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THE MOD 2 COHOMOLOGY ALGEBRAS OF SYMMETRIC GROUPS.

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1. INTRODUCTION. The purpose of the present paper is to determine the cohomology algebra $H^*(\Sigma_m) = H^*(\Sigma_m; \mathbf{Z}_2)$ for arbitrary symmetric group Σ_m. Throughout this paper, the coefficient ring is always assumed to be $\mathbf{Z}_2 = \mathbf{Z}/2\mathbf{Z}$.

In [4] M. Nakaoka has defined the homology $H_*(\Sigma_\infty)$ of the infinite symmetric group and equipped it with the structure of Hopf algebra, whose multiplication is induced by the maps $\Sigma_m \times \Sigma_n \to \Sigma_{m+n}$, m, n $\geqslant 1$, and comultiplication by the diagonals $\Sigma_m \to \Sigma_m \times \Sigma_m$, $m \geqslant 1$. He also has determined $H_*(\Sigma_\infty)$ as algebra. In this paper, we shall first describe the structure of the Hopf algebra $H_*(\Sigma_\infty)$ via the Dickson elements, which we construct by means of invariant theory. Then passing to the dual, we obtain the algebra $H^*(\Sigma_\infty)$ as the polynomial algebra with free generators being the « *Dickson classes* ». Finally, the algebra $H^*(\Sigma_m)$ is expressed as the quotient $H^*(\Sigma_\infty)/\text{Ker Res}(\Sigma_m, \Sigma_\infty)$, where Res $(\Sigma_m, \Sigma_\infty)$ denotes the restriction homomorphism from Σ_∞ to Σ_m. From this result, we can obtain easily that due to Nakaoka [5] on $H^*(\Sigma_4)$.

The details of the paper will be given elsewhere.

2. THE HOPF ALGEBRA $H_*(\Sigma_\infty)$.

For the notion of algebra with multiplicity one can refer to T. Nakamura [3]. In particular, $H_*(\Sigma_\infty)$ is a an algebra with multiplicity, where

$$ {}_nH_*(\Sigma_\infty) = H_*(\Sigma_n, \Sigma_{n-1}), $$

(cf. Nakamura [3]. Remember that, if A is an algebra with multiplicity. then ${}_nA$ denotes the submodule consisting of all homogenous elements of multiplicity n. We set $A(m) = \bigoplus_{n \leqslant m} {}_nA$. Particularly $H_*(\Sigma_m) =$

$$= H_*(\Sigma_\infty)\ (m).$$

We are going to define the Dickson elements,

Let us consider Σ_{2^n} as the symmetric group on (the point set of) the vertor space $\mathbf{Z}_2^n$ of dimension n over $\mathbf{Z}_2$. Let $E^n = E_1 \times \ldots \times E_n$ denote the subgroup consisting of all translations on $\mathbf{Z}_2^n$ where E_i is the cyclic group of order 2 generated by the translation defined by the i-th unit vertor e_i of $\mathbf{Z}_2^n$. Then, as well known, we have

$$H^*(E^n) = \mathbf{Z}_2 [y_1, \ldots, y_n] ,$$

where $y_1, \ldots, y_n$ are the elements of $H^1(E^n) = Hom(E^n, \mathbf{Z}_2)$ given by $y_i (e_j) = \delta_{i,j}$ for $1 \leqslant i, j \leqslant n$. Here $\delta_{i,j}$ denote the Kronecker symbols. In [2; Theorem 6.2] Huỳnh Mùi has proved

(2.1) $\quad \text{Im Res}(E^n, \Sigma_{2^n}) = \mathbf{Z}_2[y_1, \ldots, y_n]^{GL(n, \mathbf{Z}_2)}$

where $GL(n, \mathbf{Z}_2)$ operates naturally on $\mathbf{Z}_2 [y_1, \ldots, y_n]$. The invariants of $GL(n, \mathbf{Z}_2)$ have been determined by L.E. Dickson [1] as follows.

(2.2) $\quad \mathbf{Z}_2[y_1, \ldots, y_n]^{GL(n, \mathbf{Z}_2)} = \mathbf{Z}_2[Q_{n,0}, \ldots, Q_{n,n-1}]$,

where $Q_{n,0}, \ldots, Q_{n,n-1}$ are the Dickson invariants defined in [1].

