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ON COHOMOLOGY OF HOMOGENEOUS SPACE

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Let G be a real connected Lie Group left — acting on smooth manifold X. For eaeh element $g \in G$ there is a diffeomorphism which is also denoted by $g: X \to X$ $(x \to gx, x \in X)$.

Let $\Omega^*(X, \mathbb{R})$ be the space of all differential forms on X. Then G is naturally acting on $\Omega^*(X, \mathbb{R})$ by $g: \omega \to g\omega = (g^{-1})^*(\omega)$, where $\omega \in \Omega^*(X, \mathbb{R})$ and $(g^{-1})^*$ is the induced transformation defined by the diffeomorphism $g^{-1}: X \to X$.

DEFINITION: A differential form $\omega \in \Omega^*(X, \mathbb{R})$ is called invariable (or spherical), if $g\omega = \omega$, $\forall g \in G$ (respectively if the subspace generated by all forms $g\omega$, $g \in G$ has a finite dimension).

We denote the space of invariable differential forms by $\Omega_I^*(X, \mathbb{R})$ and the space of spherical differential forms by $\Omega_0^*(X, \mathbb{R})$.

On the other hand if we consider $\Omega^*(X, \mathbb{R})$ with the operator d [1], then it becomes a cochain complex, and since $g^*d = dg^*$ $\Omega_I^*(X, \mathbb{R})$ and $\Omega_0^*(X, \mathbb{R})$ are its subcomplexes.

We shall denote the cohomology groups of these complexes respecting by $H^*(X, \mathbb{R})$, $H_I(X, \mathbb{R})$ and $H_0^*(X, \mathbb{R})$. The purpose of this note is to prove the following result:

THEOREM I: *Let $X = G/U$, where G is a real connected semisimple Lie Group and U is a closed connected subgroup of G. Then $H_I^*(X; \mathbb{R}) \cong H_0^*(X\mathbb{R})$. Moreover, if G is compact, then $H_I(X, \mathbb{R}) \cong H_0^*(X, \mathrm{R}) \cong H^*(X, \mathrm{R})$.*

I. For arbitrary smooth manifold X we shall denote the algebra of smooth functions on X by Λ, and the Lie Algebra of all differentiations of Λ by $\mathcal{V}$. It is well—known that $\mathcal{V}$ is the Lie Algebra of all smooth vector fields on X, and it is a Λ - module. Let $A_\Lambda^*(\mathcal{V}, \Lambda)$ be the graded Λ—module of all Λ—linear antisymmetric functions from $\mathcal{V}$ into Λ. The operator δ, which is defined usually by $(\delta\tau)(X_1, X_{n+1}) = \sum_{i=1}^{n+1} (-1)^{i+1} X_i \, \tau(X_1, \dots \widehat{X}_i, \dots, X_{n+1}) +$

$+ \sum_{i<j} (-1)^{i+j} \tau([X_i, X_j], \dots, \widehat{X}_i, \dots, \widehat{X}_j, \dots X_{n+1})$ where $\tau \in A_\Lambda^n(\mathcal{V}, \Lambda)$; $X_i \in \mathcal{V}$, $i - 1, \dots, n+1$, is a linear map from A_Λ^n) into $A_\Lambda^{n+1}(\mathcal{V}, \Lambda)$ satisfying $\sigma^2 = 0$. Therefore $A_\Lambda^*(\mathcal{V}, \Lambda)$ with δ will be a cochain complex. We denote the cohomology group of this complex by $H_\Lambda^*(\mathcal{V}, \Lambda)$. From [2] (Chapter I § 2) we see that $A_\Lambda^*(\mathcal{V}, \Lambda)$ is exactly $\Omega^*(X, \mathbf{R})$ and σ is exactly d. So, $H_\Lambda(\mathcal{V}, \Lambda)$ is identically $H^*(X\, \mathbf{R})$ in De Rham's usual mean. Therefore we can consider the De Rham's cohomology as the cohomology of the Lie Algebra of smooth vector fields with values in the space of smooth functions. This fact is very important for considering properties of exterior differential forms.

