t_o, T]$. Therefore, by the compactness of $[t_o, T]$ there exists $\overline{t} \in [t_o, T]$ such that $0 < \delta(\overline{t}) = \min \{\delta(t) : t \in [t_o, T]\}$. It implies that $S_{\delta(\overline{t})}(z(.)) \subset \mathcal{N}$ and hence $z(.) \in \text{int } \mathcal{N}$.

It remains to prove the lower semicontinuity of $\delta(t)$ on $[t_o, T]$. Let $\varepsilon > 0$ and $t_f \in [t_o, T]$ be given. Since $N(t)$ is lower semicontinuous on $[t_o, T]$, there exists $\gamma_1 > 0$ such that

$$N(t_f) \subset N(t) + S_\varepsilon \text{ whenever } |t - t_f| < \gamma_1. \tag{12}$$

Since $z(t)$ is continuous, there exists $\gamma_2 > 0$ such that

$$\|z(t) - z(t_f)\| < \min\{\varepsilon, \delta(t_f)\} \text{ whenever } |t - t_f| < \gamma_2. \tag{13}$$

Taking $\gamma = \min\{\gamma_1, \gamma_2\}$, for every $t \in [t_o, T]$ satisfying $|t - t_f| < \gamma$ and every $y \in \partial N(t_f)$ we have $\delta(t_f) = \rho(z(t_f), \partial N(t_f)) \leqslant \|z(t_f) - y\| \leqslant \|z(t_f) - z(t)\| + \|z(t) - y\| < \varepsilon + \|z(t) - y\|$. Consequently,

$$\delta(t_f) < \varepsilon + \rho(z(t), \partial N(t_f)) \text{ whenever } |t - t_f| < \gamma. \tag{14}$$

On the other hand, from the definition of $\delta(t)$ and (12) (13) we obtain

$$z(t) \in S_{\delta(t_f)}(z(t_f)) \subset N(t_f) \subset N(t) + S_\varepsilon,$$

which implies, in connection with (14)

$$\delta(t_f) \leqslant \varepsilon + \rho(z(t), \partial(N(t) + S_\varepsilon)) \tag{15}$$

of each t satisfying $|t - t_f| < \Upsilon$. Now, set

$$\delta_\varepsilon(t) = \rho(z(t), \partial(N(t) + S_\varepsilon)).$$

If $\delta_\varepsilon(t) \leqslant \varepsilon$ then obviously, by (15) $\delta(t_f) \leqslant 2\varepsilon + \rho(z(t), \partial N(t))$. Let suppose $\delta_\varepsilon(t) > \varepsilon$. Then we obtain $z(t) + S_{\delta_\varepsilon(t)} \subset N(t) + S_\varepsilon$, or, equivalently' $z(t) + S_{\delta_\varepsilon(t)-\varepsilon} + S_\varepsilon \subset N(t) + S_\varepsilon$. Since N(t) is convex and closed it follows from the above inclusion that $z(t) + S_{\delta_\varepsilon(t)-\varepsilon} \subset N(t)$ and so $\rho(z(t), \partial N(t)) \geqslant \delta_\varepsilon(t) - \varepsilon$, or, $\delta_\varepsilon(t) \leqslant \varepsilon + \rho(z(t), \partial N(t))$. Therefore, in all cases, from (15) we obtain

$$\delta(t_f) \leqslant 2\varepsilon + \delta(t) \quad \text{whenever} \quad |t - t_f| < \Upsilon.$$

This proves the lower semicontinuity of $\delta(t)$ and completes the proof of Lemma.

Proof of Theorem 2. Clearly, by (10), (11)
$L(x_0) = G(x_0) \cap \mathcal{N}$ and $L_0(x_0) = G(x_0) \cap \mathcal{N}_0$ where $\mathcal{N}$ and $\mathcal{N}_0$, as above denote the sets of all continuous sections and continuous interior sections of $N(t)$ $(t_0 \leqslant t \leqslant T)$, respectively. Observe that $G(x_0)$ is a densely convex set in $C([t_0, T], X)$ by Theorem 1, and $\mathcal{N}_0 = \mathrm{int}\mathcal{N}$, by Lemma 3. Moreover, by virtue of the convexity of N(t), $\mathcal{N}$ is a convex subset of $C([t_0, T], X)$. Theorem 2 now readily follows from Proposition 1.