Further, let $\rho_{2^n} : \Sigma_{2^n} \to 0(2^n)$ denote the natural representation of the symmetric group in the orthogonal group $0(2^n)$. As it is well known $H^*(0(2^n)) = \mathbf{Z}_2 [W_1, \ldots, W_{2^n}]$, where W_i is the i-th universal Stiefel-Whitney class of dimension i. We let

$W_{n,s} = \rho^*_{2^n} (W_{2^n - 2^s}) \in H^*(\Sigma_{2^n}), 0 \leqslant s < n.$

According to Huỳnh Mùi, Quillen and Milgram (see[2; Appendix]), we have

(2.3) $\quad \text{Res}\ (E^n, \Sigma_{2^n})\, W_{n,s} = Q_{n,s}\ , 0 \leqslant s < n.$

Combine 2.1, 2.2 and 2.3, we observe

$H^*(\Sigma_{2^n}) = \text{Ker Res}\ (E^n, \Sigma_{2^n}) \oplus \mathbf{Z}_2 [W_{n,0}, \ldots, W_{n,n-1}].$

This allows us to define

2.4. DEFINITION. The Dickson elements $D_{k_0,\dots,k_{n-1}} \in H_*(\Sigma_2 n)$, $k_0,\dots,k_{n-1} \geqslant 0$, are defined by the conditions

$$\left\langle D_{k_0,\dots,k_{n-1}}, \operatorname{Ker} \operatorname{Res}(E^n, \Sigma_2 n) \right\rangle = 0$$

$$\left\langle D_{k_0,\dots,k_{n-1}}, \prod_{s=0}^{n-1} W_{n,s}^{h_s} \right\rangle = \begin{cases} 1 & (k_0,\dots,k_{n-1}) = (h_0,\dots,h_{n-1}) \\ 0 & \text{otherwise,} \end{cases}$$

where $\langle\, .\,,\,.\,\rangle : H_*(\Sigma_2 n) \times H^*(\Sigma_2 n) \to \mathbf{Z}_2$ denotes the dual pairing.
For $K = (k_0,\dots,k_{n-1})$, the image of $D_K = D_{k_0,\dots,k_{n-1}}$ under the injection $i(\Sigma_\infty, \Sigma_2 n) : H_*(\Sigma_2 n) \to H_*(\Sigma_\infty)$ will be denoted simply by D_K.

Because the Dickson elements are defined in close relation to the invariants, they seem to be more useful than the Nakaoka elements in determination of the Hopf algebra $H_*(\Sigma_\infty)$. This notice is proved by the following (compare with Nakaora [5; §2]).

2.5. THEOREM. The structure of $H_*(\Sigma_\infty)$ as Hopf algebra and as algebra with multiplicity are described as follows.

(i) $H_*(\Sigma_\infty) = \mathbf{Z}_2 [D_K;\, K \in J^+]$ *as algebras with multiplicity.*

Here $J^+ = \{K = (k_0,\dots, k_{n-1});\ n > 0,\ k_0 > 0,\ k_0,\dots,k_{n-1} \in \mathbf{Z}_+\}$, *and multiplicity of* $D_{k_0,\dots,k_{n-1}}$ *is defined to be* 2^n.

So we get the isomorphism of $\mathbf{Z}_2$*-modules for arbitrary m*

$$H_*(\Sigma_m) = \mathbf{Z}_2 [D_K;\, K \in J^+]\,(m)$$

The basis of $H_*(\Sigma_m)$ *consisting of all monomials in* $\mathbf{Z}_2 [D_K;\, K \in J^+]$ *of multiplicities* $\leqslant m$ *will be called the Dickson basis.*

(ii) *The comultiplication* $\Delta : H_*(\Sigma_\infty) \to H_*(\Sigma_\infty) \otimes H_*(\Sigma_\infty)$ *satisfies the formula*

$$\Delta D_{k_0,\dots,k_{n-1}} = \sum_{l_i + m_i = k_i} D_{l_0,\dots,l_{n-1}} \otimes D_{m_0,\dots,m_{n-1}}$$

for $k_i, l_i, m_i \geqslant 0$, $0 \leqslant i < n$.

The above formula can be reduced by the relation

$$D^{2^s}_{k_0,\dots,k_{n-1}} = D_{\underbrace{0,\dots,0}_{s},\, k_0,\dots,k_{n-1}}$$

for $s, k_0,\dots,k_{n-1} \geqslant 0$.

3. THE ALGEBRAS $H^*(\Sigma_m)$.

Let $J = \{K = (k_0,\dots, k_{n-1}) \neq 0,\ n > 0,\ k_0,\dots,k_{n-1} \in Z_+\} \supset J^+$.

If $K = (k_0,\dots,k_{n-1})$, then n is called the length of K and denoted by $l(K)$.

For each $(\mathscr{H}. T) = (H_1,\dots, H_r) \times (t_1,\dots, t_r) \in J^r \times |N^r$ we define the Dickson element

(3.1) $W^{\mathscr{H}}_T = W^{H_1,\dots,H_r}_{t_1,\dots,t_r} = (D^{t_1}_{H_1} \dots D^{t_r}_{H_r})^* \in H^*(\Sigma_\infty)$,

where * denotes the dual defined by the Dickson basis.

