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SOME RESULTS ON LOCALLY LIPSCHITZIAN MAPPINGS.

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INTRODUCTION. Interior Mapping Theorem and Inverse Mapping Theorem for locally lipschitzian mappings in finite dimentional spaces were obtained by many authors by different methods, (see in Clarke [3], Pourciau [5]). Halkin in [4] gave a Interior Mapping Theorem for continuous mappings after giving a notion of Screen, he also showed that the theorem is not true in the case of infinite-dimentional spaces.

In this paper we present the Interior Mapping Theorem and the Inverse Mapping Theorem for locally Lipschitzian mappings from a Banach space into another one. In the case of finite-dimentional spaces these theorems give us wellknown Clarke's and Pourciau's results. For other essential results concerning the problem in question we can refer to Ioffe [10] and Aubin [11].

Some concepts of approximations of locally lipschitzian mappings are given in the first part of the paper and main results — in the remaining one.

We first give some preliminary definitions and properties.

1. SOME DEFINITIONS AND PROPERTIES.

Throughout this paper X,Y will denote real Banach spaces, $L(X,Y)$ — the Banach space of linear continuous mappings from X into Y with the norm $\| . \|$ defined by

$$\| A \| = \sup \{ \mid Ax \mid / x \in X, \mid x \mid \leqslant 1 \},$$

where A is an element of $L(X,Y)$.

If ε is a positive real number and, if $\bar{x}$ (respectively $\bar{y},\bar{A}$) is an element of X (resp. $Y,L(X,Y)$), we denote by $B(\bar{x},\varepsilon)$ (resp. $S(\bar{y},\varepsilon)$, $G(\bar{A},\varepsilon)$) the open ball of X (resp. $Y,L(X,Y)$) with the centre at $\bar{x}$ (resp. $\bar{y},\bar{A}$) and with the radius ε, i. e.

$$B(\bar{x},\varepsilon) = \{x \in X / \mid x-\bar{x} \mid < \varepsilon\} .$$

The closure of $B(\bar{x},\varepsilon)$ is denoted by $\overline{B}(\bar{x},\varepsilon)$.

If $\mathcal{A}$ is an arbitrary set in a Banach space, then $N_\varepsilon(\mathcal{A})$ denotes the ε-neighborhood of $\mathcal{A}$ which is equal to $\bigcup\limits_{a\in\mathcal{A}} B(a,\varepsilon)$.

DEFINITION 1. 1. A mapping f from X into Y is said to be locally lipschitzian at a point $\bar{x}$ in X if there exist a neighborhood U of $\bar{x}$ and a positive real number α such that

$$\mid f(x_1) - f(x_2) \mid \leqslant \alpha \mid x_1 - x_2 \mid$$

for every $x_1, x_2 \in U$.

DEFINITION 1. 2, [1]. We shall say that a closed convex set $\Delta \subset L(X,Y)$ is approximating a mapping f at a point $\bar{x} \in X$ if for any positive real number ε there exists a neighborhood U of $\bar{x}$ such that if $x_1, x_2 \in U$ there exists an element $A \in \Delta$ satisfying the following condition

$$\mid f(x_1) - f(x_2) - A(x_1 - x_2) \mid \leqslant \varepsilon \mid x_1 - x_2 \mid .$$

DEFINITION 1. 3, [2]. A closed convex set $\Delta \subset L(X,Y)$ is called a Shield for the mapping f at a point $\bar{x} \in X$ if for any positive real number ε there exists a neighborhood U of $\bar{x}$ such that if $x_1, x_2 \in U$ then there exists an element $A \in N_3(\Delta)$ satisfying the following condition

$$f(x_1) - f(x_2) = A(x_1 - x_2) .$$

From Haln-Banach Extension Theorem we easily see the following

PROPOSITION 1. 1. *For every $\bar{x} \in X$, $x \neq 0$, and for every $\bar{y} \in Y$ there exists an element $A \in L(X, Y)$ such that:*

$$A\bar{x} = y, \text{ and } \| A \|' = \mid \bar{y} \mid / \mid \bar{x} \mid$$

REMARK 1.1:

1) Proposition 1. 1 shows that a set $\Delta \subset L(X,Y)$ is approximating f at $\bar{x}$ iff it is a Shield for f at $\bar{x}$. We denote a Shield for f at $\bar{x}$ by $\Delta f(\bar{x})$.

