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ON THE SPACE FILTRATION OF A HOMOGENEOUS DAM

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1. PCOLLEM STATEMENT

The aim of this paper is to give a numerical analytic solution to a free boundary problem for a space homogeneoas dam, by using the method of right lines.

Denote by D the parallepiped

$$\{(x, y, z) \in R^3: \ 0 < x < a, \ \ 0 < y < b, \ \ 0 < Z < H\}$$

where a, b and H are given positive constants. D is considered as a homogeneous dam while H is the water level of its right reservoir. Let h, B, T be the water level of the left reservoir, the base and the top of D, respectively, such that

$$B = \{(x, y, z) \in \overline{D}: 0 < x < a, \ \ 0 < y < b, \ \ Z = 0\}$$

$$T = \{(x, y, z) \in \overline{D}: 0 < x < a, \ \ 0 < y < b, \ \ Z = H\}$$

where $\overline{D}$ is the closure of D in R^3.

Denote by Ω the wet part of the dam and by μ the interface (so-called «free boundary») between the wet region and the dry region of the dam. The equation of the free boundary is given by

$$z = \Psi(x, y), \qquad (x, y) \in B,$$

where $\Psi(x, y)$ is a continuous funetion, such that

$$\Psi(O, y) = H, \ \ \Psi(a, y) \geqslant h, \ \ 0 < y < b. \tag{1}$$

The domain Ω is determined by

$$\Omega = \{(x, y, z) \in D: \ \ 0 < y < \Psi(x, y)\}$$

We adopt the following notations for the portions of the dam

$$G_1 = \{(x, y, z) \in \overline{D}: x = 0, \ 0 < y < b, \ 0 < z < H\},$$

$$G_2 = \{(x, y, z) \in \overline{D}: x = a, \ 0 < y < b, \ 0 < z < h\},$$

$$G_2^+ = \{(x, y, z) \in \overline{D}: x = a, \ 0 < y < b, \ h < z < \Psi(b, y)\},$$

$$G_2^{++} = \{(x, y, z) \in \overline{D} : x = a,\ 0 < y < b,\ h < z < H\},$$

$$S^+ = \{(x, y, z) \in \overline{D} : 0 < x < a,\ y = 0,\ 0 < z < \Psi(x, 0)\},$$

$$S^- = \{(x, y, z) \in \overline{D} : 0 < x < a,\ y = b,\ 0 < z < \Psi(x, b)\}.$$

Assume that the parts of the dam B, S^+ and S^- are impervious. The problem we are concerned with can be stated as follows (cf. [5], [2]).

PROBLEM A: *Find a function* $\psi(x, y)$, $(x, y) \in B$, *satisfying the condition* (1) *and a function* φ *defined in* Ω *such that.*

$$\Delta\varphi = 0 \quad \text{in } \Omega,$$

$$\varphi = H \text{ on } G_1, \qquad \varphi = h \text{ on } G_2,$$

$$\varphi = z \text{ on } G_2^+, \qquad \frac{\partial\varphi}{\partial n} = 0 \text{ on } B \cup S^+ \cup S^-$$

$$\varphi = z, \text{ and} \qquad \frac{\partial\varphi}{\partial n} = 0 \text{ on } \Gamma.$$

Using the Baiocchi's transformation

$$u(x, y, z) = \int_z^H [\varphi(x, y, t) - t]\, dt,$$

we can reduce the above problem to the following [3]

PROBLEM B: *Find a function* $u \in C^1(D) \cap H^2(D)$ *satifsying*

$$\Delta u = \chi_\Omega \textit{ in the sense of } \mathcal{D}'(D), \tag{2}$$

where χ_Ω *is the characteristic function of* Ω *and*

$$u = \frac{1}{2}(H - z)^2 \text{ on } G_1, \quad u = \frac{1}{2}(h - z)^2 \text{ on } G_2,$$

$$u = 0 \quad \text{on} \quad G_2^{++} \cup T,\ u_y = 0 \text{ at } y = 0 \text{ and } y = b,$$

$$u = \frac{1}{2a}(h^2 - H^2)x + \frac{1}{2}H^2 \quad \text{on} \quad B.$$

The existence and uniqueness of solutions for Problems A and B have been considered in [3], where one has shown that if u is a solution to Problem B, then the pair $\{\varphi, \Psi\}$, with

$$\varphi(x, y, z) = z - u_z(x, y, z), \tag{3}$$

$$\psi(x, y) = \inf_{z \in [0, H]} \{z : u(x, y, z) = 0,\ (x, y) \in B\}$$

is a solution to Problem A.