2. Let G be a connected Lie Group, Λ be the space of smooth functions on G, and $\mathcal{J}$ be its Lie Algebμa. If the Lie Algebra of all smooth vector fields on G is denoted by $\mathcal{V}$ as in I., and the natural inclusion of $\mathcal{J}$ into $\mathcal{V}$ — by $\rho : \mathcal{J} \to \mathcal{V}$ then ρ extends to a Λ — linear homomorphism denoted also by

$$\rho : \Lambda \otimes \mathcal{J} \to \mathcal{V}$$

Considering $\mathcal{V}$ as a Λ-module, it is clearly that ρ is an isomorphism. So, ρ induces the following isomorphism:

$$\rho^* : A_\Lambda^*(\mathcal{V}, \Lambda) \to A^*(\mathcal{J}, \Lambda) \quad .$$

where $A^*(\mathcal{J}, \Lambda)$ is the graded $\mathbf{R}$-module of all linear antisymmetric functions from $\mathcal{J}$ into Λ. It is well-known that $A^*(\mathcal{J}, \Lambda)$ is a cochain complex [3], [4], and ρ^* is a cochain isomorphism. Thus we obtain:

$$H_\Lambda^*(\mathcal{V}, \Lambda) \cong H^*(\mathcal{J}, \Lambda)$$

where $H^*(\mathcal{J}, \Lambda)$ is the cohomology group of Lie Algebra $\mathcal{J}$ with values in Λ [3], [4], [5]. Combining with the remarks of I. we have:

PROPOSITION I: $\quad H^*(G, \mathbf{R}) \cong H^*(\mathcal{J}, \Lambda)$

This proposition is generalised for homogeneous space $X = G/U$ by the following:

PROPOSITION 2: *If U is a closed connected subgroup of G, and $\breve{u}$ is the Lie Algebra of U, then we have:*

$$H^*(X, \mathbf{R}) \cong H^*(\mathcal{J}, \breve{u}, \Lambda)$$

here $H^*(\mathcal{J}, \breve{u}, \Lambda)$ is the relative cohomology group of $\mathcal{J}$ acoding to $\breve{u}$ [3], [5].

Proof. Let us consider the natural projection

$$p : G \to X = G/U$$

and the induced map

$$p^* : \Omega^* (X, \mathbb{R}) \to \Omega^* (G, \mathbb{R})$$

Since p is surjective, so p^* is injective. Moreover, because P^* commutes with d, P^* is a cochain monomorphism.

Let $A^*(\mathcal{J}, \check{u}, \wedge)$ be the submodule of $A^*(\mathcal{J}, \wedge) \cong A^*_\lambda(\mathcal{U}, \wedge) \cong \Omega^*(G, \mathbb{R})$ (see I.) consisting of functions $\tau \in A^S(\mathcal{J}, \wedge)$ $(S = 0,1,...)$ satisfying two properties:

1) $\tau(X_1,..., X_s) = 0 \quad$ if $\exists X_i \in \check{u}$

2) $(\theta(X)\tau)(X_1,..,X_s) \overset{\text{def}}{=} X\tau(X_1,...,X_s) - \sum_{i=1}^{S} \tau(X_1,...,[X, X_i]...X_s) = 0$

for every $X \in \check{u}$ and $X_1,.., X_s \in \mathcal{J}$.

It is easy to check that $A^*(\mathcal{J}, \check{u}, \wedge)$ is a subcomplex of $A^*(\mathcal{J}, \wedge)$, and it gives the relative cohomology group $H^*(\mathcal{J}, \check{u}, \wedge)$ [3.]

Now we will prove the following: $p^*(\Omega^*(X\cdot\mathbb{R}) \cong A^*(\mathcal{J}, \check{u},$

Suppose $\tau = p^*\omega$, $\omega \in \Omega^S(X, \mathbb{R})$. Then τ satisfies two following properties:

a) $\tau(X_1,.,X_s) = (p^*\omega)(X_1,.., X_s) = 0$ if $\exists X_i$, which is tangent to U (it means that $(X_i)_u \in T_u(U) \subset T_u(G), \forall_u \in U$

b) $R^*_u \ \tau = \tau, \forall_u \in U$ (R_u is the right translation corresponding to u).