2) Proposition 1.1 also shows that a mapping f is locally Lipschitz continuous a point $\bar{x}$ iff there exists a bounded Shield for f at $\bar{x}$. Throughout this paper,

if the mapping f is locally Lipschitz continuous at $\bar{x}$, then we always suppose that $\Delta f(\bar{x})$ is bounded. It is easily seen that $\bar{G}(0, \alpha)$ is a Shield for f at $\bar{x}$ iff f is locally Lipschitz continuous at $\bar{x}$ with Lipschitz constant α.

3) Definition 1. 2 implies that a mapping f is strongly differentiable at a point $\bar{x}$ iff there exists a Shield for f at $\bar{x}$ consisting of one element. (For the definition of strong derivatives, see [3], [5]).

4) Pourciau's Generalized Derivative is a particular example of Shield. Indeed, this fact is easily implied from Semi-Upper-Continuity of the generalized derivative and the Mean Value Theorem in [5].

5) The Generalized Jacobien and the Generalized Gradient in the sense of Clarke are also particular examples of Shields. (see [1]).

The demonstration of the next three propositions is trivial and hence is omitted.

PROPOSITION 1.2: *If f, g are mappings from X into Y and $\Delta f(\bar{x})$, $\Delta g(\bar{x})$ are their Shields at a point $\bar{x} \in X$, then the closure of the set $(\Delta f(\bar{x}) + \Delta g(\bar{x}))$ is a Shield for the mapping $(f + g)$ at $\bar{x}$.*

PROPOSITION 1.3: *If Y_1, Y_2 are Banach spaces, if f_i is a mapping from X into Y_i $(i = 1, 2)$, and if $\Delta f_i(\bar{x})$ is a Shield of f_i at $\bar{x}$, then the closure of the Cartesian product $\Delta f_1(\bar{x}) \times \Delta f_2(\bar{x})$ is a Shield for the mapping $f = (f_1, f_2)$: $X \longrightarrow Y_1 \times Y_2$ at the point $\bar{x}$.*

PROPOSITION 1.4: *If $\Delta f(x)$ is a Shield for $f: X \to Y$ at apoint $\bar{x} \in X$, then for every real number λ the set $\lambda \Delta f(\bar{x})$ is a Shield for the mapping λf at $\bar{x}$.*

The simple results below will prove to be very useful in the subsequent section.

PROPOSITION 1.5: *If a mapping $f: X \to Y$ is locally Lipschitz continuous at $\bar{x} \in X$, if g is a mapping from Y to a Banach space Z, which is strongly differentiable at the point $\bar{y} = f(\bar{x})$, if $\Delta f(x)$ is a Shield for f at $\bar{x}$, and if $A \in L(X,Y)$ is the derivative of g at $\bar{y}$, then the closure of the set $A.\Delta f(\bar{x}) = \{A.A' / A' \in \Delta f(\bar{x})\}$ is a Shield for the mapping $F: X \to Z$, defined by $F(x) = g(f(x))$, at $\bar{x}$.*

Proof. We denote by α the Lipschitz constant of the mapping f on some neighborhood U of x. Since A is the strong derivative of g at $\bar{y}$, then for every $\varepsilon > 0$ we can find a neighborhood V of $\bar{y}$ $(V \subset Y)$ such that for arbitrary Y_1, $Y_2 \in V$ the following inequality is hold

$$|g(y)_1) - g(y_2) - A(y_1 - y_2)| \leqslant \frac{\varepsilon}{2\alpha} |y_1 - y_2|$$