By the method of right lines we can find an approximate solution to problem B, hence to problem A.

2. NUMERICAL ANALYTIC SOLUTION

Let us introduce two families of parallel lines

$$x_i = ih_1,\ h_1 = \frac{a}{m+1},\ i = 0, 1, \ldots, m+1,$$

$$y_i = ih_2,\ h_2 = \frac{b}{n+1},\ i = 0, 1, \ldots n+1,$$

where m and n are positive integers.

Set

$$u_{ij}(z) = u(x_i, y_j, z),\ z_{ij} = \psi(xi, y_j)$$

Consider the m—dimensional vectors

$$u_j(z) = \{u_{1j}(z), u_{2j}(z), \ldots, u_{mj}(z)\}$$

and the solution m×n matrix

$$u(t) = [u_1(z), u_2(z), \ldots, u_n(z)]$$

Discreting the equation (2) with respect only to x and y we obtain the following system of difference — differential equations

$$\frac{d^2u_{ij}(z)}{dz^2} + \frac{1}{h_1^2}(u_{i+1,j} - 2u_{ij} + u_{i-1j}) + \frac{1}{h_2^2}(u_{ij+1} - 2u_{ij} + u_{ij-1})$$

where $= \chi(z_{ij})$,

$$\chi(\xi) = \begin{cases} 1 & \text{if} \quad z \leqslant \xi \\ 0 & \text{if} \quad z > \xi. \end{cases} \tag{4}$$

Now, using the technique of summary representations (cf. [6], [2]), we can reduce the system (4) to a system of differential equations.

Set

$$Ru_{ij}(z) = u_{ij+1}(z) - 2(1 + \Upsilon^2)\, u_{ij}(z) + u_{ij-1}(z),\ \Upsilon = \frac{h_2}{h_1}.$$

Then, for i = 1, 2,..., n in (4) we have

$$\begin{aligned}
&\frac{d^2u_{1j}(z)}{dz^2} + \frac{1}{h_2^2} Ru_{1j}(z) + \frac{1}{h_1^2} u_{2j}(z) &&= \chi(z_{1j}) - \frac{1}{h_1^2} u_{oj}(z),\\
&\frac{d^2u_{2j}(z)}{dz^2} + \frac{1}{h_2^2} Ru_{2j}(z) + \frac{1}{h_1^2} (u_{1j}(z) + u_{3j}(z)) &&= \chi(z_{2j}),\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots && \qquad (5)\\
&\frac{d^2u_{m-1j}(z)}{dz^2} + \frac{1}{h_2^2} Ru_{m-1j}(z) + \frac{1}{h_1^2} (u_{m-2j}(z) + u_{mj}(z)) &&= \chi(z_{m-1j}),\\
&\frac{d^2u_{mj}(z)}{dz^2} + \frac{1}{h_2^2} Ru_{mj}(z) + \frac{1}{h_1^2} u_{mj}(z) &&= \chi(z_{mj}) - \frac{1}{h_1^2} u_{mj}(z).
\end{aligned}$$

where

$U_{oj}(z)=\frac{1}{2}(H-z)^2,\ U_{m+1j}(z)=\frac{1}{2}(h-z)^2\chi(h),\ j=1,2,\dots,n.$

Consider the m—dimensional vectors

$$f_j(z) = \{\chi(z_{1j}),\ \chi(z_{2j}),\dots,\ \chi(z_{mj})\}$$

$$\vec{\omega}_j(t) = \{u_{oj}(z),\ 0,\ \dots,\ ,u_{m+1j}(z)\},$$

and the mxm matrix

$$T=\begin{bmatrix} 0 & 1 & 0 & & \\ 1 & 0 & 1 & 0 & \\ & \dots\dots & & & \\ & & 1 & 0 & 1 \\ & 0 & & 1 & 0 \end{bmatrix}$$