Indeed, a) is directly deduced from the fact that the differential dp of p vanishes on $\check{u}$. That is $(dp)_g(X_g) = (dp)_g(dL_g Xe) = (d_g)_g U((dp)_e(X_e)) = 0$ if $X \in \check{u}$. (Lg is the left translation corresponding to g). b) is deduced from $p^0 R_u = p \Rightarrow R^*_{u^0} p^* = p^* \forall_u \in U$.

Let $u(t)$ be a curve in U corresponding to $X \in u$. Putting $\check{u} = u(t)$ into the equation b) we have:

b') $R^*_{u(t)} \tau = \tau$

It is easily proved that the equation b') is equivalent to the equation $\theta(X)\tau = 0$ for each vector field $X \in \check{u}$. So, τ satisfies the properties 1) and 2) that is, $\tau \in A^*(, \check{u}, \wedge)$.

Conversely, (f $\tau \in A^*(J, \check{u}, \wedge)$, then it satisfies 1) and 2), and so, it also satisfies a) and b). Because U is connected, τ satisfies b). Then by [3] there is a $\omega \in \Omega^*(X, \mathrm{IR})$ such that $\tau = p^*\omega$. This means that.

$$\Omega^*(X.R) \cong P^*(\Omega^*(G, \mathbb{R})) \cong A^*(J, u, \wedge)$$

from which:

$$H^*(X, \mathbb{R}) \cong H^*(J, \check{u}, \wedge)$$

3. Consider the space $\wedge$ of all smooth functions on G. Because G left-acts on itself by left-translations Lg, G naturally acts on $\wedge$ by:

$(gf)(\tilde{g}) = f(Lg^{-1}(\tilde{g})) = f(g^{-1}\tilde{g})$, $f \in \wedge$; $g, \tilde{g} \in G$.

Put $\wedge_0 = \{f \in \wedge : \dim [gf : g \in G]_{\mathbf{RI}} < \infty\}$ and call the elements of $\wedge$ spherical functions. It is obvious that $\wedge_0$ is an invariant subspace of $\wedge$. Using the semisimplicity of G and the Zorn's Lemma, we can prove that $\wedge_0 = \bigoplus_i V_i$ where V_i are irreducible invariant subspaces of finite dimension.

It is clear that $A^*(J, \check{u}, \wedge_0)$ is a subcomplex of $A^*(J, \check{u}, \wedge)$, and it gives the cohomology group $H^*(J, \check{u}, \wedge_0)$.

PROPOSITION 3. *The map p^* is a cochain isomorphism from $\Omega_0^*(X, \mathbf{IR})$ into $A^*(J, \check{u}, \wedge_0)$.*

Proof. Suppose that $\omega \in \Omega^*(X, \mathbf{IR})$ and let $P: G \to X = G/_U$ be the natural projection. It is obvious that for $g, \tilde{g} \in G$ we have:

$p(Lg(\tilde{g})) = p(g\tilde{g}) = (g\tilde{g})U = g(\tilde{g}U) = g(p(\tilde{g})) \Rightarrow p^\circ Lg = g^\circ p \Rightarrow Lg_\circ^* p^* = p_\circ^* g^*$
from which:

$p^*(g\omega) = p^*((g^{-1})^*(\omega)) = Lg^{*-1}(p^*(\omega)) = gp^*(\omega)$, This means that p^* is a G — homomorphism if $\Omega^*(X, \mathbf{R})$ and $\Omega^*(G, \mathbf{R})$ are considered as G— modules.

Suppose $\omega \in \Omega_0^*(X, \mathbf{R})$. Then $\tau = p^*(\omega) \in A^*(\mathcal{J}, \check{u}, \wedge)$ is a spherical form on G. So we have an inclusion:

$p^*(\Omega_0^*(X, \mathbf{R})) \subset A^*(\mathcal{J}, \check{u}, \wedge) \cap \Omega_0^*(G, \mathbf{R})$, where $\Omega_0^*(G, \mathbf{R})$ is the space of spherical forms on G. (Here we identified $A^*(\mathcal{J}, \wedge)$ with $\Omega^*(G, \mathbf{R})$ as in 1. and 2.).