On the other hand, by the definition of Δf there exists a neighborhood U_1 of $\overline{x}(U_1 \subset U)$ with $f(U_1) \subset V$ such that for every $x_1, x_2 \in U_1$ there exits an element $A' \in \Delta f(\overline{x})$ satisfying

$$|f(x_1) - f(x_2) - A(x_1 - x_2)| \leqslant \frac{\varepsilon}{2\|A\|} |x_1 - x_2|$$

Therefore,

$$|g(f(x_1)) - g(f(x_2)) - A.A'(x_1 - x_2)| \leqslant \varepsilon |x_1 - x_2|$$

It means that the closure of $A_0 \Delta f(\overline{x})$ is a Shield for $g.f$ at $\overline{x}$.

PROPOSITION 1.6: *If X is reflexive, if a mapping $f: X \to R$ is locally Lipschitz continuous at a point $x \in \overline{X}$, if $f(\overline{x}) \leqslant f(x)$ for every x in some neighborhood U of $\overline{x}$, and if $\Delta f(\overline{x})$ is Shield for f at $\overline{x}$, then $0 \in \Delta f(\overline{x})$*

Proof. Suppose that this is not the case, i.e., $0 \overline{\in} \Delta f(\overline{x})$. Then we can find an element x_0 in X with $|x_0| = 1$ and a number $\delta > 0$ such that $\langle x_0, x^* \rangle \leqslant \delta < 0$ for every $x^* \in \Delta f(\overline{x})$. For every real number $\varepsilon < \delta$, we can find a neighborhood U_1 of $\overline{x}$ $(U_1 \subset U)$ such that for every x $\in U_1$ there exists $x^* \in \Delta f(\overline{x})$ satisfying

$$|f(x) - f(\overline{x}) - \langle x^*, x - \overline{x} \rangle| \leqslant \varepsilon |x - \overline{x}| \ ,$$

Consequently, for all positive t closed enough to 0, we can find $x^* \in \Delta f(\overline{X})$ satisfying the following condition:

$$|f(\overline{x} + tx_0) - f(\overline{x}) - \langle x^*, t x_0 \rangle| \leqslant \varepsilon |tx_0| = \varepsilon t,$$

and hence

$$f(\overline{x} + tx_0) - f(\overline{x}) \leqslant \langle x^*, t x_0 \rangle + \varepsilon t \leqslant -\delta t + \varepsilon t < 0$$

This contradicts the minimality of $\overline{x} \in U$ and concludes the proof of Proposition 1.6.

2. MAIN RESULTS.

The following lemma will be need in the sequel:

LEMMA 2.1: *Suppose that A is an element of $L(X,Y)$, that γ, r are positive real numbers $(r \leqslant 1)$, and that*

$$S(O,\gamma) \subset A[\overline{B}(O,r)]$$

Then,

$$S(O, \left(\frac{\gamma}{2}\right) \subset A'[\overline{B}(O,r)] \text{ for each } A' \in G(A, \frac{\gamma}{2}).$$

This lemma is not new. The reader will have no difficulty making his own proof for it. For example, it may be established by using Nadler's fixed point theorem (see [6]).

As an easy consequence of the preceeding Lemma, we have the following result.

COROLLARY 2.1:

If $\mathcal{A}$ is a nonempty compact subset of $L(X,Y)$, the two following conditions are equivalent:

$$(i)\ 0 \in \text{int } A[\overline{B}(0,1)] \text{ for every } A \in \mathcal{A}$$
$$(ii)\ 0 \in \text{int} \bigcap_{A \in \mathcal{A}} A[\overline{B}(0,1)]$$

Throughout this section we shall suppose that X, Y are reflexive. It is known (see [7]) that every reflexive Banach space has an equivalent norm which is strongly differentiable at every nonzero point. Assume in what follows that Y is endowed with such a norm.

THEOREM 2, 1 (Interior Mapping Theorem).

Suppose that the mapping f from an open set $U \subset X$ into Y is locally lipschitzian at a point a.