By [6]

$$T = P \wedge P \tag{6}$$

where

$$P = [a_{ik}]_1^m,\ a_{ik} = \sqrt{\frac{2}{m+1}}\sin\frac{ik\pi}{m+1},$$

$$\Lambda = \mathrm{diag}\,[\lambda_1,\ \lambda_2,\dots,\ \lambda_n],\ \lambda_i = 2\cos\frac{i\pi}{m+1}.$$

Note that $P^2 = E$ where E is the identity matrix.

With the above notation we can write the system (5) in the vector form:

$$\frac{d^2\vec{u}_j(z)}{dz^2}+\frac{1}{h_2^2}R\vec{u}_j(z)+\frac{1}{h_1^2}\vec{u}_j(z)=\vec{f}_j(z)-\frac{1}{h_1^2}\vec{\omega}_j(z). \tag{7}$$

For any m – dimensional vector $\vec{g}$ let us set

$$\hat{\vec{g}} = P\vec{g}. \tag{8}$$

Multiplying both sides of the equation (7) by the matrix P, and using (6) and (8) we have

$$\frac{d^2\hat{\vec{u}}_j}{dz^2}+\frac{1}{h_2^2}R\hat{\vec{u}}_j(z)+\frac{1}{h_1^2}\Lambda\hat{\vec{u}}_j(z)=\hat{\vec{f}}_j(z)-\frac{1}{h_1^2}\hat{\vec{\omega}}_j(z). \tag{9}$$

We now write the equation (9) in the scalar form

$$\frac{d^2\hat{u}_{ij}}{dz^2}-\frac{\sigma_i}{h_2^2}\hat{u}_{ij}+\frac{1}{h_1^2}(\hat{u}_{ij+1}-\hat{u}_{ij})=\hat{f}_{ij}-\frac{1}{h_1^2}\hat{\omega}_{ij}). \tag{10}$$

Where $\sigma_i = 2\left(1+\gamma^2-\gamma^2\cos\frac{i\pi}{m+1}\right)$

For $j = 1, 2, \ldots, n$ in (10), using boundary conditions at $y = 0$ and $y = b$ we obtain

$$\frac{d^2 \widehat{u}_{i1}}{d\tau^2} - \frac{\sigma_i}{h_2^2} \widehat{u}_{i1} + \frac{1}{h_2^2} (\widehat{u}_{i1} + \widehat{u}_{i2}) = \widehat{f}_{i1} - \frac{1}{h_1^2} \widehat{\omega}_{i1},$$

$$\frac{d^2 \widehat{u}_{i2}}{dt^2} - \frac{\sigma_i}{h_2^2} u_{i2} + \frac{1}{h_2^2} (\widehat{u}_{i1} + \widehat{u}_{i3}) = \widehat{f}_{i2} - \frac{1}{h_1^2} \widehat{\omega}_{i1}, \tag{11}$$

$$\cdots\cdots$$

$$\frac{d^2 \widehat{u}_{in}}{dt^2} - \frac{\sigma_i}{h_2^2} \widehat{u}_{in} + \frac{1}{h_2^2} (\widehat{u}_{in-1} + \widehat{u}_{in}) = \widehat{f}_{i2} - \frac{1}{h_2^2} \widehat{\omega}_{in}.$$

Set

$$\vec{v}_i = (\widehat{u}_{i1}, \widehat{u}_{i2}, \ldots, \widehat{u}_{in}),$$

$$\vec{F}_i = (\widehat{f}_{i1}, \widehat{f}_{i2}, \ldots, \widehat{f}_{in}),$$

$$\vec{\omega}_i = \widehat{\omega}_{i1}, \widehat{\omega}_{i2}, \ldots, \widehat{\omega}_{in}),$$

$$T_4 = \begin{bmatrix} 1 & 1 & 0 & & \\ 1 & 0 & 1 & 0 & \\ \cdots & & 1 & 0 & 1 \\ & 0 & & 1 & 1 \end{bmatrix} \text{ (of order n).}$$

