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REPRESENTATIONS AND REGULARITY OF MULTIVALUED MARTINGALES

ĐINH QUANG LƯU
Institute of Mathematics,
Hanoi.

INTRODUCTION. The theory of Multi-functions and vector-valued Asymptotic Martingales (Amarts) has been developed and extentively studied in recent years by K.Kuratowski and Ryll—Nardzewski C. [9], C. Castaing and M. Valadier [2], A.Bellow[1], G.A.Edgar and L.Sucheston [6] among others. One of most interesting results has been shown by F. Hiai and H. Umegaki in [8] that one can define a conditional expectation for an integrably bounded multi-function with closed bounded, non-empty set values in a separable B-space. This result leads to the study of some classes of multivalued amarts in [4], [7] and [5]. In the present paper we want to give some representation theorems for multi-valued martingales. In Section 1 we recall some notations, definitions and basis properties of multi-functions which will be used in the next sections. In Section 2 we study the existence of martingale selections of a sequence of multi-functions and at the same time prove some representation theorems for multivalued martingales. In Section 3 we present a necessary and sufficient condition for the regularity of multivalued martingales. In particular some results of [4] and [13] can be recovered by our results. In Section 4 we prove some invariance properties of the class of multivalued martingales w.r.t. the set of all bounded stopping times and some results related to the RN-property in B-spaces.

1. Notations, definitions and basis properties of multi-functions.

Throughout the paper B always denotes a separable B-space with some norm $\|\cdot\|$, $L_1(B,E)$ the B-space of all B-valued Bochner integrable functions defined on some probability space $(S.E.P)$. We will consider the class K of all closed bounded non-empty subsets of B with the usual Hausdorff s metric $h(\ldots)$. Thus $|X| = h(X, \{o\})$ is well-defined for all $X \in K$.

DEFINITION 1.1. A function $X: S \to K$ is called measurable, if for every open subset V of B the set $\{S; X(S) \cap V \neq \phi\}$ is measurable. If this occurs, we write $X \in \mu(K,E)$ with $S_X(F) = \{f \in \mu(B,F); f(s) \in X(s), a.e.\}$, where F denotes always a sub σ-field of E.

PROPERTY 1.1. ([2], theor. III . 9) Let $X: S \to K$. Then X is F-measurable iff there is a sequence $\langle f(n) \rangle$ of $S_X(F)$ s.t. $X(s) = cl\{f(n,s); n \in N\}$, a.e.in $\|\cdot\|$-norm, where N is the set of all positive integers. If this occurs then we write

$$X \xleftrightarrow{\|\cdot\|} \langle f(n) \rangle_{n=1}^{\infty} \qquad (w.r.t.F)$$

DEFINITION 1.2. (see [8]) A function $X \in \mu(K,E)$ is called integrably bounded. If the real-valued function $s \mapsto |X(s)|$ is integrable. If X is integrably bounded we will write $X \in L_1(K,E)$, and denote

$$\int_A X\,dP = \{\int_A f dP : f \in S_X(E)\}$$

In the case, if $\langle f(n) \rangle \subset (S_X(F)$ and $S_X(F) = cl\{f(n); n \in N\}$ in L_1-norm, then

we write $X \underset{L_1}{\leftrightarrow} \langle f(n) \rangle_{n=1}^{\infty}$ $(w.r.t.F)$.

Note that, if $X \underset{L_1}{\longleftrightarrow} \langle f(n) \rangle_{n=1}^{\infty}$ then $X \xleftrightarrow{\|\cdot\|} \langle f(n) \rangle_{n=1}^{\infty}$

PROPERTY 1.2. (see [8]) Let $X,Y \in L_1(K,E)$. Define

$H(X,Y) = \int_S h(X(s), Y(s))\,dP$; $K_c = \{A \in K; A$ is convex $\}$ and $K_{cc} = \{A \in K;$ A is convex compact$\}$, then $\langle L_1(K,E), H \rangle$ is a complete metric space and $L_1(K_c, E)$ and $L_1(K_{cc}, E)$ are closed subspaces of $L_1(K,E)$.