From proposition 2 and the fact that P^* is a G — homomorphism we have a reversed inclusion:

$$p^*(\Omega_0^*(X, \mathbf{R})) \supset A^*(\mathcal{J}, \check{u}\wedge) \cap \Omega_0^*(G, \mathbf{R}),$$

from which:

$$p_*[\Omega_0^*(X, \mathbf{R})) = A^*(\mathcal{J}, \check{u}, \wedge) \cap \Omega_0^*(G, \mathbf{R}).$$

To finish the proof of Proposition 3 we only have to check the following equation: $\Omega_0^*(G, \mathbf{R}) = A^*(\mathcal{J}, \wedge_0)$

Suppose $\tau \in \Omega_0^s(G, \mathbf{R})$ then $\exists \tau_1, \ldots, \tau_n, \in \Omega_0^s(G, \mathbf{R})$ such that

$$g\tau = \sum_{i=1}^{n} c^i(g)\tau_i \ , \ c^i(g) \in \mathbf{R}.$$

Put:

$$f = (X_{1,\dots,} X_s)$$
$$f_i = \tau_i (X_{1,\dots,} X_s),\ i = 1,\dots, n$$

where $X_{1,\dots,} X_s$ are arbitrary fixed vector fields from J. We have $f, f_i \in \wedge$.

Let g and $\tilde{g}$ be two arbitrary elements of G. Then: $(gf)(\tilde{g}) = f(g^{-1}\tilde{g}) =$ $= (\tau(X_{1,\dots,} X_s))(g^{-1}\tilde{g}) = \tau_{g^{-1}\tilde{g}}((dL^{-1})_{\tilde{g}}((X_1)_{\tilde{g}}),\dots,(dLg^{-1})_{\tilde{g}}((X_s)_{\tilde{g}})) =$ $=(g\tau)_{\tilde{g}}((X_1)_{\tilde{g}},\dots, = (X_s)_{\tilde{g}}) = ((g\tau)((X_{1,\dots,} X_s))(\tilde{g})$

Put $g\tau \sum_{i=1}^{n} c^i(g)\tau_i$ into the right part of this equation, we have:

$$(gf)\ (\tilde{g}) = \sum_{i=1}^{n} c^i(g)\ \tau_i (X_{1,\dots,} X_s))(\tilde{g}) = \sum_{i=1}^{n} c^i(g) f_i(\tilde{g})$$

It signifies that $f_{1,\dots,} f_n$ generate the space $[gf,\ g \in G]_{\mathbf{R}}$ and $f \in \wedge_0$. Therefore we have $\tau \in A^s(\mathcal{J}\ \wedge_0)$ and $\Omega_0^*(G, \mathbf{R}) \subset A^*, (\mathcal{J}\ \wedge_0)$.

If conversely $\tau \in A^*(\mathcal{J}, \wedge_0)$, then $f = \tau(X_{1,\dots,} X_s) \in \wedge_0\ (X_{1,\dots,} X_s \in \mathcal{J})$. Repeating the same argument we obtain:

$$gf = (g\tau)(X_{1,\dots,} X_s)\ (g \in G)$$

Because $f \in \wedge_0$, so $\exists_{g_1,\dots g_n} \in G$ such that $g_1 f,\dots, g_n f$ generate space $[gf,\ g \in G]_{\mathbf{R}}$. Then

$$gf = \sum_{i=1}^{n} d^i(g)\ g)_i f),\ d^i(g) \in \mathbf{R}.$$

Combining these two equations we have

$$(g\tau)\ X_{1,\dots,} X_s) = \sum_{i=1}^{n} d^i(g)(g_i\tau)(X_{1,\dots,} X_s)$$

This equation holds for every $X_{1,\dots,} X_s \in \mathcal{J}$ so we get:

$$g\tau = \sum_{i=1}^{n} d^i(g)(g_i\ \tau)$$

Thus $\tau \in \Omega_0^s(G, \mathbf{R})$ and $A^*(\mathcal{J}, \wedge_0) \subset \Omega_0^*(G, \mathbf{R})$ from which $\Omega_0^*(G, \mathbf{R}) = A^*(\mathcal{J}, \wedge_0)$.