If for every x belonging to some neighborhood of the point a there exists a Shield $\Delta f(x)$ for f at x satisfying the following conditions:

(1) The set-valued mapping $x \to \Delta f(x)$ is semi-upper-continuous at the point a,

(2) $$0 \in int \bigcap_{A \in \Delta f(a)} A[\overline{B}(0,1)],$$

then $f(a) \in int\ J(\bigcup)$.

REMARK 2. 1: *if $\Delta f(a)$ is compact then condition (2) is equivalent to the following condition:*

(2') every A in $\Delta f(a)$ is a surjection, because of the above Corollary.

Proof of theorem 2.1:

For some positive real number $r \leqslant 1$ we have $\overline{B}(a, r) \subset U$.

From condition (2) we can find a positive real number γ such that

$$S(0,2\gamma) \subset A[B(0,1)] \text{ for every } A \in \Delta f(a).$$

By the upper-semi-continuity of the mapping $\Delta f(.)$ we choose a number $\delta > 0$ such that

$$\Delta f(x) \subset N_\gamma(\Delta f(a)) \text{ for every } x \in B(a, \delta),$$

and by Lemma 2. 1 we conclude that

$$S(0, \gamma) \subset A[\overline{B}(0,1)] \quad \text{for every } A \in \mathcal{A},$$

where $\mathcal{A} = \{A/A \in \Delta f(x),\ x \in B(a, \delta)\} \subset N\gamma(\Delta f(a))$.

If we denote the transpose of A by A^*, we then have that for every $A \in \Delta f(x)$ and every $x \in B(a, \delta)$ the inequality

$$\| A^* y^* \| \geqslant \gamma \tag{1}$$

is hold for every $y^* \in Y^*$, $|y^*| = 1$, (where Y^* stands for the dual Y^* of Banach space Y), since $\| A^* y^* \| = \sup\limits_{x \in B(0,1)} \langle A^* y^*, x \rangle = \sup\limits_{x \in B(0,1)} \langle y^*, Ax \rangle \geqslant$

$\geqslant \sup\limits_{y \in S(0,\gamma)} \langle y^*, y \rangle \geqslant \gamma.$

Now we claim that

$$f(a) \in \operatorname{int} f(B(a,r)) \quad .$$

Indeed, assume the contrary, then we can choose a sequence $\{y_n\} \subset Y$ such that:

$$y_n \notin f(B(a,r)) \quad \text{for every } n, \text{ and } y_n \xrightarrow[n \to \infty]{} f(a), \tag{2}$$

For every number n we difine the function $\varphi(x) = |y_n - f(x)|$.

Note that $\varphi(a) \leqslant \inf\limits_{x \in \overline{B}(a,r)} \varphi(x) + |y_n - f(a)|$, and that $\overline{B}(a,r)$ is a complete metric space, from Ekeland's variational principle we can assert that: for the positive real number $\varepsilon_n = |y_n - f(a)|$ there exists a point $v_n \in \overline{B}(a,r)$ such that

(i) $|v_n - a| \leqslant \sqrt{\varepsilon_n}$,

(ii) $\varphi(v_n) \leqslant \varphi(x) - \sqrt{\varepsilon_n}\,|x - v_n|$, for every $x \in B(a,r)$ (3)

If n is large enough we shall have $\sqrt{\varepsilon_n} < r$, because of (2), and hence $v_n \in \operatorname{int} B(a,r)$, and (3)-(ii) means that the function $F(x) = \varphi(x) + \sqrt{\varepsilon_n}\,|x - v_n|$ is actually minimized at v_n over a full neighborhood of v_n, and by Proposition 1. 6 we conclude that

$$0 \in \Delta F(v_n).$$

From Propositions 1. 2 , 1. 5, and Remark 1. 1 we have that

$$0 \in \Delta^* f(v_n) y_n^* + \sqrt{\varepsilon_n} B', \tag{4}$$

where $\Delta^* f(v_n)$ stands for the set $\{A^*/A \in \Delta f(v_n)\}$, y_n^* for the derivative of the norm $|\,.\,|$ in space Y at the point $(y_n - f(v_n))$ and B' for the ball $\{x^* \in X^* / |x^*| \in 1\}$ in the dual X^* of Banach space X.