DEFINITION 1. 3. (see [8], theor. 5.1) Let $X \in L_1(K, E)$ then there is a unique function $E(X,F)$ of $L_1(K,F)$ s.t.

$$S_{E(X,F)}(F) = cl\{E(f,F); f \in S_X(E)\} \tag{1.1}$$

Such a function $E(X,F)$ will be called a conditional expectation of X (given F).

PROPERTY 1.3. (see [8]) Let $X \in L_1(K_c, E)$ and $Y \in L_1(K_{cc}, E)$ then $E(X,F) \in L_1(K_c, F)$ and $E(Y.F) \in L_1(K_{cc}, F)$.

For further informations we refer to [2] and [8].

2. Representations of Multivalued Martingales.

In the rest of the paper, it is supposed that we are given an increasing sequence $\langle E(n) \rangle$ of sub σ-fields of E with $\sigma(VF(n)) = E$. A sequence $\langle X(n) \rangle$ of multi-functions is said to be adapted to $\langle E(n) \rangle$ if $X(n)$ is $E(n)$ — measurable for all $n \in N$. Unless other-wise mentioned all our considered sequences are assumed adapted to $\langle E(n) \rangle$

DEFINITION 2. 1. Let $X(n)$ be a sequence of multi-functions. A sequence $\langle f(n) \rangle$ is called a *Martingale Selection* of $\langle X(n) \rangle$ is $f(n) \in S_{X(n)}(E(n))$ for all $n \in N$ and the sequence $\langle f(n) \rangle$ is a martingale in $L_1(B,E)$. If this occurs, we write $\langle f(n) \rangle \in MS(\langle X(n) \rangle)$.

DEFINITION 2. 2. A sequence $\langle X(n) \rangle$ is called a martingale in $L_1(K_c, E)$ if $X(n) \in L_1(K_c, E(n))$ for all $n \in N$ and $X(n) = E(X(m), E\ n))$ $(m > n \in N)$

Example 2.3. Let $\langle f(n) \rangle$ be a martingale in $L_1(B, E)$ and $\langle r(n) \rangle$ a non-negative martingale in $L_1(R, E)$. Define

$$X(n) = F(n) + r(n)\, U \quad (n \in N) \tag{2.1}$$

where $U = \{x \in B;\ \|x\| \leqslant 1\}$. Then by [10], $\langle X(n) \rangle$ is a martingale in $L_1(K_c, E)$ with closed ball values. It is easy to see that in the case every sequence $< g(n) >$ defined by $g(n) = f(n) + r(n)x$ for some $x \in U$ is a martingale selection of $\langle X(n) \rangle$. Moreover, it has been shown in [10] that every martingale with closed ball values can be written in the form (2.1). Thus the natural question arises whether there is always a martingale selection of a multivalued martingale? The following theorem, in particular, give us a positive answer to this question.

THEOREM 2.4. *Let $\langle X(n) \rangle$ be a sequence in $L_1(K_c, E)$, then the following conditions are equivalent:*

(1) $\langle X(n) \rangle$ is a martingale

(2) $X(k) = X_k(k+1)$ $(k \in N)$, where $X_n(m) = E(X(m), E(n))$

(3) $S_{X(k)}(E(k)) = cl\{E(g, E(k));\ g \in S_{X(k+1)}(E(k+1)\}$ $(k \in N)$

(4) $S_{X(k)}(E(k)) = cl\{g(k);\ \langle g(n) \rangle \in MS(\langle X(n) \rangle)\}$ $(k \in N)$

Proof. The equivalences (1) $\leftrightarrow$ (2) $\leftrightarrow$ (3) can be established easily from theorem 5.3 (2,3) in [8]. The major part of a proof consists in showing last equivalence (4) $\leftrightarrow$ (3). Suppose first thus that (3) holds. To show (4), we fix $\varepsilon > o$; $k \in N$ and $f(k) \in S_{X(k)}(E(k))$. Hence by (3) we can choose