It is obvious that

$$A^*(\mathcal{J}, \check{u}, \Lambda_0) = A\ (\mathcal{J}, \check{u}, \Lambda) \cap A^*(\mathcal{J}, \Lambda_0) =$$

$$= A^*(\mathcal{J}, \check{u}\ \Lambda) \cap \Omega_0^*(G, \mathbf{R}) = P^*(\Omega_0^*(X, \mathbf{R}).$$

We have thus proved Proposition 3.

Corollary I: $H_0^*(X, \mathbf{R}) \cong H^*(\mathcal{J}, \check{u}, \Lambda_0)$

4. Consider the cohomlogy group $H^*(\mathcal{J}, \check{u}, \Lambda_0)$. Because $\Lambda_0 = \bigoplus_i V_i$ we have: $H^*(\mathcal{J}, \check{u}, \Lambda_0) = \bigoplus_i H^*(\mathcal{J}, \check{u}, V_i)$

From the results of Chevalley and Eilenberg [3] and because G is semisimple, we have $H^*(\mathcal{J}, \check{u}, V_i) = 0$ if the representation of G on V_i is not trivial.

Now suppose $f \in \Lambda$ and $gf = f$, $\forall g \in G$. Since G is transitive on itself, so $f =$ constant. It means that there is only one component corresponding to the trivial representation in $\Lambda_0 = \bigoplus_i V_i$. This component contains all constant functions. From that we have: $H^*(\mathcal{J}, \check{u}, \Lambda_0) \cong H^*(\mathcal{J}, \check{u}, \mathbf{R})$.

Combining with Corollary I we obtain:

PROPOSITION 4. $H_0^*(X, \mathbf{R}) \cong H^*(\mathcal{J}, \check{u}, \mathbf{R})$

To prove Theorem I we only have to observe some followings.

From [3] we have known that $H_I^*(X, \mathbf{R}) \cong H^*(\mathcal{J}, \check{u}, \mathbf{R})$ Using Proposition 4 we get: $H_I^*(X, \mathbf{R}) \cong H_I^*(X, \mathbf{R})$. Moreover, if G is compact, then from [3] too, we obtain $H_I^*(X, \mathbf{R}) \cong H^*(X, \mathbf{R})$. Finally, if G is compact and semisimple too, then $H_I^*(X, \mathbf{R}) \cong H_I^*(X, \mathbf{R}) \cong H^*(X, \mathbf{R})$ namely, $H_I^*(G/U, \mathbf{R}) \cong$ $\cong H_0^*(G/U, \mathbf{R}) \cong H^*(G/U, \mathbf{R})$. and with this we completed the proof of Theorem I.

Remark: It is easy to see that our results are also true when $\mathbf{R}$ is replaced by $\mathbf{C}$.

5. In this part we shall show that if G is not compact, then in general $H_0^*(G/U, \mathbf{R})$ isnt isomorphic to $H^*(G/U, \mathbf{R})$. First we prove

THEOREM 2: *Let $\mathcal{J}$ be a complex semisimple Lie Algebra and $\check{u}$ be a parabolic subalgebra of $\mathcal{J}$. Then*

$$H^0(\mathcal{J}, \check{u}\ \mathbf{C}) = \mathbf{C}$$

$$H^i(\mathcal{J}, \check{u}, \mathbf{C}) = 0 \qquad \forall\, i \geqslant 1$$

Proof.

Let us remember the definition of $H^*(\mathcal{J}, \breve{u}, \mathbf{R})$.