As previously, by using the evaluation operator H_T we obtain.

Corollary 3. If the system (1)—(2)—(3) possesses an interior trajectory from $x_0 \in X$ then its reachable set $R_c(x_0)$ is densely convex in X and, moreover, $\overline{R_c(x_0)} = \widetilde{R}_c(x_0)$ where $\widetilde{R}_c(x_0)$ denotes the reachable set of the system (1) — (2)' — (3).

This result has been obtained in [11] for a finite-dimensional system with state constraints of a less general type.

THEOREM 3. *Assume that all stated hypotheses hold for the system (1)—(2)—(3) Let, in addition, the spaces X and U be reflexive and X be sparable. The mapping* $L : x_0 \rightarrow L(x_0)$ *defined by (10) and regarded as a multifunction from X into* $C([t_0, T], X_W)$ *is continuous at any point* $x_0 \in X$ *whenever the system possesses an interior trajectory from this point. Moreover, L is lower semicontinuous when regarded as a multifunction from X into* $C([t_0, T], X)$.

To prove Theorem 3 we need the following

LEMMA 4. Let $\widetilde{\Omega}(t)$ be a measurable multifunction from $[t_0, T]$ with nonempty closed convex values in a separable reflexive Banach space U such that for every t, $\widetilde{\Omega}(t)$ is contained in some given ball S_r of U. Then the set $\widetilde{U}(t_0, T)$ of all measurable sections of $\widetilde{\Omega}(t)$ on $[t_0, T]$ is compact and metrisable for theeweak topology $\sigma_0 = \sigma(L^\infty([t_0, T], U), L^1([t_0, T], U^*))$.

Proof. Denote $E = U^*$, then, since U is separable and reflexive, it follows that E is separable and $E^* = U$. It is well known that in this case $L^\infty([t_o, T], E^*_W) = L^\infty([t_o, T], E^*) = (L^1([t_o, T], E))^*$ (see e.g. [12]). Further, for every t, $\widetilde{\Omega}(t)$ being closed convex bounded subset of U $(= E^*)$ is compact for the weak topology $\sigma(E^*, E)$. Now, according to Theorem V–1 of [15] we conclude that $\widetilde{U}(t_o, T)$ is convex and compact for the weak topology $\sigma(L^\infty[t_o, T], E^*), L^1([t_o, T], E))$, which, clearly, is equal to σ_o. Furthermore, since $L^1([t_o, T], E)$ is separable due to the separability of E, it follows that $U(t_o, T)$ is metrisable for the weak topology σ_o. The proof is complete.

Proof of Theorem 3. Assuming $L_o(\bar{x}_o) \neq \phi$ we shall prove that L is simultaneously upper and lower semicontinuous at $\bar{x}_o$. Note first that $L(x_o)$ is not empty for all x_o in some neighbourhood of $\bar{x}_o$. Indeed, let $\bar{x}(.)$ be an interior trajectory of (1) – (3) satisfying $\overline{x}(t_o) = \overline{x}_o$ (i.e. $\overline{x}(.) \in L_o(\overline{x})$) and $\overline{u}(.)$ be an admissible control generating $\overline{x}(.)$. Since $\overline{x}(t) \in \text{int } N(t)$ for every $t \in [t_o, T]$, it follows from Lemma 3 that there exists $\delta > 0$ such that $S_\delta(x(.)) \subset \mathcal{N}$. Then for each x_o in the ball $S_{\delta'}(\overline{x_o})$ with $\delta' = \delta / M$ (M is defined by (7)) the trajectory

$$x(t) = E(t, t_o)\, x_o + \int_{t_o}^{t} E(t, s)\, B(s)\, \overline{u}(s)\, ds, \quad t_o \leqslant t \leqslant T,$$

as easily seen, is an element of $L(x_o)$. We note additionally that for each such $x(.)$ we have $\| x(.) - \overline{x}(.) \|_C < \delta$.