We shall consider J^r as a subset of $J^r \times |N^r$ by the injection

$$J^r \subset J^r \times |N^r, \qquad (H_1,\dots,H_r) \to (H_1,\dots,H_r) \times (\underbrace{1,\dots,1}_{r}).$$

For $(\mathscr{H}, T) = (H_1,\dots,H_r) \times (\underbrace{1,\dots,1}_{r})$ we write simply $W^{\mathscr{H}} = W^{H_1,\dots,H_r}$ instead of $W^{\mathscr{H}}_T$.

Note that, if $H = (h_0, \dots, h_{n-1})$ we get

(3.2) $\quad \mathrm{Res}\,(\Sigma_{2^n}, \Sigma_\infty)\, W^H = \prod_{s=0}^{n-1} W^{h_s}_{n,s}$.

As easily seen, $H^*(\Sigma_\infty)$ has the additive basis consisting of the elements

(3.3) $\quad W^{\mathscr{H}}_T,\ (\mathscr{H}, T) = (H_1,\dots, H_r) \times (t_1,\dots, t_r) \in (J^+)^r \times |N^r$,

with $H_1 < \dots < H_r$, $r \geqslant 0$. Here $<$ denotes the order defined in J by lengths and by lexicographic order for elements of the same length. Again, this basis is called the *Dickson basis* for $H^*(\Sigma_\infty)$.

We are now ready to state the main result of this paper.

3.4 THEOREM. $H^* (\Sigma_\infty) = \mathbf{Z}_2 [W^H ; H \in J_{odd}]$ *as algebras. Here*

$J_{odd} = \{ (h_o ,..., h_{n-1}) \in J$; *there exists i such that h_i is odd*, $n > 0 \}$.

The comultiplication $\triangle$ of $H^* (\Sigma_\infty)$ is formulated via the Dickson basis by the formula

$$\triangle W^{H_1, \ldots, H_r}_{t_1, \ldots, t_r} = \sum_{u_i + v_i = t_i} W^{H_1, \ldots, H_r}_{u_1, \ldots, u_r} \otimes W^{H_1, \ldots, H_r}_{v_1, \ldots, v_r}$$

for $H_1 ,..., H_r \in J^+$ and $H_1 < \ldots < H_r$.

Remark. By use of the Borels theorem on structure of Hopf algebra, Nakaoka has shown in [4; Theorem 5.3] that $H^* (\Sigma_\infty)$ is a polynomial algebra. But he has not obtained any knowledge about generators of the algebra except an information on the dimensions of generators.

Now we prepare to state the final theorem.

Set $J^\infty = \coprod_{r \geqslant 0} J^r$, $F(J, \infty) = \coprod_{r \geqslant 0} F(J, r)$,

where $F(J, r) = \{ (H_1 ,..., H_r) ; H_i \in (J, H_i \neq H_j \text{ if } i \neq j, 1 \leqslant i, j \leqslant r \}$.

We define the product $* : J^\infty \times J^\infty \to J^\infty$ by putting

$(L_1,..., L_m) * (M_1,..., M_n) = (L_1,..., L_m, M_1,..., M_n)$.

Let $\mathcal{L} = (L_1 ,..., L_m)$, $\mathcal{M} = (M_1 ,..., M_n)$. We denote by $R (\mathcal{L}, \mathcal{M})$ the set of all bijective relations h: $\{ 1, ..., m \} \to \{ 1, ..., n \}$ such that $l(L_i) = l(M_{h(i)})$ for $i \in$ Def (h). Here, Def (h) means the domain of definition of a relation h. We write Def (h) $= (j_1 , ..., j_u)$ as an ordered subset of the ordered set (1,...,m), and its ordered complement $(1, ..., m) \setminus$ Def $(h) = (i_1 , ..., i_t)$. Further let $(1, ..., n) \setminus$ Im $(h) = (k_1 , ..., k_v)$. Then we define the element of J^∞, which is called the join of $\mathcal{L}$ and $\mathcal{M}$ by h:

$$\mathcal{L} \vee_h \mathcal{M} = (L_{i_1} , ..., L_{i_t}) * (L_{j_1} + M_{h(j_1)} ,..., L_{j_u} + M_{h(j_u)}) * (M_{k_1} ,..., M_{k_v}),$$

where the partial addition «+» is defined in terms of the coordinates for only elements of the same length in J.