Let $A^n(\mathcal{J}, \mathbf{C})$ be the space of all antisymmetric n-linear functions from $\mathcal{J}$ into $\mathbf{C}$. Then $A^*(\mathcal{J}, \mathbf{C}) = \sum_{n=0}^{\infty} A^n(\mathcal{J}, \mathbf{C})$ is a cochain complex with differential operator δ, which is defined as following:

$$(\delta f)(X_1,\ldots, X_{n+1}) = \sum_{i<j} (-1)^{i<j} f([X_i, X_j], \ldots, \widehat{X}_i, \ldots, \widehat{X}_j, \ldots, X_{n+1})$$

$$(f \in A^n(\mathcal{J}, \mathbf{C});\ X_1 \ldots, X_{n+1} \in \mathcal{J}).$$

Let $A^n(\mathcal{J}, \breve{u}, \mathbf{C})$ be the subspace of $A^n(\mathcal{J}, \mathbf{C})$, containing functions f satisfying two following properties:

$$1)\ f(X_1,\ldots, X_n) = 0 \quad \text{if} \quad \exists\, X_i \in \breve{u}$$

$$2)\ (\mathcal{O}(X) f)(X_1,\ldots, X_n) \overset{\text{def}}{=} \sum_{i=1}^{n} f(X_1,\ldots, [X, X_i], \ldots, X_n) = 0$$

where $X \in \breve{u}$, $\quad X_1,\ldots, X_n \in \mathcal{J}$.

Then $A^*(\mathcal{J}, \breve{u}\ \mathbf{C}) = \sum_{n=0}^{\infty} A^n(\mathcal{J}, \breve{u}, \mathbf{C})$ is a subcomplex of $A^*(\mathcal{J}, \mathbf{C})$ and it gives us the relative cohomology group $H^*(\mathcal{J}, \breve{u}, \mathbf{C})$.

Lei $\mathcal{J}/\breve{u}$ be the factor-space. Then the adjoint representation ad of $\mathcal{J}$ will induce the representation φ of $\breve{u}$ in $\mathcal{J}/\breve{u}$ as following:

$$\varphi(X)(Y) + \breve{u}) \overset{def}{=} ad_X(Y) + \breve{u}\,(Y \in \mathcal{J})$$

On other hand, if we denote

$$(\wedge^n \mathcal{J})^* \cong \wedge^n(\mathcal{J}^*) = \underbrace{\mathcal{J}^* \wedge \cdots \wedge \mathcal{J}^*}_{n}$$

and

$$(\Delta^n(\mathcal{J}/u)^*) \cong \wedge^n(\mathcal{J}/\breve{u}^*) = \underbrace{(\mathcal{J}/\breve{u})^* \wedge \cdots \wedge \mathcal{J}/\breve{u})^*}_{n}$$

and $\wedge^n \varphi^* = \underbrace{\varphi^* \wedge \cdots \wedge \varphi^*}_{n}$ where $\mathcal{J}^*$ and $(\mathcal{J}/\breve{u})^*$ are the dual spaces of $\mathcal{J}$ and $\mathcal{J}/\breve{u}$ respectively, and φ^* is the dual representation of φ, then we can identify $A^n(\mathcal{J}, \mathbf{C})$ with $\wedge^n(\mathcal{J}^*)$ and $A^n(\mathcal{J}, \breve{u}, \mathbf{C})$ with:

$$(\wedge^n(\mathcal{J}/\breve{u})^*)_I \overset{def}{=} \{f \in \wedge^r(\mathcal{J}/\breve{u})^* : (\wedge^n \varphi^*) f = 0 \quad \forall\, X \in \breve{u}\}$$

Theorem 2 is proved, if we show that

$$\wedge^n (\mathcal{J}/\breve{u})^*)_I = 0 \ \forall \ n \geqslant 1$$

Since $\breve{u}$ is a parabolic subalgebra, there in a Cartan's Subalgebra $f \subset \mathcal{J}$ ich that:

$$\breve{u} = f + \sum_{\alpha \in \Sigma_1} \mathcal{J}_\alpha$$

where Σ_1 is a closed subsystem of rooted system Σ of f, which contains all ositive roots from Σ^+, and $\mathcal{J}_\alpha$ is the root-subspace corresponding to $\alpha \in \Sigma$ see [6]). Then

$$\mathcal{J}/\breve{u} \cong \sum_{\alpha \in \Sigma \setminus \Sigma_1} \mathcal{J}_\alpha$$