To show the upper semicontinuity of L, we note that $L(x_o) \subset C([t_o, T], K)$ for all x_o in a neighborhood of $\overline{x}_o$ where K is a fixed ball in X. As X is separable and reflexive, K is a compact metrisable set in X_W. Hence, $C([t_o, T], K)$, regarded as a subspace of $C([t_o, T], X_W)$, is metrisable. Suppose that L is not upper semicontinuous at $\bar{x}_o$. Then, by definition, there exist a abalanced open 0–neighbourhood S in $C([t_o, T], X_W)$ and sequences $\{x_{on}\} \subset X$ and $\{x_n(.)\} \subset L(x_{on})$ such that $x_n(.) \notin L(x_o) + S$ and $\| x_{on} - x_o \| \to 0$ as $\to \infty$. Let $\{u_n(.)\}$ be a sequence of admissible controls generating the trajectories $\{x_n(.)\}$, i.e.

$$x_n(t) = E(t, t_o)x_{on} + \int_{t_o}^{t} E(t, s)\, B(s)\, u_n(s)ds, \quad t_o \leqslant t \leqslant T.$$

Then it is easily seen that the sequence $\{x_n(.)\}$ is equicontinuous and, in addition, $x_n(t) \in K$ for all $t \in [t_o, T]$ and all $n = 1, 2, ...$ By virtue of the Ascoli theorem and the metrisability of $C([t_o, T]; K)$ there exists a subsequence $\{x_{n_k}(.)\}$ converging to $x_\infty(.)$ in $C([t_o, T], K)$.

It follows that

$$x_\infty(.) \notin L(\overline{x}_o) + \frac{1}{2} S. \qquad (16)$$

Also, since $x_n(t) \in N(t)$ for each $t \in [t_o, T]$ and $N(t)$ is closed and convex we have

$$x_\infty(t) \in N(t) \quad (\forall t \in [t_o, T]). \tag{17}$$

On the other hand, by Lemma 4 we can suppose that the corresponding sequence of admissible controls $u_{n_k}(.)$ converges in the weak topology $\delta(L^\infty([t_o, T], U), L^1([t_o, T], U))$ to an admissible control $u_\infty(.) \in \widetilde{U}(t_o, T)$. Thus, for every $f \in U^*$ and $t \in [t_o, T]$ we have

$$< f, x_{n_k}(t) > = < f, E(t_o, t)\, x_{on_k} > + \int_{t_o}^{T} \chi_{[o,t]}(s) < B^*(s) E^*(t, s)\, f\, u_{n_k}(s) > ds$$

$$\to < f, x_\infty(t) > = < f, E(t_o, t)\, \overline{x} > + \int_{o}^{T} \chi_{[o,t]} < B^*(s) E^*(t, s)\, f, u_\infty(s) > ds =$$

$$= < f, E(t, s)\, \overline{x}_o + \int_{o}^{t} E(t, s) B(s) u_\infty(s) ds >.$$

This with (17) shows that $x_\infty(.)$ is a trajectory of (1) — (2) — (3) satisfying $x_\infty(t_o) = \overline{x}_o$, i. e. $x_\infty(.) \in \widetilde{L}) \overline{x}_o)$. Now, since $\widetilde{L}_o(\overline{x}) \neq \phi$ we conclude, in view of Theorem 2 that there exists a trajectory $\widetilde{x}_\infty(.) \in \widetilde{L}_o(\overline{x}_o)$ such that $\widetilde{x}_\infty(.) \in x_\infty(.) + \frac{1}{2} S$. Further, since $\widetilde{G}(\overline{x}_o) = \overline{G(\overline{x}_o)}$ by Theorem 1, one can find a trajectory $x_\varepsilon(.) \in L(\overline{x}_o)$ such that $x_\varepsilon(.) \in \widetilde{x}_\infty(.) + \frac{1}{2} S$. Consequently $x_\infty(.) \in L(\overline{x}_o) + S$ which contradicts (16). Thus L is semicontinuous at x_o.

