ACTA MATHEMATICA VIETNAMICA

Vol.7 N° 1 (1982)

THE MOD p COHOMOLOGY ALGEBRA OF THE GROUP $M(p^n)$

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INTRODUCTION

The purpose of the present paper is to determine the cohomology algebras of the groups

$$M(p^n) = \langle a, b;\ a^{p^{n-1}} = b^p = 1,\ b^{-1}ab = a^{1+p^{n-2}} \rangle,\ n \geq 3$$

with coefficients in Z_p (the prime field of p elements). As is well known, $M(p^n)$ is the unique non—abelian p—group of order p^n having a maximal subgroup which is cyclic if $p > 2$ and $n \geq 3$. The case $p = 2$ has been considered in [2; §3] by the second author. In this paper, we shall study the remaining case by use of a similar argument, so from now on, we shall assume $p > 2$ and $H^*(G) = H^*(G, Z)$ unless otherwise specified.

Long ago, G. Lewis [4] has computed the integral cohomology ring $H^*(M(p^3), Z)$ by means of the Hochschild—Serre spectral sequence (H—S spectral sequence for short) of the group extension

$$1 \to \langle a \rangle \to M(p^3) \to \langle b \rangle \to 1$$

and by use of the additive structure of $H^*(M(p^3), Z)$ obtained by C.T.C Wall [6] to estimate the spectral sequence.

Here we shall use the H—S spectral sequence of the central group extension.

$$(M) \qquad 1 \to Z \to M(p^n) \to C_{p^{n-2}} \times C_p \to 1$$

where $Z = \langle a^{p^{n-2}} \rangle \cong Z_p$, $C_{p^{n-2}} \times C_p = \langle \underline{a}, \underline{b} \rangle$ with $\underline{a} = aZ$, $\underline{b} = bZ$, and C_n a cyclic group of order n

The paper contains 3 sections. In §1, we shall recall the H—S spectral sequence with some modifications in such a way that it is compatible with respect to the sign convention. In particular, we shall compute the terms E_3, E_4

of the spectral for a central extension of a cyclic group of order p. In §2, we compute the term E_4 for (M), and show that $E_4 = E_\infty$ in this case. Finally, we determine the algebra $H^*(M(p^n))$ in §3.

§1. THE H–S SPECTRAL SEQUENCES.

Suppose that we are given a group extension

$$1 \to N \to G \to G/N \to 1,$$

and a G–module A. Let B(G) denote the (normalized) bar resolution of the trivial G–module $\mathbb{Z}$, and $B^*(G, A) = \mathrm{Hom}_G(B(G), A)$ the complex of the normalized cochains for G in the G–module A. Then, we recall that an n–cochain $f : B_n(G) \to A$ may be identified with a function f of n arguments g_i in G, with valued in A, which satisfies the conditions

$$f(g_1, \ldots, \overset{i}{\check{1}}, \ldots, g_n) = o, \quad 1 \leq i \leq n.$$

The coboundary homomorphism $\delta : B^n \to B^{n+1}$ is defined by

(1.1) $\qquad (\delta f)(g_1, \ldots, g_{n+1}) =$

$$= (-1)^{n+1}\left(g_1 f(g_2, \ldots, g_{n+1}) + \sum_{i=1}^{n} (-1)^i f(g_1, \ldots, g_i g_{i+1}, \ldots, g_{n+1}) + (-1)^{n+1} f(g_1, \ldots, g_n)\right).$$

For each normal subgroup N of G, $B^*(G, A)$ is filtered by Hochschild–Serre as follows. We write $B^* = B^*(G, A)$ for a moment. We define $F^iB^* = B^*$ for $i \leqslant 0$, and

$$F^iB^* = \sum_{n=1}^{\infty} F^iB^* \cap B^n \text{ for } i > 0,$$

where $F^iB^* \cap B^n = 0$ for $i > n$; and for $0 < i < n$, $F^iB^* \cap B^n$ is the group of all n – cochains f for which $f(g_1, \ldots, g_n) = 0$ whenever $n - i + 1$ of the arguments belong to the subgroup N. This filtration is compatible with the cup products, if A is a G – ring.