$f(k+1) \in S_{X(k+1)}(E(k+1))$ s.t.

$$E(\|f(k) - f_k(k+1)\|) \leqslant \frac{\varepsilon}{2^k}$$

Therefore by induction we can construct a sequence $\langle f(n) \rangle_{n \geqslant k}$ s.t

$$f(n) \in S_{X(n)}(E(n)) \text{ and } E(\|f(n) - f_n(n+1)\|) \leqslant \frac{\varepsilon}{2^n} \qquad (n \geqslant k)$$

It is easy to see that, in the case $\langle f(n) \rangle_{n \geqslant k}$ is a quasimartingale in $L_1(B, E)$. Thus by the same arguments given in a proof of theorem 1. 1. in [12], *we* have.

$\lim_{m \to \infty} E(\|f_n(m) - g(n)\|) = 0$ for some martingale $\langle g(n) \rangle_{n \geqslant k}$ in $L_1(B, E)$.

Moreover

$$E(\|f(k) - g(k)\|) \leqslant \sum_{j \geqslant k} E(\|f_{(j)} - f_j(j+1)\|) \leqslant \varepsilon.$$

But since $\langle f_n(m) \rangle_{m \geqslant n} \subset S_{X(n)}(E(n))$ and $S_{X(n)}(E(n))$ are closed $(n \geqslant k)$, then $g(n) \in S_{X(n)}(E(n))$ $(n \geqslant k)$. Now put

$g(m) = E(g(k), E(m))$ $(m \leqslant k-1)$. Then it is clear that $\langle g(n) \rangle \in MS(\langle X(n) \rangle)$ and $E(\|f(k) - g(k)\|) \leqslant \varepsilon$. It completes a proof (3 $\rightarrow$ 4). Suppose conversely that (4) holds, and $k \in N$ is any but fixed. On the one hand by (4) we have

$$S_{X(k)}(E(k)) = cl\{g(k);\ \langle g(n) \rangle \in MS(\langle X(n) \rangle)\}$$
$$= cl\{g_k(k+1);\ \langle g(n) \rangle \in MS(\langle X(n) \rangle)\}$$
$$\subset cl\{E(g, E(k));\ g \in S_{X(k+1)}(E(k+1))\}$$

On the other hand, if $g \in S_{X(k+1)}(E(k+1))$, then by (4) there is a sequence

$\{\langle g^i(n) \rangle\}_{i=1}^{\infty}$ of $MS(\langle X(n) \rangle)$ *s. t.*

$$\lim_{i\to\infty} E(\| g^i(k+1) - g \|) = 0 \text{ hence}$$

$$\lim_{i\to\infty} E) \| g^i (k) - E(g, E(k) \|) = 0$$

But $\langle g^i(k) \rangle_{i=1}^{\infty} \subset S_{X(k)}(E(k))$ and $S_{X(k)}(E(k))$ is closed, then

$$E(g, E(k)) \in S_{X(k)}(E(k)).$$

It follows that $cl \{ E(g, E(k)) ; g \in S_{X(k+1)}(E(k+1)) \} \subset S_{X(k)}(E(k))$

It means that (3) holds. A proof of Theorem 2. 4 is thus complete.

In the connection with property 1.1 given in previous Section we obtain the following result:

COROLLARY 2. 5. Let $\langle X(n) \rangle$ be a martingale in $L_1(K_c, E)$, then there is a sequence $\{ \langle g^i(n) \rangle \}_{i=1}^{\infty}$ of $MS(\langle X(n) \rangle)$ s.t.

$$X(k) \overset{\| \quad \|}{\longleftrightarrow} \langle g^i(k) \rangle_{i=1}^{\infty} \quad (k \in N) \tag{2.2}$$

Proof. Let $\langle X(n) \rangle$ be a martingale in $L_1(K_c, E)$. Thus in particular in view of property 1.1 there is a sequence $\langle f^{k,i} \rangle$ s. t.