To prove the lower semicontinuity of L at x_o, let $\overline{x}(.)$ be an arbitrary element of $L(\overline{x}_o)$. By Theorem 2, for every $\varepsilon > 0$ there exists $\overline{x}_\varepsilon(.) \in L_o(\overline{x}_o)$ such that $\| \overline{x}(.) - \overline{x}_\varepsilon(.) \|_C < \varepsilon/2$. Then, in view of the remarks preceding the proof of the upper semicontinuity of L, one can find $\delta > 0$ such that for every $x_o \in S_\delta(\overline{x}_o)$ there exists a trajectory $x(.) \in L(x_o)$ satisfying $\| x(.) - \overline{x}_\varepsilon(.) \|_C < \varepsilon/2$. Therefore $\| \overline{x}(.) - x(.) \|_C < \varepsilon$ and so $\overline{x}(.) \in L(x_o) + S_\varepsilon$. So we have shown that for every $\varepsilon > 0$ there exists $\delta < 0$ such that $L(x_o) \subset L(\overline{x}_o) + S_\varepsilon$ whenever $\| x_o - \overline{x}_o \| < \delta$. Thus L is lower semicontinuous at $\overline{x}_o$ when regarded as a multifunction from X into $C([t_o, T], X)$. The proof of Theorem is complete.

COROLLARY 4. Under the hypotheses of Theorem 3, the mapping $R_c : x_o \to R_c(x_o)$ ($R_c(x_o)$ is the reachable set of the system (1)—(3)) regarded as a multifunction from X to X_W is continuous at point any $\overline{x_o}$ whenever the system (1) — (3) possesses an interior trajectory from this point. Moreover R_c is lower semicontinuous when regarded as a multifunction from X into X.

CORROALRY 5. The reachable set $R_c(x_o)$ of the system (1) — (3) with $X = R^n$ and $U = R^m$ determines a mutifunction which is continuous at any proint x_o whenever the system possesses an in interior trajetory from this point.

4. APPLLICATIONS

In this concluding paragraph we use the results obtained in the previous sections to study several optimization and controllability problems for the systems (1) — (3).

Assume that all hypotheses stated in Section 1 for the system (1) – (3) are satisfied.

Given an initial state $x_o \in X$, let us consider the optimziation problem

$$\begin{cases} \text{Minimize } \varphi\,(x_o,\, x(.)) \\ \text{subject to } x(.) \in L(x), \end{cases} \tag{18}$$

where $L(x_o)$, as previously, denotes the set of all trajectories of (1)—(3) satisfying $x(t_o) = x_o$, and the cost functional φ is assumed to be continuous and convex on $X \times C([t_o,\ T]X)$.

Let us denote by $\partial\varphi(x(.))$ the subgradient of the function $x(.) \to \varphi(x_o, x(.))$.

THEOREM 4. *Let the system (1) – (3) prossess an interior trajectory with a gien initial point* $\overline{x}_o \in X$ *(i. e.* $L_o(\overline{x}_o) \neq \Phi$*).*

(i) The necessary and sufficient condition fo a trajectory $x^o(.) = x^o(.\,; x_o, u^o)$ *to be a solution of the problem (18) is that there exists a functional* $f \in \partial\varphi(x^o(.))$ *such that,* $\langle f, x^o(.) \rangle = min \{ \langle f, x(.) \rangle : x(.) \in L(\overline{x_o}) \}$

(ii) Under the usumptions of Theorem 3, the Belman function of the prodlem (18) which is defined by

$$J(x_o) = inf\,\{\varphi(x_o,\ x(.)) : x(.) : x(.) \in) Lx_o)]$$

is continuous at $\overline{x}_o$.