On the other hand, for each $i > 0$, we have the homomorphism

(1.2) $$Sh^i : B^{i+j}(G, R) \to B^i(G, B^j(N, A))$$

defined by

$$((Sh^i f)(g_{j+1}, \ldots, g_{j+i}))(x_1, \ldots, x_j) = (-1)^{ij} f((x_1, \ldots, x_j) * (g_{j+1}, \ldots, g_{j+i})).$$

Here $* : B(N) \otimes B(G) \to B(G)$ is the shuffle product of B(N) and B(G) into B(G) given explicitely as follows:

$$(x_1, \ldots, x_j) * (g_{j+1}, \ldots, g_{j+i}) = \sum \mathrm{sgn}(s)\,(c_s(s^{-1}(1)), \ldots, c_s(s^{-1}(i+j))$$

where the summation $\sum$ runs over the set of all (j, i) – shuffles i.e. permutations s of degree $j + i$ such that

$$s(1) < \ldots < s(j)$$
$$s(j+1) < \ldots < s(j+i);$$

and, for each (j, i) – shuffle s,

$$c_s(k) = \begin{cases} x_k^{g_{k+1}\dots g_{s(k)}} & 1 \leqslant k \leqslant j \\ g_k & j+1 \leqslant k \leqslant j+i \end{cases}$$

with $x^g = g^{-1}xg$. (Note that if G is abelian and $N = G$, the above product is the usual shuffle product).

As it is easily seen, for each i, the map Sh^i induces the homomorphism

$$F^iB^{i+j}(G, A) \to B^i(G/N, B^j(N, A))$$

such that we have the commutative diagram

$$\begin{array}{ccc} B^{i+j}(G, A) & \to & B^i(G, B^j(N, A)) \\ \uparrow & & \uparrow \\ F^iB^{i+j}(G, A) & \to & B^i(G/N, B^j(N, A)). \end{array}$$

Now, let A be a G—ring, and let E_r, $r \geqslant 2$ denote the spectral sequence associated with the Hochschild – Serre filtration of $B^*(G, A)$ with respect to the normal subgroup N. Then each E_r is a bigraded ring, each d_r satisfies the product rule, and E_{r+1} is the cohomology ring of E_r. Further, E_∞ is isomorphic to $E_0(H^*(G, A))$ as a ring.

Since $H^*(N, A)$ is a graded G/N – ring, $H^*(G/N, H^*(N, A))$ is a bigraded ring. From [1] and (1.1), (1.2), we have the following

THEOREM 1.3. (*Hochschild—Serre*). *The homomorphisms* Sh^i *in (1.2) induces the isomorphism of bigraded rings* $E_2 \cong H^*(G/N, H^*(N, A))$.

We are interested in the case where A is the trivial G – ring $\mathbf{Z}_p$ and in the central extension

$$\text{(E)} \qquad 1 \to Z \to G \to G/Z \to 1,$$

where Z is a cyclic group of order p. In this case, the H–S spectral sequence is of the form

$$E_2 \cong H^*(G/Z) \otimes H^*(Z) \Rightarrow H^*(G).$$

We shall compute the terms E_3 and E_4 of this spectral sequence.

As it is well known, we have

$$H^*(Z) = \mathbf{Z}_p[u, v]/(u^2).$$

Where $u: Z \simeq \mathbf{Z}_p$ is a non zero element of $H^1(Z) = \mathrm{Hom}(Z, \mathbf{Z}_p)$ and $v = \beta u \in H^2(Z)$. Here and in what follows, β denotes the Bockstein operator.