$$X(k) \overset{\| \quad \|}{\longleftrightarrow} \langle f^{k,i} \rangle_{i=1}^{\infty} \qquad \text{(w. r. t. } E(k)) \ (k \in N)$$

But since $\langle f^{k,i} \rangle_{i=1}^{\infty} \subset (S_{X(k)}(E(k))$, then by theorem 2.4 (4) there is a sequence

$$\{ \langle h^{k,i,j}(n) \rangle \}_{i=1}^{\infty} \text{ of } MS(\langle X(n) \rangle) \text{ s. t.}$$

$$\lim_{i\to\infty} E(\| h^{k,i,j}(k) - f^{k,i} \|) = 0 \qquad (k, i \in N)$$

hence
$$X(k) \overset{\| \quad \|}{\longleftrightarrow} \langle h^{k,i,j}(k) \rangle_{i,j=1}^{\infty} \qquad (k \in N)$$

Finally, if $\{ \langle g^i(n) \rangle \}_{i=1}^{\infty}$ denotes the sequence $\{ \langle h^{k,i,j}(n) \rangle \}_{k,i,j=1}^{\infty}$ then (2.2) is automatically satisfied. The proof of Corollary 2.5 is thus complete.

In the connection with the considerations of the RN—property in B — spaces one can suppose without loss of generality that E is separable (σ – generated). Therefore we give a following corollary which present a martingale in $L_1(K_c, E)$ in the case.

COROLLARY 2.6. Let $\langle X(n)\rangle$ be a sequence in $L_1(K_c, E)$ with a separable E. Then $\langle X(n)\rangle$ is a martingale iff there is a sequence $\left\{\langle g^i(n)\rangle\right\}_{i=1}^{\infty}$ of $MS(\langle X(n)\rangle)$ s. t.

$$X(k) \underset{L_1}{\longleftrightarrow} \langle g^i(k)\rangle_{i=1}^{\infty} \qquad (k \in N) \tag{2.3}$$

Proof. The necessarity can be established from Theorem 2.4 (4) and the fact that E is separable. The condition (2.3) implies condition 4 in theorem 2.4 thus we get the sufficiency A proof of Corollary 2.6 is complete.

3. *The Regularity of Multivalued Martingales.*

DEFINITION 3.1. Let $X \in L_1(K_c, E)$. Define $X(n) = E(X, E(n))$ $(n \in N)$. In view of Property 1.3 and Theorem 5,3 (3) in [8] such a sequence is a martingale in $L_1(K_c, E)$. We shall call it a regular martingale.

If $\langle g(n)\rangle \in MS(\langle X(n)\rangle)$ and $\langle g(n)\rangle$ is regular, then we write $\langle g(n)\rangle \in RMS(\langle X(n)\rangle)$

The following theorem gives us a necessary and sufficient condition for the regularity of multivalued martingales. Note that this result is independent of that given in Section 2.

THEOREM 3. 2. Let $\langle X(n)\rangle$ be a sequence in $L_1(K_c, E)$. *Then* $\langle X(n)\rangle$ *is a regular martingale iff the following conditions hold:*

(1) $\langle |X(n)|\rangle$ *is* L_1*—bounded*

$$(2)\; S_{X(k)}(E(k)) = cl\,\{g(k)\,;\,\langle g(n)\rangle \in RMS(\langle X(n)\rangle)\} \tag{3.1}$$

Proof. Suppose first that $\langle X(n)\rangle$ is a regular martingale in $L_1(K_c, E)$; *i.e.* $X(n) = E(X, E(n))$ for some $X \in L_1(K_c, E)$ (n∈N) then by proposition 4.1 in [7] condition (1) holds. Further, by definition 3.1 and 1.3 (1.1) we get (2). It is more interesting to show the converse implication. Suppose thus condition (1),(2) hold for some sequence $\langle X(n)\rangle$ in $L_1(K_c, E)$.