Then by [6]

$$T_4 = P_4 \ \Lambda_4 \ P_4',$$

where

$$P_4 = [b_{jl}]_1^n \quad \text{with}$$

$$b_{jl} = \begin{cases} \dfrac{1}{\sqrt{n}} & j = 1,\ l = l,\ 2, \ldots, m, \\ \sqrt{\dfrac{2}{n}} \cos\left[\dfrac{(2j-1)(l-1)}{2n} r\right] & j = 2, 3, \ldots, n, \quad l = 1, 2, \ldots, m, \end{cases}$$

P_4' is the transpose of P_4. and

$$\Lambda_4 = \operatorname{diag}\, [\bar{\lambda}_1, \bar{\lambda}_2, \ldots, \bar{\lambda}_n] \text{ with}$$

$$\bar{\lambda}_j = 2 \cos \frac{(j-1)\pi}{n}, \ j = 1, 2, \ldots, n. \tag{12}$$

Note that $P_4 P_4' = P_4' P_4 = E$.

So we can write the system (11) in the following form

$$\frac{d^2 \vec{v}_j}{dz^2} - \frac{\sigma_i}{h_2^2} \vec{v}_j + \frac{1}{h_2^2} T_4 \vec{v}_i = \vec{F}_i - \frac{1}{h_1^2} \vec{\omega}_i. \tag{13}$$

Set

$$V(z) = [v_1(z), v_2(z), \ldots, v_m(z)] \tag{14}$$

$$\widehat{U}(z) = P \cup (f) = [\vec{\widehat{u}}_1(z), \widehat{u}_2(z), \ldots, \vec{\widehat{u}}_n(z)].$$

Then we have

$$U(t) = V'(z), \tag{15}$$

where $V'(z)$ is the transpose of $V(z)$.

Now let us introduce the transformation

$$\hat{\vec{t}} = P_4 \vec{t}, \tag{16}$$

where $\vec{t}$ is a n−dimensional vector.

Multiplying both sides of the equation (13) by P_4' and using (12) we have

$$\frac{d^2 \tilde{\vec{v}}_i(z)}{d\tau^2} - \frac{\sigma_i}{h_2^2} \tilde{\vec{v}}_i(\tau) + \frac{1}{h_2^2} \Lambda_4 \tilde{\vec{v}}_i(\tau) = \tilde{\vec{F}}_i - \frac{1}{h_1^2} \tilde{\vec{w}}_i. \tag{17}$$

Writing (17) in scalar form we obtain the following differential equation

$$\frac{d^2 \widetilde{v_{ij}}(\tau)}{d\tau^2} - V_{ij}^2 \widetilde{v_{ij}}(\tau) = \widetilde{\Phi}_{ij}(z), \tag{18}$$

where

$$\nu_{ij}^2 = \frac{4}{h_2^2}\left[\sin^2\frac{(j-1)\pi}{2n} + \Upsilon^2 \sin^2\frac{i\pi}{2(m+1)}\right],$$

$$\Phi_{ij}(\tau) = \sum_{s=1}^{n}\sum_{l=1}^{m} a_{il}\, b_{sj}\, \chi(z_{ls}) - \frac{1}{h_1^2}\sum_{s=1}^{n} b_{sj}\left[\alpha_{il}\, u_{os} + a_{jm} u_{m+ls}\right],$$