u is transgressive, hence so is $v = \beta u$. If we denote by $\tau: H^*(Z) \to H^{*+1}(G/Z)$ the transgression as usual, we have (1.4)

$$\tau u = z_E \in H^2(G/Z),$$
$$\tau v = -\beta\tau u = -\beta z_E.$$

(see [1: Chap. III,3]. Here, z_E is the cohomology class corresponding to the extension(E) via the isomorphism $u_*: H^2(G/Z, Z) \simeq H^2(G/Z, \mathbf{Z}_p)$. Further we have $v^{p^k} = P^{p^{k-1}} \dots P^pP^1v$, so v^{p^k} is also transgressive, and we have

$$\tau v^{p^k} = -P^{p^{k-1}} \dots P^pP^1\beta z_E.$$

As in the case $p = 2$ (see Huỳnh Mùi [2; 2.1]), we have the following

PROPOSITION 1.5. *In the H—S spectral sequence for the central extension (E), we have*

$$E_3 \simeq H^*(G/Z)/(z_E) \otimes Z_p[x] \oplus \mathrm{Ann}_H(z_E) \otimes Z_p[v]u,$$

$$\begin{aligned} E_4 \simeq H/(z_E,\ \beta z_E) &\otimes Z_p[v^p] \\ &\oplus \mathrm{Ann}_{H/z_E}(\beta z_E) \otimes Z_p[v^p]v^{p-1} \\ &\oplus \mathrm{Ann}_{H/(z_E)}(\beta z_E)/\beta z_E) \otimes \left(\sum_{i=1}^{p-2} Z_p[v^p]v^i\right) \\ &\oplus \mathrm{Ann}_H((z_E)/(\beta z_E)) \otimes Z_p[v^p]u \\ &\oplus \mathrm{Ann}_H(z_E,\ \beta z_E) \otimes Z_p[v^p]v^{p-1}u \\ &\oplus \mathrm{Ann}_H(z_E, \beta z_E)/((\beta z_E)) \otimes \left(\sum_{i=1}^{p-2} Z_p[v^p]v^i u\right). \end{aligned}$$

Here $H = H^*(G/Z)$ *and* $((\beta z_E)) = (\beta z_E)\mathrm{Ann}_H(z_E)$.

Proof. Identify $E_2 = H \otimes Z_p[u;\ v]/(u^2)$. Let $x \otimes v^n \in E_2$ and $x \otimes v^n u$. We have then $x \otimes v^n = (x \otimes 1)(1 \otimes v)^n$ and $x \otimes v^n u = (x \otimes v^n)(1 \otimes u)$ in E_2. v is transgressive and $|v| = \dim v > 1$, so $d_2(1 \otimes v) = 0$. Hence

$$d_2(x \otimes v^n) = (-1)^{|x|} n(x \otimes 1)(1 \otimes v)^{n-1} d_2(1 \otimes v) = 0.$$

Moreover, $d_2(1 \otimes u) = z_E \otimes 1$ by 1.4, so

$$d_2(x \otimes v^n u) = (-1)^{|x|}(x \otimes v^n)(z_E \otimes 1) = (-1)^{|x|}(xz_E \otimes v^n).$$

From this, compute $\ker d_2/\mathrm{im}\ d_2$, we obtain easily E_3.

Let $x \otimes v^n \in E_3$ and $y \otimes v^n u \in E_3$. Then $x \in H/(z_E)$ and $y \in \mathrm{Ann}_H(z_E)$. Clearly, in E_3, we have

$$x \otimes v^n = (x \otimes 1)(1 \otimes v)^n \text{ and } y \otimes v^n u = (y \otimes u)(1 \otimes v)^n.$$

Since $d_3(1 \otimes v) = -\beta z_E \otimes 1$, we have

$$\begin{aligned} d_3(x \otimes v^n) &= (-1)^{|x|}(x \otimes 1)n(1 v)^{n-1}(\beta z_E \otimes 1) \\ &= (-1)^{|x|+1} n(x\beta z_E \otimes v^{n-1}), \\ d_3(y \otimes v^n u) &= (-1)^{|y|+1}(y \otimes u)n(1 \otimes v)^{n-1}(-\beta z_E \otimes 1) \\ &= (-1)^{|y|+1} n(y\beta z_E \otimes v^{n-1}u). \end{aligned}$$

Obviously,

$$d_3(x \otimes v^n) = 0 \Leftrightarrow n = kp \text{ or } x \in \mathrm{Ann}_{H/(z_E)}(\beta z_E),$$

$$d_3(y \otimes v^n u) = 0 \Leftrightarrow n = kp \text{ or } y \in \mathrm{Ann}_H(\beta z_E)$$

Compute $\ker d_3/\mathrm{im}\ d_3$, we obtain the formula for E_4. The proposition follows

§ 2- COMPUTATION OF $E_0(H^*(M(p^n))$.