Define

$M = \{f \in L_1 (B, E) ; \langle E (f, E (n)\rangle \in RMS (\langle X (n)\rangle)\}$
Therefore M is convex closed and decomposable. Hence by (1) and (2) M is also bounded and non-empty Consequently, by Theorem 3.1) and Corollary 1.6 in /8/ $M = S_X (E)$ for some $X \in L_1 (K_c, E)$ Finally by (3.1) and definition of X and M we get for every $k \in N$

$$S_{X(k)} (E (k) = d \{E (f, E (k) ; f \in S_X (E) (= M)\}, \text{ hence}$$

by Theorem 5. 1 in /8/ or Definition 1. 3 in Section 1, $\langle X (n)\rangle$ must be regular, more precisely $X (k) = E (X, E (k)) (k \in N)$, where X has been contructed above. A proof of Theorem 3. 2 is complete.

In the connection with Corollary 2. 5 we have the following

COROLLARY 3. 3. Let $\langle X (n)\rangle$ be a regular martingale in $L_1 (K_c, E)$ i. e. $X (k) = E (X, E (k))$ for some $X \in L_1 (K_c, E) (k \in N)$ then there is a sequence $\langle g^1\rangle_{i=1}^{\infty}$ of $S_X (E)$ s. t.

$$X \xleftrightarrow{\|\cdot\|} \langle g^i \rangle_{i=1}^{\infty} \text{ and } X (k) \xleftrightarrow{\|\cdot\|} \langle E (g^i, E (k)\rangle_{i=1}^{\infty} (k \in N)$$

Proof. Let $\langle X (n)\rangle$ be a regular martingale in $L_1 (K_c, E)$ i. e. $X (k) = E (X, E (k))$ $(k \in N)$ for some $X \in L_1 (K_c, E)$

On the one hand, by Property 1. 1 we can choose some sequence $\langle p^i \rangle_{i=1}^{\infty}$ of $S_X (E)$ s. t.

$$X \xleftrightarrow{\|\cdot\|} \langle p^i \rangle_{i=1}^{\infty}$$

On the other hand, by Theorem 3. 2 (2) and the same technique used in 'he proof of Corollary 2. 5 we can contruct a sequence $\{\langle q^i (n)\rangle\}_{j=1}^{\infty}$ of $RMS (\langle X (n)\rangle)$ s. t.

$$X (k) \xleftrightarrow{\|\cdot\|} \langle q^j (k)\rangle_{j=1}^{\infty} \qquad (k \in N)$$

Now let $q^j (k) = E(q^j, E(k) (j, k \in N)$ for some sequence $\langle q^j \rangle_{j=1}^{\infty} (L_1 B, E)$. It is not hard to show that in the case $\langle q^i \rangle_{j=1}^{\infty} \subset S_X (E)$,

Thus, if $\langle g^i \rangle_{i=1}^{\infty} = \langle q^j \rangle_{j=1}^{\infty} \cup \langle p^k \rangle_{k=1}^{\infty}$ then the sequece $\langle g^i \rangle_{i=1}^{\infty}$ satisfies all conditions required in Corollary 3. 3.

COROLLARY 4. 3. Let $\langle X(n)\rangle$ be a sequence in $L_1(K_c, E)$ with a separable E. Then $\langle X(n)\rangle$ is a regular martingale iff there is a sequence $\{\langle g^i(n)\rangle\}_{i=1}^{\infty}$ of RMS $(\langle X(n)\rangle$ s. t.

(1) $\langle |g^i(n)|\rangle_{i,\, n\in N}$ is L_1 bounded

(2) $X(k) \xleftrightarrow[L_1]{} \langle g^i(k)\rangle_{i=1}^{\infty} \qquad (k \in N)$

Proof. The necessarity can be established from Theorem 3. 2 and the assumption that E is separable. Conversely conditions (1) and (2) in the Corollary 3. 4 imply conditions (1) and (2) in Theorem 3. 2. Thus the sufficiency is obtained.

The following results give us some sufficient conditions for the regularity of multivalued martingales.