For each $H = (h_0, \dots h_{n-1}) \in J$ we set

$$H^{2^s} = (h_0, \dots, h_{n-1})^{2^s} = (\underbrace{0, \dots, 0}_{s}, h_0, \dots, h_{n-1}).$$

Further, for every sequence of positive integers $f = (v_0, \dots, v_\alpha)$, we put

$$H^f = (\underbrace{H, \dots, H}_{v_0}, \underbrace{H^2, \dots, H^2}_{v_1}, \dots, \underbrace{H^{2^\alpha}, \dots, H^{2^\alpha}}_{v_\alpha}).$$

For each natural number t, we denote by $F(t)$ the set of all sequences of integers $f = (v_0, \dots, v_\alpha)$, where $\alpha \geqslant 0$, $v_i \geqslant 0$, $v_\alpha > 0$, such that $t = v_0 + v_1 2^1 + \dots + v_\alpha 2^\alpha$. Let

$$(\mathcal{H}, T) = (H_1, \dots, H_r) \times (t_1, \dots, t_r) \in J^r \times |\mathbf{N}^r,$$

$$(\mathcal{K}, U) = (K_1, \dots, K_s) \times (u_1, \dots, u_s) \in J^s \times |\mathbf{N}^s,$$

$$f = (f_1, \dots, f_r) \in F(t_1) \times \dots \times F(t_r) = F(T)$$

$$g = (g_1, \dots, g_s) \in F(u_1) \times \dots \times F(u_s) = F(U).$$

Put $\mathcal{H}^f = H_1^{f_1} * \dots * H_r^{f_r}$, $\mathcal{K}^g = K_1^{g_1} * \dots * K_s^{g_s}$. We define

(3.5) $(\mathcal{H}, T) \vee (\mathcal{K}, U) = \left\{ \mathcal{H}^f \underset{h}{\vee} \mathcal{K}^g ; f \in F(T), g \in F(U), h \in R(\mathcal{H}^f, \mathcal{K}^g) \right\} \cap$
$\cap F(J, \infty)$.

Passing Theorem 2.5 to the dual we obtain

3.6 LEMMA.

$$W_T^{\mathcal{H}} \cdot W_U^{\mathcal{K}} = \sum_{(\mathcal{X}, Y)} W_Y^{\mathcal{X}}$$

where the summation runs over the representatives of Σ_* *-orbits of* $(\mathcal{H}, T) \vee (\mathcal{K}, U)$. *Here* $\Sigma_* = \coprod_{r \geqslant 0} \Sigma_r$ *operates on* $\coprod_{r \geqslant 0} J^r \times |\mathbf{N}^r$ *by*

$\sigma(H_1, \dots, H_r) \times (t_1, \dots, t_r) = H_{\sigma^{-1}(1)}, \dots, H_{\sigma^{-1}(r)}) \times (t_{\sigma^{-1}(1)}, \dots t_{\sigma^{-1}(r)})$,

for $\sigma \in \Sigma_r$.

3.7 DEFINITION. (i) The depth $\theta(z)$ of an element z in the Dickson basis of $H^*(\Sigma_\infty)$ is defined by the formulas

$$\theta(1) = 0, \qquad \theta(W^{H_1,\dots,H_r}_{t_1,\dots,t_r}) = \sum_{i=1}^{r} t_i \, 2^{l(H_i)}.$$

(ii) Suppose $z = \sum_{(\mathcal{H},T)} W^{\mathcal{H}}_T$, the linear decomposition of $z \in H^*(\Sigma_\infty)$ via the Dickson basis, then we put $\theta(z) = \min_{(\mathcal{H},T)} \theta(W^{\mathcal{H}}_T)$.

Note that, by use of Lemma 3.6 one can compute depth of every element $z \in \mathbf{Z}_2\left[W^H;\ H \in J_{odd}\right] = H^*(\Sigma_\infty)$.

3. 8 *THEOREM. We have the isomorphism of algebras*

$$H^*(\Sigma_m) = Z_2\,[W^H;\ H \in J^m_{odd}]/I_m$$

for arbitrary natural number m. Here

$$J^m_{odd} = \{H \in J_{odd} : \theta(W^H) = 2^{l(H)} \leqslant m\},$$

$$J_m = \{z \in \mathbf{Z}_2\,[W^H;\ H \in J^m_{odd}] : \theta(z) > m\}.$$

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Added in proof. In the case where p is an odd prime, we have determined the algebra $H^*(\Sigma_\infty ; Z_p)$ by use of the Dickson-Huỳnh Mùi $GL(n, Z_p)$ — invariants by a similar way as in Definition 2. 4. of this paper. This algebra is isomorphic to the tensor product of a polynomial algebra, an exterior algebra and a truncated polynomial algebra of height p. The generators of this polynomial algebra are corresponding to the Dickson invariants as seen in Theorem 3. 4. Meanwhile, the generators of the exterior and the truncated polynomial ones are corresponding to the Huỳnh Mùi nilpotent invariants (see [2]).