Proof. To prove (i) we first note that a trajectory $x^o(.)$ is a solution of he problem (18) if and only if it is a solution of the following problem

$$\begin{cases} \text{Minimize } \varphi(\overline{x}_o,\ x(.)) \\ \text{subject to } x(.) \in \overline{L(\overline{x}_o)}. \end{cases}$$

Since $\overline{L(\overline{x}_o)}$ is convex (by Theorem 2), the last property is equivalent to the existence of a functional $f \in \partial\varphi(x^o(.))$ satisfying

$$\langle f, x^o(.) \rangle = \min\{\langle f, x(.)\rangle : x \in \overline{L(\overline{x}_o)}$$

(see, for example, [16]).

Thus $\qquad \langle f, x^o(.)\rangle = \min\,\{\langle f, x(,)\rangle : x \in L(\overline{x}_o)\}$

For the assertion (ii) we shall prove that J(.) is simultaneously upper and lower semicontinuous at $\overline{x}_o$. The upper semicontinuity follows from Theorem 2 of § 2.5 ([17], Chapter 2) and the fact that the multifunction $L: C([t_o, T],X) \to \to X$ is lower semicontinuous at x_o by virtue of Theorem 3 above. For the lower semicontinuity we first prove that the function φ is lower semicontinuous when regarded as a function from $X \times C([t_o t\ T],\ X_W)$ into X. The main points of the proof is due to [13].

Choose $\delta > 0$ so that $L(x_0)$ is not empty for every x^0 in $S\delta(\bar{x})$ and define

$$\tilde{L} = \cup \{ \widetilde{L}(x_0) ; x_0 \in s_\delta (\overline{x_0}) \}$$

and

$$Z = S_\sigma (\overline{x_0}) \times \bar{L}.$$

where the closure of L is with respect to the topology of uniform convergence of $C([t_0, T], X_W)$. Note that Z is closed and equicontinuous when regarded as a subset of $X_W \times C([t_0, T, X_W)$ as well as a subset of $X \times C([t_0, T], X)$.

Since Z is closed in $X \times C([t_0, T], X_W)$ is suffices to show that φ is lower semicontinuous on Z, or, equivalently, that the set

$$S(\varphi, \lambda) = \{ z \in Z : \varphi(z) \leqslant \lambda \}$$

is closed for every λ. By hypotheses, it is clear that $S(\varphi, \lambda)$ is convex and closed in $X \times X^I$ where X^I denotes the space of all functions from $I = [t_0, T]$ in to X endowed with the topology of uniform convergence. Since $S(\varphi, \lambda)$ is equicontinuous. Corollary 0.4.10 of [14] shows that it is also closed in the topology of simple convergence of $X \times X^I$ and hence closed in the product topology of $V = \dot{X} \times \prod_{t \in I} X^t$. It follows by the convexity of $S(\varphi,\lambda)$ that it is closed in the topology $\sigma(V, V_W)$. In view of the identity

$$\sigma(V, V_W) = \sigma(X, X_W \times \prod_{t \in I} \sigma(X^t, X^t_W)$$

(by Theorem 4.3, Chapter IV, [18]) we conclude that $S(\varphi, \lambda)$ is closed when regarded as a subset of the space $X_W \times \prod_{t \in I} X^t_W$ (endowed with the product topology). Again, the equicontinuity of $S(\varphi, \lambda)$ implies that it is closed in $X_W \times C(t_0, T], X_W)$, which is obviously equivalent to φ being lower semicontinuous on Z.

Further, according to Theorem 3, the multifunction $L : X \times C(t_0, T], X_W)$ is upper semicontinuous at $\overline{x}_0$ and $L(x_0)$ is nonempty compact with respect to the topology of uniform convergence of $C([t_0, T], X_W)$. The lower semicontinuity of J(.) is now a direct consequence of Theorem 1 of § 2.5 ([17], Chapter 2), This concludes the proof.

COROLLARY 6. Assume the hypothese of theorem 4; in addition assume that the control set $\Omega(t)$ is convex for every $[t \in t_0, T]$. Then the solution set of the problem (18) is not empty.