In this section we compute the H — S spectral sequence for the central extension (M) given in the introduction.

Let u_a, u_b denote the elements of $H^1(C_{n-2} \times C_p) = \mathrm{Hom}(C_{p^{n-2}} \times C_p,\ Z_p)$ with $u_a(\underline{a}) = u_b(\underline{b}) = 1$, $u_a(\underline{b}) = u_b(\underline{a}) = 0$.

Let $u'_a \in H^1(C_{p^{n-2}} \times C_p, Z_{p^{n-2}})$ be the element with $u'_a(a) = 1$, $u'_a(b) = 0$. Let $\beta_n : H^*(., Z_{p^n}) \to H^{*+1}(.; Z_p)$ denote the Bockstein operator associated to the exact sequence of the coefficients.

$$o \to Z_p \to Z_{p^{n+1}} \to Z_{p^n} \to 0.$$

Let $v_a = \beta_{n-2}u'_a$, $v_b = \beta u_b$. Then, as it is well known, we have

(2.1) $$H^*(C_{p^{n-2}} \times C_p) = Z_p[u_a, u_b, v_a, v_b] / (u_a^2, u_b^2);$$

In particular, $H^2(C_{p^{n-2}} \times C_p) = Z_p u_a u_b \oplus Z_p v_a \oplus Z_p v_b$.

LEMMA 2.2. *Let* $z = [f] \in H^2(C_{p^{n-2}} \times C_p)$ *be represented by a 2 – cocycle f. Then we have*

$$z = q(a)v_a + q(b)v_b + (f(a, b) - f(a, b))\, u_a u_b$$

where $q(x) = \sum_{i=1}^{ord(x)} f(x, x^i)$ *for* $x \in C^*_{p^{n-2}} \times C_p$, *In particular, we have* $z_M = v + u_a u_b$ *up to a non zero constant multiple).*

Proof. By definition of the Bockstein operator, v_a can be represented by the 2–cocycle given as follows.

$$v_a(a^i b^k, a^j b^l) = 0 \quad \text{if} \quad 0 \leqslant i,j,\ i + j < ord(a)$$
$$= 1 \quad \text{if} \quad 0 \leqslant i,j < ord(a) \leqslant i + j.$$

Similarly, we have a 2–cocycle for v_b. Write $f = m_a v_a + m_b v_b + m_{ab} u_a u_b + \delta g$

A direct computation shows $(\delta g)(x,y) - (\delta g)(y,x) = 0$, $\sum_{i=1}^{ord(x)} (\delta g)(x, x^i) = 0$. From this we obtain easily the first part of the lemma.

For the later part, we observe that the projection $M(p^n) \to C_{p^{n-2}} \times C_p$ has an obvious inverse map

$t : C_{p^{n-2}} \times C_p \to M(p^n)$ given by $t(a^i b^j) = a^i b^j$ for $0 \leqslant i < p^{n-2}$, $0 \leqslant j < p$. By definition, the cohomology class corresponding to the extension (E) is the class $[\tilde{f}] \in H^2(M(p^n) / Z, Z)$ with $\tilde{f}$ being the 2–cocycle given by $\tilde{f}(x, y) = t(x)t(y)t(xy)^{-1}$ for $x, y \in M(p^n) / Z = C_{p^{n-2}} \times C_p$. Remind that $Z = \langle a^{p^{n-2}} \rangle$. We have $\tilde{f}(a, a^{-1}) = a^{p^{n-2}}$, $\tilde{f}(b, b^{-1}) = \tilde{f}(ab, a^{-1}b^{-1}) = 1$.

Now, let $u \in H^1(Z)$ be a non zero element. Then, we have $u : Z \cong Z_p$ and the isomorphism $u_* : H^2(M(p^n) / Z, Z) \cong H^2(M(p^n)/Z, Z_p)$. We have immediately $u_*[\tilde{f}] = u(a^{p^{n-2}})(v_a + u_a u_b)$. The lemma follows.