COROLLARY 3.5: (see [4], p.954)

Let B be a separable B-space with the RN-property. Then for every uniformly integrable and L_1-bounded martingale $\langle X(n)\rangle$ in $L_1(K_c, E)$ there is a (unique) function $X \in L_1(K_c, E)$ s.t.

$$X(n) = E(X, E(n)) \qquad (n \in N)$$

In other words, every uniformly integrable and L_1-bounded martingale in $L_1(K_c, E)$ is regular.

Proof. It follows immediately from theorem 2.4 (4), in Section 2; Theorem 6 in [3] and Theorem 3.2. in this section. Note that by the limit projective method, A.Costé ([4], p.954) has also obtained this result. The author should like to express many thanks to Doctors N.X.Loc and N.D. Tien for this useful information. Although the author has proved and informated this result in one of seminars of Institute of Mathematics, Hanoi before knowing [4].

COROLLARY 3.6. (see [13], theor. 2)

Every martingale $\langle X(n)\rangle$ in $L_1(K_c, E)$ with the following properties

(1) $\langle |X(n)|\rangle$ is uniformly integrable and L_1-bounded.

(2) $\forall_{\alpha>0} \exists$ a convex compact subset C of B s.t.

$$\forall_{\beta>0} \exists_{n_0} \exists_{A_0 \in E(n_0)}\ P(A_0) > 1-\alpha \quad \forall_{n\geq n_0} \forall_{A \in E(n)}$$

$$\text{if } A \subset A_0 \text{ then } \int_A X(n)\, dp \subset p(A).\, C + \beta U$$

is regular

Proof. It follows immediately from Theorem 2.4 (4) in Section 2; Theorem 2 of Uhl. J.r. in [13] and Theorem 3.2 in this section. Note that, using the same method of A.Costé in [4] and Theorem 2 of Uhl J.r. in [13] one can also prove Corollary 3.6.

4. Invariability of Multivalued Martingales and the RN-property in B-spaces.

DEFINITION 4.1. A function $\tau : s \to N$ is called a bounded stopping time *w.r.t.* $< E(n) >$, if $\{\tau = n\} \in E(n)$ for all $n \in N$ and $\tau(s) \leqslant n$ a.e. for some $n \in N$. If this occurs we write $\tau \in T$. By $\eta \geqslant \sigma$ we mean $\eta(s) \geqslant \sigma(s)$, a.e. For every $\tau \in T$, we define $F(\tau) = \{A \in E ; A \cap \{\tau = n\} \in E(n) \ \ \forall_{n \in N}\}$

Then by ([11], p.19-21) $< E(\tau) >$ is an increasing generalized sequence of sub σ-fields of E with $\sigma\text{-}(VE(\tau)) = \sigma\text{-}(VE(n)) = E$

Similarly, for $\tau \in T$, define $X(\tau, s) = X(\tau(s), s)$ $(s \in s)$ then $X(\tau)$ is $F(\tau)$ — measurable. The following proposition gives us further informations about it.

PROPOSITION 4. 2. Let $\langle X(n) \rangle$ be a sequence in $\mu(K,E)$.

$$\text{Then } \forall_{k \in N} \quad X(k) \xleftrightarrow{\|\cdot\|} \langle \overset{i}{f}(k) \rangle_{i=1}^{\infty} (w.r.t. E(k)) \quad iff$$

$$\forall_{\eta \in T} \quad X(\eta) \xleftrightarrow{\|\cdot\|} \overset{i}{f}(\eta) \ >_{i=1}^{\infty} (w.r.t. E(\eta)) \ \textit{Hence if}$$

$\langle X(n) \rangle$ is adapted to $\langle E(n) \rangle$, then $\langle X(\tau) \rangle$ is adapted to $\langle E(\tau) \rangle$

Proof. It follows from definitions of $\langle E(\tau) \rangle$; $\langle X(\tau) \rangle$ and property **1.1.**

PROPOSITION **4.3.** *Let $\langle X(n) \rangle$ be a sequence in $L_1(K_c, E)$ Then the following conditions are equivalent*:

$$\underset{X(k)}{S}(E(k)) = cl\{g(k) ; \langle g(n) \rangle\rangle \in MS(\langle X(n) \rangle)\} \ (k \in N) \tag{4.1}$$

$$\underset{X(\eta)}{S}(E(\eta)) = cl\,\{g(\eta) : \langle g(n) \rangle \in MS\,(\langle X(n) \rangle)\}$$

$$= cl\{g(\eta) : \langle\{g(\tau)\rangle \in MS\,(\langle X(\tau) \rangle)\} \ (\tau \in T) \tag{4.2}$$

It follows that, is $\langle X(n) \rangle$ is a martingale then so is $X(\tau)$.