COROLLARY 7. Let $h(x)$ be a conxex and continuous functional on X and suppose that the system (1)–(3) possesses an interior trajectory satisfying $x(t_0) = \overline{x}_0$. A trajectory $x^0(.)$ is a solution of the terminal control problem

$$\begin{cases} \text{Minimize } h(x(T)) \\ \text{Subject to } x(.) \in L(\overline{x_0}) \end{cases}$$

if and only if there exists a functional $g \in \partial h(x^0(T))$ such that

$$\langle g, x^0(T) \rangle = \min \{ \langle g, x \rangle : x \in R_c(\overline{x_0}) \}$$

(where $R_c(\overline{x}_o)$ denotes the reachable set from $\overline{x}_o$ of the system (1) – (3) and $\partial h(x)$ denotes the subdifferential of h at $x_o \in X$.

Proof. Setting $\varphi_h(x.)) = h(x(T)) = h(H_T x(.))$, we observe, by the calculus of subdifferentials, that $\varphi_h(x(.)) = H_T^* h(H_T x(.)) = H_T^* \partial h(x(T))$. Thus, such functionat $f \in \partial\varphi_h(x(.))$ can be written as $f = H_T^* g$ with some $g \in \partial h(x(T))$. Now applying the above theorem to the cost functional $\varphi_h(x(T))$ and taking into account that

$$\langle H_T^* g, x(.) \rangle = \langle g, H_T x(.) \rangle = \langle g, x(T) \rangle$$

we obtain the assertion.

As an another application of the preceding results we consider the following controllability problem for the system (1)–(3).

Let M_o, M_1 be compact convex sets in X such that $M_o \subset N(t_0)$.

DEFINITION 2. The system (1)–(3) is said to be approximately M_oM_1—controllable on the time interval $[t_o, T]$ if for every $\varepsilon > 0$ there exists an initial-state $x_0^\varepsilon \in M_o$ and an admissible control $u^\varepsilon(.) \in U(t_o, T)$ such that $x(T) = x(T, x_0^\varepsilon, u_0^\varepsilon) \in M_1 + S_\varepsilon$.

THEOREM 5. *Assume that the system (1) – (3) possesses an interior trajectory from avery point $x_o \in M_o$. The the system is approximately M_oM_1 – controllable if and only if*

$$\sup_{x_o \in M_o} \min_{f \in S_1^*} \left\{ \sup_{x \in R_c(x_o)} \langle f, x \rangle + \max_{x_1 \in M_1} \langle f, -y \rangle \right\} \geq 0 \tag{19}$$

(where $R_c(x_o)$ denotes the reachable set of the system (1) – (3) from x_o).

Proof. Necessity: It follows from the approximate M_oM_1—controllability of the system (1)—(3) that there exists a sequence $(x_{on}) \in M_o$ such that

$$R_c(x_{on}) \cap (M_1 + S_{1/n}) \neq \phi$$

for n = 1, 2, ... Therefore, for every $f \in S_1^*$ we have

$$\sup_{x \in R_c(x_{on})} \langle f, x \rangle \geq \min_{y \in M_1} \langle f, y \rangle + \min_{e \in S_{1/n}} \langle f, e \rangle,$$

or

$$\sup_{x \in R_c(x_{on})} \langle f, x \rangle + \max_{y \in M_1} \langle f, -y \rangle + 1/n \geq 0.$$

By the compactness of M_o we can assume $x_{on} \to \overline{x}_o \in M_o$. Then, letting $n \to \infty$ we oltain from the above inequality, and the continuity of the multifunction $R_c(x_o)$ (by Corollary 4),

$$\sup_{x \in R_c(x_o)} \langle f, x \rangle + \max_{y \in M_1} \langle f, -y \rangle \geq 0.$$

This now really implies (19).