We note that the first part of the lemma can be generalized easily for an arbitrary finite abelian p–group (refer to Huỳnh Mùi [2; 1,5]).

Now we compute the H–S spectral sequence

$$H^*(C_{p^{n-2}} \times C_p) \otimes H^*(Z) \Rightarrow H^*(M(p^n))$$

for the extension (M) by means of Proposition 1.5. In what is to follow, we let $u : Z \cong Z_p$ with $u(a^{p^{n-2}}) = 1$, and $v = \beta u$. We have from 1.4 and the proof Lemma 2.2:

(2.2') $$z_M = v_a + u_a u_b,$$

and we have $H^*(Z) = Z_p[u, v] / (u^2)$.

LEMMA 2.3. *We have*

$$E_3 = Z_p[u_a, u_b, v_a, v_b]/(u_a^2, u_b^2, z_M) \otimes Z_p[v],$$

$$E_4 = Z_p[u_a, u_b, v_a, v_b]/(u_a^2, u_b^2, z_M, \beta z_M) \otimes Z_p[v^p]$$
$$\oplus (Z_p[v_b]u_a + Z_p v_b u_a u_b \text{ modulo } (z_M)) \otimes Z_p[v^p]v^{p-1}$$
$$\oplus (\sum_{i=1}^{p-2} Z_p u_a + Z_p u_a u_b \text{ modulo } (z_M, \beta z_M)) \otimes Z_p[v^p]v^i.$$

Here we remind that $z_M = v_a + u_a u_b$, and $\beta z_M = v_a u_b - u_b u_b$ if $n = 3$, $z_M = -v_b u_a$ if $n > 3$. Therefore,

$$\beta z_M = -v_b u_a \ (z_M) \qquad n \geqslant 3.$$

So the computation of E_r is similar for every $n \geqslant 3$.

Proof. We write $H = H^*(C_{p^{n-2}} \times C_p)$. Clearly, $\mathrm{Ann}_H(z_M) = 0$, so we obtain the formula for E_3. To compute E_4, we first note that the last assertion is true since we have

$$\beta u_a = v_a \text{ if } n = 3, \text{ and } \beta u_a = 0 \text{ if } n > 3.$$

Next, we show

(i) $\mathrm{Ann}_{H/(z_M)}(\beta z_M) = Z_p[v_b]u_a \oplus Z_p[v_b]u_a u_b \quad (z_M)$,

(ii) $(\beta z_M)\underline{H}/(z_M) = Z_p[v_b]v_b u_a \oplus Z_p[v_b u_a]v_b u_a u_b \quad (z_M)$

(iii) $\mathrm{Ann}_{H/(z_M)}(\beta z_M)/(\beta z_M) = Z_p u_a \oplus Z_p u_a u_b \quad (z_M, \beta z_M)$.

These results can he obtained easily by a direct computation. From 1.5 follows the lemma.

LEMMA 2.4. $E_4 = E_5 = ... = E_{2p-1}$.

Proof. Fromm Lemma 2.3, $E_4^{i,j} = 0$ if j is odd. Hence, we have $d_{2k} = 0$, and so $E_{2k} = E_{2k+1}$ for $k \geqslant 2$. Now we prove $d_{2k+1} = 0$ for $k < p-1$ by induction. Suppose $E_4 = \ldots = E_{2k-1}$. If $1 \leqslant j < k$, $2k+1 > \max(1, 2j+1)$, so d_{2k+1} $1>2.(E^{i,2j}_{2k+1}) = 0$. If $k < j \leqslant p$, $d_{2k+1}(E^{i,2j}_{2k+1}) \subset E^{i+2k+1,2(j-k)}_{2k+1} = 0$ since $i+2k+$ Therefore, we need only to consider $d_{2k+1}(u_a \otimes v^k)$ and $d_{2k+1}(u_a u_b \otimes v^k) = -u_b d_{2k+1}(u_a \otimes v^k)$.