To solve the equation (18) we need the following

LEMMA [4]: *The differential equation*

$$\frac{d^2v}{dy^2} - a^2 v(y) = g(y),\ 0 < y < b, \tag{20}$$

where

$$g(y) = \alpha_i y^2 + \beta i y + \Upsilon i,\ y_{j-1} \leqslant y \leqslant y_i,\ i = 1, 2, \ldots, m+1 \tag{21}$$

$$y_o = 0,\ y_{m+1} = b$$

has the following solution in the class C^1 (0,b) H^2 (0, b)

$$v(y) = A\, e^{\alpha y} + B^{-\alpha y} - \frac{1}{2a^2} R(y), \tag{22}$$

where A and B are arbitrary constants and R(y) is determined by the following formula:

$$R(y) = 2\alpha_i y^2 + 2\beta_i y + \frac{4\alpha_i}{a^2} + 2\Upsilon_i + \sum_{j=1}^{i-1}\Big\{ e^{a(y-y_j)}\Big[(\alpha_i - \alpha_{j+1})\left(y_j + \frac{2y_i}{a} + \frac{2}{a^2}\right) +$$
$$+ (\beta_j - \beta_{j+1})\left(y_i + \frac{1}{a}\right) + \Upsilon_j - \Upsilon_{j+1}\Big] + e^{-(y-y_j)}\Big[(\alpha_i - \alpha_{j+1})\left(y_j - \frac{2y_i}{a} + \frac{2}{a^2}\right) +$$
$$+ (\beta_j - \beta_{j+1})\left(y_j - \frac{1}{a}\right) + \Upsilon_j - \Upsilon_{j+1}\Big]\Big\}, \tag{23}$$

for $y \in [y_{i-1}, y_i]$, $i = 1, 2, \ldots, m+1$.

Applying the above lemma to (18) yields

$$\widetilde{v}_{ij}(\tau) = A_{ij}\, e^{v_{ij}(\tau)} + B_{ij}\, e^{-v_{ij}\tau} - \frac{1}{2v_{ij}} R_{ij}(\tau,\ \tau_{11}, \tau_{12}, \ldots, \tau_{mn}), \tag{24}$$

where A_{ij}, B_{ij} are constants, R_{ij} have the form (23) and τ_{1j} ar parameters to be determined later.

Now let us find the constants A_{ii} and B_{ij}. Set

$$\vec{A}_i(Z) = (A_{i1}\, e^{v_{i1}Z},\ A_{i2}\, e^{v_{i2}Z}, \ldots,\ A_{in}\, e^{v_{in}Z})$$

$$\vec{B}_i(Z) = (B_{i1}\, e^{-v_{i1}Z}.\ B_{i2}e^{-v_{i2}Z}, \ldots,\ B_{in}e^{-v_{in}Z})$$

$$\vec{Q}_j(Z) = \left(-\frac{R_{i1}}{2v_{i_1}^2}, -\frac{R_{i2}}{2v_{i_2}^2}, \ldots, -\frac{R_{in}}{2v_{i_n}^2}\right).$$

Then we have

$$\vec{\widetilde{v}}_i(z) = \vec{A}_i(z) + \vec{B}_i(z) + \vec{Q}_i(z)$$

which together with (12) yields

$$\vec{v}_i(z) = P_4(\vec{A}_i(z) + \vec{B}_i(z) + \vec{Q}_i(z))$$

From this it follows that

$$V(z) = P_4(A'(z) + B'(z) + Q'(z)) \tag{25}$$

where

$$A' = [\vec{A}_1(z), \vec{A}_2(z), \ldots, \vec{A}_m(z)],\ B' = [\vec{B}_1(z), \vec{B}_2(z), \ldots, \vec{B}_n(z)].$$

By (15) and (25) we obtain

$$Q' = [\vec{Q}_1(z), \vec{Q}_2(z), \ldots, \vec{Q}_n(z)],$$

$$U(z) = (A(z) + (B(z) + Q(z))Q'_4,$$

hence

$$U(z) = P(A(z) + B(z) + Q(z)P'_4 \tag{26}$$

where

$$A(z) = \left[Akle^{vklz} \right]_{kl=1}^{m,n};\ B(z) = \left[B_{kl}e^{-vklz} \right]_{k,l=1}^{m,n};\ Q(t) = \left[-\frac{R_{kl}}{2v_k^2 l} \right]_{k,l=1}^{m,n}$$