Since $(y_n - f(v_n)) \neq 0$, because of (2), we have $|y_n^*| = 1$.

In n is large enough v_n belongs to $B(a,\delta)$, and from (1) we can assert that

$$|A^* y_n^*| \geqslant \gamma \text{ for every } A \in \Delta f(v_n) .$$

But on the other hand $\sqrt{\varepsilon_n} B'$ is infinitely small when n tends to infinity, that contradicts (4).

The proof is thus complete.

REMARK 2. 2, If X,Y are finite-dimentional spaces, if $f : X - Y$ is a locally Lipschitz continuous funetion, and if $\Delta f(x)$ sands for $\partial f(x)$, where $\partial f(x)$ is either a generalized gradient in the Clarke's sense [3], or a generalized derivative in the Pourciau's sense [5], then the condition (1) of Theorem 2. 1 is satisfied and hence we have Theorem 2. 1 as Interior Mapping Theorem for generalized gradient and for generalized derivative.

It should be noted that our method in the proof of Theorem 2.1 can be applied to any concept of approximations for which Propositions 1. 2, 1. 5, 1. 6 are valid. For example it can be applied to the case of the Warga's derivative container [12].

THEOREM 2.2 (Inverse Mapping Theorem)

If under the assumption of Theorem 2. 1 the condition (2) is replaced by the following condition :

(2") every mapping $A \in \Delta f(a)$ has an inverse mapping A^{-1} and the set $\{A^{-1} / A \in \Delta f(a)\}$ is bounded,

then there exist a neighborhood U_1 of the point a, and a neighborhood V_1 of the point $f(a)$ and a locally lipschitzian mapping $g : V_1 \to U_1$ such that $f_0 g$ is the identical mapping on V_1.

REMARK 2. 3: 2 If $\Delta f(a)$ is compact then the set $\{A^{-1} / A \in \Delta f(a)\}$ is bounded, because the function $A \to \| A^{-1} \|$ is continuous.

PROOF OF THEOREM 2. 2. Suppose that $\Delta f(a)$ is bounded by some positive real number β. From the condition (2") and theorem 4 in[9]it follows that for every positive real number $\varepsilon < \dfrac{1}{2\beta}$ the set $N_\varepsilon(\Delta f(a))$ consists of only invertible elements.

By Definition 1. 3 we can find a neighborhood U'_1 of the point a such that for every $x_1, x_2 \in U'_1$ there exists an element $A \in N_\varepsilon(\Delta f(a))$ satisfying the following condition

$$f(x_1) - f(x_2) = A(x_1 - x_2).$$

By the invertibility, of A, $f(x_1) \neq f(x_2)$ if $x_1 \neq x_2$. Condition (2'') means that there exists a positive real number γ such that $\|A^{-1}\| \leqslant \gamma$ for every $A \in \Delta f(a)$. It is easily seen that

$$S\left(0, \frac{1}{\gamma}\right) \subset A[B(0, 1)] \quad \text{for every } A \in \Delta f(a) .$$

Applying Theorem 2. 1 we can find a neighborhood U_1 of the point a $(U_1 \subset U'_1)$ and a neighborhood V_1 of $f(a)$ such that $V_1 \subset f(U_1)$.

Now we have that for every $y \in V_1$ there exists an unique element $x \in U_1$ such that $y = f(x)$. The mapping $g: V_1 \to U_1$ defined by $g(y) = x$, where $f(x) = y$, is satisfying the following condition $f_0 g = E$, where E stands for the Identical mapping on V_1.