Clearly $v_b d_{2k+1}(u_a \otimes v^k) = 0$ because $u_a v_b \otimes v^k = 0$ in E_{2k+1}. However, $E^{2i,0}_{2k+1} = Z_p v_b^i \otimes 1 \ (z_M, \beta z_M)$, so $v_b d_{2k+1}(u_a \otimes v^k) = v_b(m v_b^{k+1} \otimes 1) = m v_b^{k+2} \otimes 1 = 0$ with $m \in Z_p$. Hence $m = 0$. The lemma follows

LEMMA 2.5. $d_{2p-1}(v_b u_a \otimes v^{p-1}) = 0$.

Proof. Following P. May [5; 3.4], the Kudo's transgression theorem can be applied to the H–S spectral sequence : if $x \in E_2^{0,2k}$ is transgressive, and if $y \in E_2^{2k+1,0}$ is given so that $d_{2k+1}(1 \otimes x) = y \otimes 1$, then $y \otimes x^{p-1} \in E_{2k(p-1)+1}$ and $d_{2p(p-1)+1}(y \otimes x^{p-1}) = -\beta P^1 y \otimes 1$.

Using this theorem to our case: $x = 1 \otimes v$, $y = -\beta z_M \otimes 1$, we have $d_{2p-1}(\beta z_M \otimes v^{p-1}) = \beta P^1 \beta z_M \otimes 1$. Clearly,

$$\begin{aligned}\beta P^1 \beta z_M &= v_a^p v_b - v_b^p v_a \\ &= u_a u_b v_b^p \ (z_M, \beta z_M) \\ &= -u_b(u_b v_a) v_b^{p-1} \ (z_M, \beta z_M).\end{aligned}$$

Therefore, $d_{2p-1}(\beta z_M \otimes v^{p-1}) = 0$. But, as readily seen in Lemma 2.3, $\beta z_M \otimes v^{p-1} = v_b u_a \otimes v^{p-1}$ in $E_4 = (E_{2p-1}$ by 2.4). The lemma follows.

LEMMA 2.6. $E_{2p-1} = E_{2p} = E_{2p+1} = E_\infty$.

Proof. By means of Lemma 2.5 and the argument in proving 2.4, we have $d_{2p-1}(u_a \otimes v^{p-1}) = 0$. From this follows immediately $d_{2p-1} = 0$, and so $E_{2p-1} = E_{2p}$. Further, we have seen in the proof of 2.4 that $E_{2p} = E_{2p+1}$. Now it is clear that we need only to prove $d_{2p+1}(1 \otimes v^p) = -p^1 \beta z_M \otimes 1 = 0 \, (z_M, \beta z_M)$ We have

$$P^1 \beta z_M = -v_b^p u_a \doteq -x_b^{p-1}(v_b u_a) = 0 \ (z_M, \beta z_M).$$

The lemma follows.

Combine the above lemmata, we have reach to the following

PROPOSITION 2.7. *In the $H-S$ spectral sequence for the extension* (*M*), *we have* $E_4 = E_0(H(M(p^n)))$. *Moreover, the algebra* $E_0(H^*(M(p^n))$ *is generated by the following elements*

$$u_a \otimes 1,\ u_b \otimes 1,\ v_b \otimes 1,\ 1 \otimes v^p,\ u_a \otimes v^i (1 \le i \le p-1).$$

Remark. We have proved Proposition 2.7 by using only the $H-S$ spectral sequence for (M). If we remark also to the extension

$$9 \to \langle a \rangle \to M(p^n) \to \langle b \rangle \to 1,$$

the lemma can be obtained also by the fact that this extension is split, so clearly $Z_p[v_b] \otimes Z_p$ in the $H-S$ spectral sequence for (M) never can be hitted.

§ 3. COHOMOLOGY ALEBRA $H^*(M(p^n))$

According to Proposition 2.7, the algebra $H^*(M(p^n))$ is generated by the following elements

(3.1) $\qquad u_a,\ u_b,\ v_b,\ v(2p),\ z_{a,1}, \dots z_{a,p-1}$

where $w_x = \inf(M(p^n), C_{p^{n-2}} \times C_p) w_x$ with $w = u$, v and $x = a, b$; $v(2p)$ is any element of $H^*(M(p^n))$ which restricts to $H^*(Z)$ equal to v^p; and $z_{a,i} (1 \le i \le p-1)$ are certain elements of $F^1H(M(p^n))$ such that

$$z_{a,i} \in F^1H^*(M(p^n)) \longmapsto u_a \otimes v^i \in E^{1,2i},\ 1 \le i \le p-1$$

We shall show how to choose the elements $z_{a,i}$ and formulate the algebra $H^*(M(p^n))$ in terms of the system of the generators 3.1.