Sufficiency: Suppose (19) is satisfied, but the system is not approximately M_0M_1 – controllable. This means that

$$\overline{\bigcup_{x_0 \in M_0} R_c(x_0)} \cap M_1 = \phi$$

Since $\overline{\bigcup_{x_0 \in M_0} R_c(x_0)}$ is convex, by Corollary 3, and M_1 is compact and convex, by hypotheses, the Hahn–Banach separation theorem shows that there exist $\delta > 0$ and a non-zero $f_0 \in S_1^*$ such that

$$\sup_{x \in \overline{R}_c(x_0)} \langle f_0, x\rangle < -\delta + \min_{y \in M_1} \langle f, y\rangle.$$

Thus, for every $x_0 \in M_0$,

$$\min_{f \in S_1^*} \left\{ \sup_{x \in R_c(x_0)} \langle f, x\rangle + \max_{y \in M_1} \langle f, -y\rangle \right\} < \delta,$$

and so we reach a contradiction.

COROLLARY 8. Let $L_0^-(x_0) \neq \phi$. Then the system (1) – (3) is approximately M_1 – controlable from $\overline{x}_0$ if and only if

$$\min_{f \in S_1^*} \left\{ \sup_{x \in R_c(x_0)} \langle f, x\rangle + \max_{y \in M_1} \langle f, -y\rangle \right\} \geq 0.$$

The system (1) – (3) is approximatety null controllable from $\overline{x}_0$ is and only if

$$\min_{f \in S_1^*} \sup_{x \in R_c(\overline{x})} \langle f, x\rangle \geq 0.$$

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REFERENCES

[1] Fattoriny H.O. *Some remarks on complete controllability.* SIAM J. control, 4(1966), 686–694.

[2] Kurjanskii A.B. *On controllability in Banach space.* Differen. Uravnenija, 5(1969), 1269–1271. (in Russian).

[3] Triggiani R. *Controllability and observability in Banach space with bounded aperators.* SIAM J. Control, 13(1975), 462–491.

[4] Triggiany R. *On the lack of exact controllability for mil sclations in Banach space.* J. Math. Anal. Appl.. 50(1975), 438–446.

[5] Curtain R.F., Pritchard A.I. *Infinite demensional linear system theory.* Lecture Notes in Control and Information Sciences V.8. Springer – Verlag, Berlin, 1978.

[6] Korobov V.I., Rabakh R. *Exact controllability in Banach space.* Differen. Uranenija, 15(1979), 2142–2150. (in Russian).

[7] Korobov V.I, Nguyen Khoa Son. *Controllability of linear systeme in Banach spaee with Contraints on controls*, I, II. Differen. Uravnenija, 16(1980), 806–817, 1010–1022. (in Russianh)

[8] Nguyen Khoa Son. *Local controllability of linear systems with restrained controls in Banach space*. Acta Math. Vietnamica, 5(1980),

[9] Arkin V.I., Levin V.L. *Extreme point of some set of measurable vectoh–functions and convexity of the range of vector integrals*. DAN, 199: 6(1971), 1223 – 1226. (in Russian)

[10] Hermes H. *On the closure and convexity of attainable sets in finite and infinite dimensions*. SIAM J. Control, 5(1967), 409–418.

[11] Levitin A.V. *On the optimal control problem with phase contraints*. Vestnik Mosk. Universiteta, matem., mekh., 3(1971), 59–68.

[12] Phan Van Chuong. *Densite des sélections extremates d'une multifonction mesurable*. Travaux du Séminare d'Analyse Convexe. Montöelier, 1979, Exp. n5, 5.1 – 1.15.

[13] Phan Van Chuong. *Existence of solutions for random multivalued of Voltersa. integral equations*, (to appear)

[14] Edwards R.E. *Functional Analysis.. Theory and Applicaltons*. Holt, Rinehard. and Winston. New York, 1965.

[15] Castaing Ch., Valadier M. *Convex Analysis and measurable Multifunctions*. Lecture Notes N° 580, Springer–Verlage Berlin, 1977.

[16] Ioffe A. D., Tikhomirov V.M *Theory of Extremal Problems*. Nauka, Moskva 1974. (in Russian)

[17] Aubin J.–p, *Muthematical Methods of Game and Economic Theory*. North – Holland, Ansterdam, 1979

[18] Schaefer H.H. *Topological Vector Spaces*. Macmillan Co., New York, 1966.