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A NUCLEARITY CRITERION FOR LOCALLY CONVEX SPACES HAVING SCHAUDER BASES

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1. INTRODUCTION

Our purpose in this paper is to provide an alternative proof of the nuclearity criterion of a locally convex space equipped with an equicontinuous Schauder base and thereafter deduce the well–known Grothendieck — Pietsch citerion for nuclearity as a very simple corollary of the former. In the course of proof of the first result we develop a general technique for estimating Kolmogorov's n — th diameters of (semi — norm) neighbourhoods in a locally convex space possessing an equicontinuous Schauder base.

2. TERMINOLOGICAL EXCERPTS

We follow [1], [7], [10] and [4]; [5] respectively for the theories of locally convex spaces, nuclear spaces, Schauder bases and sequence spaces & series; however, we recall only a few relevant portions which interest us most for this paper. Throughout we write $X \equiv (X, \mathcal{J})$ to mean a Hausdorff locally convex space (abbreviated hereafter as l. c. TVS) and we use the symbol $\mathcal{D}$ for the family of all $\mathcal{J}$–continuous semi–norms generating the topology $\mathcal{J}$. For $p \in \mathcal{D}$, let us denote by U_p the set $\{x \in X; p(x) < 1\}$. A Schauder base for $(X, \mathcal{J})$ will be designated by $\{x_n; f_n\}$ where $n = 0, 1, \ldots$ and we say that $\{x_n; f_n\}$ is *equicontinuous* provided for each $p \in \mathcal{D}$ there corresponds some $q \in \mathcal{D}$ such that

$$p^*(x) \equiv \sup | f_n(x) | p(x_n) \leqslant q(x) \forall x \text{ in } x.$$

If A and B are subsets of X such that A is absorbed by B then we can talk of the n–th Kolmogorov diameter $\sigma_n(A, B)$ of A relative to B (for various elementary and useful properties of the numbers σ_n, w_n refer to [3]; [4], Chap. 1; [6] and [7], Chap. 9).

In the sequel we shall need the followiing two interesting characterizations of nuclear spaces to be f und respectively in [7], Chap. 4 and [8], p. 15.

THEOREM 2.1 *An l. c. TVS $(X, \mathcal{J})$ is nuclear if and only if for each $p \in \mathfrak{D}$ there existsq $\in \mathfrak{D}$ and a sequence $\{g_n\} \subset X^*$ with $\{sup\{|g_n(x)| : q(x) \leq 1\}\} \in l^1$ such that*

$$p(x) \leq \sum_{n \geq 0} | g_n(x) |, \forall x \text{ in } X.$$

THEOREM 2.2. *An l.c. TVS $(X, \mathcal{J})$ is nuclear if and only if for each $p \in \mathfrak{D}$ and each $\alpha > 0$ there exists $q \in \mathfrak{D}$ such that*

$$\sum_{n \geq 0} \sigma_n(U_d, U_p)^{\alpha} < \infty.$$

3. THE MAIN RESULTS

The first main result which finds a different proof hereafter is the following theorem to be found in [2].

THEOREM 3.1: *Let $(X, \mathcal{J})$ be an l.c TVS possessing an equicontinuous Schauder base $\{x_n : f_n\}$. Then $(X, \mathcal{G})$ is nuclear if and only if fore ach $p \in \mathfrak{D}$ there exists $q \in \mathfrak{D}$ such that*

$$(*) \qquad \sum_{n \geqslant 0} p(x_n)/q(x_n) < \infty, \left(\frac{0}{0} = 0\right).$$

Proof: Let $(X, \mathcal{J})$ be nuclear and choose $p \in \mathfrak{D}$ arbitrarily. Now $(X, \mathcal{J})$ is a Schwartz space, therefore there exists $q_1 \in \mathfrak{D}$ such that

$$(3.2) \qquad p(x_n)/q_1(x_n) \to 0 \text{ as } n \to \infty.$$

In fact, from (2) of [12], p. 237, for each $\varepsilon > 0$, there exist $q_1 \in \mathfrak{D}$ and a finite dimensional subspace L of X such that

$$U_{q_1} \subset \varepsilon U_p^* + L,$$

and so

$$\sup_n | f_n(x_k/q_1(x_k)) - f_n(y_k) | p(x_n) \leq \varepsilon, \forall k \geq 0$$

where $y_k \in L$. One can find an integer N such that

$$f_n(y_k) = 0, \forall n \geq N \text{ and } k \geq 0$$
$$\Rightarrow p(x_n)/q_1(x_n) \leq \varepsilon, \quad \forall n \geq N$$

and the preceding inequality results in (3.2); for an alternative argument of the existence of (3.2), see Theorem 2.1, of [13], p. 7. Putting $\alpha = 3$ in Theorem 2.2, one can find $q_2 \in \mathfrak{D}$ such that (or, alternatively, one may use Lemma 1.5 of [9])

$$(3.3) \qquad \sum_{n \geq 0} (n+1)\, \delta_n (U_{q_2}, U_p^*) < \infty.$$

Let us also note that there exists some $q_3 \in \mathfrak{D}$ with

$$(3.4) \qquad p^*(x) \leq q_3(x), \forall x \text{ in } X.$$

If $q = \max(q_1, q_2, q_3)$, then (3.2), (3.3) and (3.4) are true with q_i's replaced by q. Define

$$\alpha_n = \begin{cases} p(n)/q(x_n), & \text{if } q(x_n) \neq 0; \\ 0 & , \text{if } q(x_n) = 0. \end{cases}$$