Now we can write

$$PU(z)P_4 = A(z) + B(z) + Q(z). \tag{27}$$

Taking $z = 0,H$ in (27)and nsing the boundary condition at $z=0,H$ we get

$$A_{ij} + B_{ij} = \alpha_{ij} - Q_{jj}(0),$$

$$A_{ij}e^{v_{ij}H} + B_{ij}e^{-ijH} = -Q_{ij}(H),$$

where

$$\alpha_{ij} = \frac{1}{2} \sum_{l=1}^{m} a_{il} \left[\frac{1}{a} (h^2 - H^2)x_l + H^2 \right] \sum_{s=1}^{n} b_{si}.$$

From this it follows that

$$A_{ij} = -\frac{1}{2sh\nu_{ji}4}[Q_{ij}(H) + e^{-\nu_{ij}H}(\alpha_{ij} - Q_{ij}(0)]$$

$$B_{ij} = \frac{1}{2sh\nu_{ij}H}[Q_{ij}(H) + e^{\nu_{ji}H}(\alpha_{ij} - Q_{ij}(0)]$$

which together with (26) yield the folloning

THEOREM: *The numerical analytic solution of Problem B has the forn*

$$u_{ij}(\tau) = \frac{1}{2}\sum_{e=1}^{m} a_{ie} \sum_{s=1}^{n} b_{is} \Big\{ -\frac{e^{\nu_{es}Z}}{sh\nu_{es}H}[Q_{es}(H) + e^{-\nu_{es}H}(\alpha_{es} - Q_{es}(0))] +$$

$$+ \frac{e^{-\nu_{es}\tau}}{sh\nu_{es}H}[Q_{es}(H) + e^{\nu_{es}H}(\alpha_{es} - Q_{es}(0))] - \frac{Q_{es}'(z)}{e_{es}^{2}} \Big\}. \qquad (28$$

3. DETERMINATION OF THE FREE BOUNDARY

The representation formula of solution (28) contains the unknown pa rameters z_{ij} $(i = 1, 2, ..., m,\ j = 1, 2, ..., n)$ which determine the free boundary Let us find these parameters.

Set $$U_{ij}(Z) = f^{i}_{j}(z, z_{11}, z_{12}, ..., z_{mn}), \qquad (29$$

where f_{ij} is the right hand side of formula (28). Taking $z = z_{ij}$ in (29) we have

$$U_{ij}(z_{ij}) \doteq \Phi_{ij}(z_{11}, z_{12}, ..., z^{m}_{mn})$$

with $$\Phi_{ij} = f_{ij}(z_{ij}, z_{11}, z_{12}, ..., z_{mn})$$

Since $U_{ij} = 0$ on the free frontier we get

$$\Phi_{ij}(z_{11}, z_{12}, ..., z_{mn}) = 0,$$
$$(i = 1, 2, ..., m,\ j = 1, 2, ..., n)$$

which is a complete system of non-linear algebraic equations for the determi nation of the parameters z_{ij}. In order to find z_{ij} we use the following iterative method of [1]. First, we choose the initial values z_{ij} of this iterative proces by the formula

$$z_{ij}^{(o)} = H + i(h - H)h_1 \text{ for } i = 1,2, ..., m,\ j = 1, 2, ..., n,$$

($z_{ij}^{(o)}$ may be also determined by the solution of the corresponding two-dimen tional problem, cf. [5]). Suppose that the values $z_{ij}^{(k-1)}$ of the $(k-1)^{th}$ step are known, then the values $z_{ij}^{(k)}$ of the k^{th} step are determined by the following formula

$$z_{ij}^{(k)} = z_{ij}^{(k-1)} + \omega_k \Phi_{ij}^{k}, \quad k = 1, 2, ...,$$

where ω_k is a parameter lying between O and 1. Note that ω_k improves the convergence of this iterative process. The stopping criterion of this iterate process is

$$\max_{ij} | z_{ij}^{(k)} - z_{ij}^{(k-1)} | < \varepsilon$$

where ε is a small enough positive numler.

Substitating the value z_{ij} found by the iterate process into (29) we can determine $U_{ij}(z)$, hence by (3) we obtain the solution of the posed filtration problem.

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