First we note that $u_a, u_b \in H^1(M(p^n)) = \mathrm{Hom}(M(p^n), \mathbf{Z}_p)$ are by definition the canonical generators of $H^1(M(p^n))$ with respect to the generators a, b of the group $M(p^n)$ (see the definition of u_a, u_b at the beginning of the section 2). Let $A = \mathrm{Ker}\ u_a$ and $B = \mathrm{Ker}\ u_b$. Then A and B are obviously two maximal subgroups of $M(p^n)$, and we have

(3.2)
$$A = \langle a^p, b\rangle \approx C_{p^{n-2}} \times C_p,\ B = \langle a\rangle \cong C_{p^{n-1}}$$
$$A \cap B = \langle a^p\rangle,$$
$$M(p^n) = AB = \bigcup_{i=0}^{p-1} Aa^iA.$$

Set $a' = a^p$. Let $u_{a'} \in H^1(A)$ with $u_{a'}(a') = 1$, $u_{a'}(b) = 0$ and let $v_{a'} = \beta_{n-2}u'_{a'}$ where $u'_{a'} \in H^1(A, Z_{p^{n-2}})$ is defined similarly as $u_{a'}$. Let

$$\mathrm{Res}(S,G): H^*(G) \to H^*(S) \text{ and } t(G, S)H^*(S) \to H(G)$$

denote the restriction and the transfer respectively for $S \subset G$ as usual. We shall consider the elements

(3.3)
$$z_{a,i} = t(M(p^n),A)(u_a, v^i_{a'}),\ 1 \le i \le p-3,\ i = p-1$$
$$= t(M(p^n), A)(u_a, v^i_{a'}) + p^{n-3}\ u_b v_b^{p-2},\ i = p-2.$$

LEMMA 3.4. *For* $1 \leqslant i, j \leqslant p-1$, *we have*

1) $\mathrm{Res}(B, M(p^n))\ z_{a,i} = u_a v^i_a$; *so* $z_{a,i}$ *are non zero*;

2) $\mathrm{Res}(A, M(p^n)z_{a,i} = 0, \qquad 1 \leqslant i \leqslant p-2$

$= p^{n-3}(u_b v_a - u_a v_b)v_b^{p-2}, \qquad i = p-1;$

3) $z_{a,i} \in F^1H^*(M(p^n)) \to m_i u_a \otimes v^i \in E^{1,2i}$ *with* $m_i \neq 0$;

4) $u_a z_{a,i} = 0$; $v_b z_{a,i} = 0$, $i \le p-2$; $z_{a,i}z_{a,j} = 0$.

Proof. 1) From 3.2, according to the double coset formula, we have

$$\mathrm{Res}(B,M(p^n))z_{a,i} = t(B,B \cap A)\ \mathrm{Res}(B \cap A,A)(u_a, u^i_{a'})$$
$$= t(B,B \cap A)(u_a, v^i_{a'}) \text{ (with some abuse of notation)}$$
$$= t(B,B \cap A)(u_a, \mathrm{Res}(B \cap A,A)\ v^i_a))$$
$$= (t(B,B \cap A)u_{a'}))v^i_a \qquad \text{(Frobenius' formula)}$$
$$= u_a v^i_a.$$

Here, $t(B,B \cap A)\ u_{a'} = u_a$ can be easily obtained by a direct computation, for instance, using the formula in E. Weiss [7; 2.5.2].