To complete the proof it remains to show that g is locally lipschitzian Take $y_1, y_2 \in V_1$, we have that $x_i = g(y_i)$, $i = 1, 2$, belong to U'_1, so we can find an element $A \in N_\varepsilon(\Delta f(a))$ such that

$$f(x_1) - f(x_2) = A(x_1 - x_2) \quad \text{, i.e.}$$

$$y_1 - y_2 = A(g(y_1) - g(y_2)) .$$

Taking $\overline{A} \in \Delta f(a)$ such that $\|A - \overline{A}\| < \varepsilon$ we have that .

$$y_1 - y_2 = \overline{A}(g(y_1 - g(y_2)) + (A - \overline{A})(g(y_1) - g(y_2)) .$$

Therefore

$$g(y_1) - g(y_2) = \overline{A}^{-1}(y_1 - y_2) - \overline{A}^{-1}(\overline{A} - A)(g(y_1) - g(y_2)),$$

and hence

$$|g(y_1) - g(y_2)| \leqslant \gamma |y_1 - y_2| + \gamma\varepsilon |g(y_1) - g(y_2)| .$$

If ε is small enough $\left(\varepsilon < \frac{1}{\gamma}\right)$ we shall have that

$$|g(y_1) - g(y_2)| \leqslant \frac{\gamma}{1 - \varepsilon\gamma} |y_1 - y_2|$$

as it was to be shown.

REMARK 2.4: If f and g are the mappings which are in theorem 2.2 then the closure of the convex hull of the set $\{A^{-1} / A \in \Delta f(a)\}$ is a Shield for g at the point $f(a)$.

Indeed, for every positive real number ε we can find a neighborhood U' of the point a such that for each $x_1, x_2 \in U'$ there exists an element $A \in \Delta f(a)$ satisfying the following condition

$$|f(x_1) - f(x_2) - A(x_1 - x_2)| \leqslant \frac{\varepsilon}{\alpha\gamma} |x_1 - x_2|,$$

where α is the Lipschitz constant of g on some neighborhood V of the point $f(a)$.

Taking a neighborhood V' of $f(a)$ such that $V' \subset V \cap f(U')$ we can assert that for each $y_1, y_2 \in V'$ there exists an element $A \in \Delta f(a)$ satisfying the condition

$$|g(y_1) - g(y_2) - A^{-1}(y_1 - y_2)| \leqslant \varepsilon |y_1 - y_2| .$$

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REFERENCES

1. Hoàng Tụy, *Micro-cours, Gradients of Lipschitz functions of Clarke*, Nha trang, 1977.

2. Sweetser T.H., *A minimal Set-valued Strong Derivative for Vector-valued Lipschitz Functions*, JOTA 23 (4), 1977.

3. Clarke F. H., *On the Inverse Functions Theorem*, Pacific Journal of Mathematics, V. 76, 1976.

4 Halkin H., *Interior Mapping Theorem with Set-valued Derivative*, J. d'Analyse Math., V.XXX, 1976.

5. Pourciau B.H., *An Analysis and Optimization of Lipschitz Continuous Mapping.*, JOTA 22 (3), 1977.

6. Nadler S., *Multivalued Contractive Mappings*, Pacific Journal of Mathematics, 30, 1969.

7. Diestel. *Geometry of Banach Spaces*, Berlin, Springer, 1975.

8. Ekeland I., *On the Variational Principle*, Journal Math. Anat., 4, 1974.

9. Kolmogorov. A.N. and Fomin C.B., *Elements of Theory of Functions and Fonctional Analysis*, Moscow, 1972, p. 215 (in Russian).

10. Ioffe A.D., *Nonsmooth Analysis and the theory of Fans*, Proc. of the Colloqium Convex Analysis And Optimization, Imperial College, London, February 28th, 1980.

11. Aubin J. P., *Contingent derivatives of Set-valued Maps and existence of solutions to non-linear inclusions and differential inclusions.*, Advances in Math. (to appear)

12. Warga J., *Necessary conditions without differentiability assumptions in optimal control*, J. Diff . Eq. 18, 1975.