For each $\alpha \in \Sigma$ we denote a root-vector corresponding to α by $E_\alpha \neq 0$, and put $\bar{E}_\alpha = E_\alpha + \breve{u}$. It is obvious that $\{\bar{E}_\alpha, \alpha \in \Sigma \setminus \Sigma_1\}$ is a basis of $\mathcal{J}/\breve{u}$. If $\{\bar{E}^*_\alpha$, $\alpha \in \Sigma \setminus \Sigma_1\}$ is its dual basis in $(\mathcal{J}/\breve{u})^*$, that is

$$\langle \bar{E}^*_\alpha, \bar{E}_\beta \rangle = \begin{cases} 0 \ ; \ \alpha \neq \beta \\ 1 \ ; \ \alpha = \beta \end{cases}$$

then $\bar{E}^*_\alpha$ is a weight-vector of corresponding to the weight $-\alpha$. Indeed, for each $H \in f$ and $\beta \in \Sigma \setminus \Sigma_1$ we have $\langle \varphi^*(H) \bar{E}^*_\alpha, \bar{E}_\beta \rangle = -\langle \bar{E}^*_\alpha, \varphi(H) \bar{E}_\beta \rangle =$

$$= -\beta(H) \langle \bar{E}^*_\alpha, \bar{E}_\beta \rangle = -\alpha(H) \langle \bar{E}^*_\alpha, \bar{E}_\beta \rangle = \langle -\alpha(H) \bar{E}^*_\varphi, \bar{E}_\beta \rangle \Rightarrow$$

$$\Rightarrow \varphi^*(H) \bar{E}^*_\alpha = -\alpha(H) \bar{E}^*_\alpha.$$

Moreover, we can choose a basis of $\wedge^n (\mathcal{J}/\breve{u})^*$ from vectors in the following forms:

$$\bar{E}^*_{i_1 i_2 \ldots i_n} \overset{def}{=} \bar{E}^*_{\alpha i_1} \wedge \ldots \wedge \bar{E}^*_{\alpha i_n}$$

where $\alpha_{i_1}, \alpha_{i_n} \in \Sigma \setminus \Sigma_1$ and these α_{i_j} are different one from another. It is easy to see that each $\bar{E}^*_{i_1 i_2 \ldots i_n}$ is a weight-vector of weights : $-(\alpha i_1 + \ldots + \alpha i_n) \neq 0$ (because $\alpha_{i_1}, \ldots, \alpha_{i_n} \in \Sigma^-$). Therefore we have:

$$(\wedge^n (\mathcal{J}/\breve{u})^*)_I = 0$$

Thus Theorem 2 is proved.

COROLLARY 2. Let G be a real non — compact connected semisimple Lie Group, and U be a connected parabolic subgroup of G. Then:

$$H^0_0 (G/U, \mathbf{R}) \cong H^0_I (G/U, \mathbf{R}) \cong \mathbf{R}$$

$$H_0^i(G/_U, \mathbf{R}) \approx H_I^i(G/_U, \mathbf{R}) = 0 \qquad \forall\, i \geqslant 1$$

Proof, If is a direct consequence of Corollary 1 and Theorem 2.

Remark. It is true that we have proved a stronger result: There are not invariable differential forms on $G/_U$.

To end this note we have to consider a simple example to see the difference between the compact and non-compact cases.

Example:

$$SO\,(n+1/_{SO(n)} \cong SL\,(n+1, \mathbf{R})/_U \cong S^n,$$

where

$$U = \left\{ \begin{pmatrix} a & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \in SL\,(n+1, \mathbf{R}), \quad a > 0 \right\}.$$

From the obtained results, we have:

$$H_0^*(SO(n+1)/_{SO(n)}, \mathbf{R}) \cong H_I^*(SO(n+1)/_{SO(n)}, \mathbf{R}) \cong H^*(S^n, \mathbf{R}).$$

and

$$H_0^*(SL(n+1, \mathbf{R})/_U, \mathbf{R}) \cong H_I^*(SL(n+1, \mathbf{R})/_U, \mathbf{R}) \cong H^*(S^n, \mathbf{R}).$$

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