2) From 3.2 and the double coset formula, we have

$$\mathrm{Res}(A, M(p^n))\ t(M(p^n), A)\ (u_a v^i_{a'}) = \sum_{k=0}^{p-1} ad_a{}^k\ (u_a, v^i_{a'} = \sum_{k=0}^{p-1} (u_{a'} - kp^{n-3} u_b)$$
$$(v_{a'} - kp^{n-3}\ v_b)$$

since the operation of a on A defined by the conjugation is as follows: $a' \mapsto a'$, $b \to ba'^{-p^{n-3}}$. If $n > 3$, the assertion is obvious. We suppose $n = 3$. By a direct computation, we have

$$\begin{aligned} \mathrm{Res}\,(A, M(p^n))\, t\,(M(p^n), A)\,(u_a v_a^i) &= 0, & i \leq p-3, \\ &= -u_b v_b^{p-2}, & i = p-2, \\ &= (u_b v_a - u_a v_b)\, v_b^{p-2}, & i = p-1. \end{aligned}$$

The assertion follows from 3.3.

3) From Lemma 2.3 and Proposition 2.7, we have

$$H^{2i+1}(M(p^n)) = \mathbf{Z}_p u_b v_b^i \oplus \mathbf{Z}_p \widetilde{z}_{a,i}$$

for $1 \leq i \leq p-1$ where $\widetilde{z}_{a,i}$ is an element such that

$$\widetilde{z}_{a,i} \in F^1H^*(M(p^n)) \to u_a \otimes v^i \in E_\infty^{1,2i}.$$

By considering the restrictions of the elements $u_b v_a^i$ and $z_{a,i}$ on $H^*(B)$, we are ready to see that $z_{a,i}$ can not be expressed only interms of $u_b v_b^i$. 3) is proved.

4) We have

$$u_a z_{a,i} = u_a\, t\,(M(p^n), A)\,(u_a' u_a') = t(M(p^n), A)\,(\mathrm{Res}\,(A. M(p^n))\, u_a \,.\, u_a v_{a'}^i) = 0,$$

since $\mathrm{Res}\,(A. M(p^n))\, u_a = 0$. Similarly we have $z_{a,i}\, z_{a,j} = 0$.

Finally, we consider $v_a z_{a,i}$. From 3), we have

$$v_a z_{a,i} \in F^3H^{2i+3}(M(p^n)) \mapsto 0 \in E_\infty^{3,2i},$$

if $i \leq p-2$. From the proof of the assertion 3), we have immediately $v_b z_{a,i} = m u_b v_b^{i+1}$. Restrict two sides on B, we have $m = 0$. The lemma is proved.

Proposition 2.7, the algebra structure of $E_4 = E_\infty$ in 2.3, and Lemma 3.4 result the following

THEOREM 3.5. *The algebra* $H^*(M(p^n))$ *is a commutative algebra generated by the elements*

$$u_a,\ u_b,\ v_b,\ v(2p),\ z_{a,1}, \ldots, z_{a,p-1}$$

and it has the algebra structure as follows.

1) As a module,

$$\begin{aligned} H^*(M(p^n)) = {} & \mathbf{Z}_p\,[u_b, v_b, v(2p)]/(u_b^2)\,\{1, z_{a,p-1}\} \\ & \oplus \mathbf{Z}_p\,[u_b, v(2p)]/(u_b^2)\,\{u_a, z_{a,1}, \ldots, z_{a,p-2}\}. \end{aligned}$$

2) The multiplication is given by the relations:

$$u_a v_b = z_{a,i}\ z_{a,j} = u_a z_{a,i} = v_b z_{a,k} = 0$$

where $1 \leq i,\ j \leq p-1,\ k \leq p-2.$

Here $R\{x, y, ..., z\}$ *means the free* R*-module generated by the free generators* $x, y, ..., z$.

Remark 3.6. As it is well known, there are only two non abelian groups of order p^3: $E(p^3)$ and $M(p^3)$. Here $E(p^3)$ is the group isomorphic to a Sylow p-subgroup of the general linear grop $GL_3(Z_p)$. The integral cohomology ring $H^*(E(p^3), \mathbf{Z})$ has been also determined by Lewis, and the mod p algebra $H^*(E(p^3))$ has been computed in Huỳnh Mùi [3]. As in the integral case, the computation of $H^*(E(p^3))$ is more complicated, however it is very illustrative of the algebra structure for the mod p cohomology of finite p-groups.

Received April 10, 1981

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