Let us observe that if the set of integers where $q(x_n) \neq 0$ is finite, then (*) readily follows. Hence assume that the set $N_1 = \{n \in N : q(x_n) \neq 0\}$ is infinite, where $N = \{0, 1, 2, ...\}$ and let $N_1 = \{n_0, n_1, ..., n_s, ...\}$.
Now

$$\lim_{i \in N_1} \frac{q(x_i)}{q(x_1)} = 0$$

and so we can find a permutation π of N_1 such that

$$\alpha_{\pi(n_m)} \leq \alpha_{\pi(n_s)}, \text{ for } n_m \geq n_s,$$

where $n_s, n_m \in N_1$. Denote by L_{s+1} the subspace of χ gennerated by $x_{\pi(n_0)}, \dots, x_{\pi(n_s)}$ and let

$$x \in \alpha_{\pi(n_s)} (U_p^* \cap L_{s+1}).$$

Then

$$x = \sum_{i=0}^{s} f_{\pi(n_i)}(x)\, x_{\pi(n_i)}$$

and consequently

$$q(x) \leq \alpha^{-1}_{\pi(n_s)} \sum_{i=0}^{s} | f_{\pi(n_i)}(x) | \, p_{\pi(n_i)}(x)$$

$$\leq (s+1)\, \alpha^{-1}_{\pi(n_s)}\, p^*(x) \leq s+1.$$

Therefore from a theorem of Tikhomirov (e.g., [10], p. 58)

$$\frac{p(x_{\pi(n_s)})}{q(x_{\pi(n_s)})} \leqslant (s+1)\, \delta_s(U_q, U_{p^*})$$

$$\Rightarrow \sum_{s \geqslant 0} p(x_{n_s}) / q(x_{n_s}) < \infty,$$

and hence (*) follows.

Conversely, let (*) hold good for a given choice of p and a corresponding q in $\mathcal{D}$. Let $g_n = p(x_n) f_n$, $n \geqslant 0$. Then

$$\sup \{ | g_n(x) | ; q^*(x) \leqslant 1 \} = p(x_n) \sup \{ | f_n(x) | : q^*(x) \leqslant 1 \}$$

$$\leqslant \frac{p(x_n)}{q(x_n)};$$

further, for x in X,

$$p(x) \leqslant \sum_{n \geqslant 0} | f_n(x) | \, p(x_n)$$

$$= \sum_{n \geqslant 0} | g_n(x) | .$$

The nuclearity of $(X, \mathcal{G})$ now follows by applying Theorem 2.1.

The Grothendieck—Pietsch Criterion: Let P be a power set (*Stufensystem*) and $\lambda(P)$ the corresponding Köthe sequence space (see for instance [7], p. 97 and [10], p. 77), then the following result ([7], p. 98) is usually known as the Grothendieck – Pietsch criterion for the nuclearity of $\lambda(P)$. We deduce this result as a simple corollary of Theorem 3.1.

THEOREM 3.2. *The space $\lambda(P)$ is nuclear if and only if for each α in P there corresponds β in P with*

$$\sum_{i > 0} \alpha_i / \beta_i < \infty.$$

Proof : Let us recall that the natural locally convex topology $\mathcal{T}$ on $\lambda(P)$ is generated by the family $\{p_a : a \in p\}$ of semi-norms on $\lambda(P)$ where

$$p_a(x) = \sum_{n \geqslant 0} |x_i| \, a_i, \; x \in \lambda(P).$$

It is clear that ($\lambda(P)$) is an AK-space and so the mappings $x \to x_i$ can be identified with the sequences e^i (cf. Exercise 3.8. Chap 2 of [4]) where

$$e^i_j = \begin{cases} i, & j = i ; \\ 0, & j \neq i. \end{cases} , \quad i, j = 0, 1, \ldots,$$

It is now simple to verify that (e^i, e^i) is a Schauder base for $(\lambda(P), \mathcal{T})$. If $\alpha \in P$ then

$$\sup_i |e^i(x)| \, p_\alpha(e^i) = \sup_i |x_i| \, \alpha_i \leqslant p_\alpha(x), \; \forall x \in \lambda(P)$$

and so $\{s^i ; e^i\}$ is an equicontinuous Schauder base for $(\lambda(P), \mathcal{T})$. Since

$$\sum_{i \geqslant 0} \frac{\alpha_i}{\beta_i} = \sum_{i \geqslant 0} \frac{p_\alpha(e^i)}{p_\beta(e^i)},$$

the result follows from Theorem 3.1.

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