Proof. Let $\langle X(n) \rangle$ be a sequence in $L_1(K_c, E)$. Then by ([11], Prop. IV. 3 — 12)

$\langle g(\eta) \rangle \in MS(\langle X(n) \rangle$ iff $\langle g(\tau) \rangle \in MS\,(\langle X(\tau) \rangle)$. Thus (4.1) $\Rightarrow$ (4.2). Suppose now that $\langle X(n) \rangle$ is a martingale then by Theorem 2.4 (4) in Section 2 we get (4.1), hence (4.2). It remains to show that $\langle X(\tau) \rangle$ is a martingale, i, e,

$$X(\sigma) = E(X(\eta), E(\sigma)) \; (= X_\sigma(\eta)) \; (\eta \geqslant \sigma \in T) \qquad (4.3)$$

Let $\eta > \sigma \in T$ be any but fixed. By Theorem 5. 3 (2) in [8], (4.3) is equivalent to

$$\underset{X(\sigma)}{S}(E(\sigma)) = cl\{E(g, E(\sigma)) ; \; g \in \underset{X(\eta)}{S}(E(\eta))\} \qquad (4.3)$$

On the on hand, by (4.2) we have

$$\underset{X(\sigma)}{S}(E(\sigma)) = cl \; \{g(\sigma) ; \langle g(\tau) \rangle \in MS(\langle X(\tau) \rangle)\}$$

$$= cl \; \{ g_\sigma(\eta) ; \langle g(\tau) \in MS \; (\langle X(\tau) \rangle)\}$$

$$\subset (cl\{E(gE(\sigma)) ; \; g \in \underset{X(\eta)}{S}(E(\eta)\}$$

On the other hand, if $g \in \underset{X(\eta)}{S}(E(\eta))$ then by (4.2) there

is a sequence $\{\langle g^i(n)\rangle\}_{i=1}^{\infty}$ of $MS(\langle X(n)\rangle)$ s.t.

$$\lim_{i\to\infty} E(\| g^i (\eta) - g \|) = 0, \text{ hence}$$

$$\lim_{i\to\infty} E \; (\| g^i (\sigma) - E(g, E(\sigma)) \|) = 0$$

It follows that $E \; (g, E(\sigma)) \in \underset{X(\sigma)}{S}(F(\sigma))$, hence

$$cl \; \{ E(g, E(\sigma)) ; \; g \in \underset{X(\eta)}{S}(E(\eta))\} \subset \underset{X(\sigma)}{S}(E(\sigma))$$

Therefore (4.3)' holds, equivalently, $X(\tau)$ is a martingale in $L_l(K_c, E)$ The proof of Proposition 4.3 is complete.

By the same technique and using Theorem 3.2 in Section 3 we can prove the following proposition.

PROPOSITION 4.4. Let $\langle X(n)\rangle$ be a sequence in $L_1 \; (K_c, E)$ *then the following conditions are equivalent*:

$$\underset{X(k)}{S}(E(k)) = cl \; \{g(k), \langle g(n)\rangle \in RMS \; (\langle X(n)\rangle\} \; (k \in N) \qquad (4.4)$$

$$\underset{X(\eta)}{S}(E(\eta)) = cl\{g(\eta), \langle g(n)\rangle \in RMS \; (\langle X(n)\rangle)\}$$

$$= cl \; \{g(\eta) ; \langle g(\tau)\rangle \in RMS \; (\langle X(\tau)\rangle)\} \quad (\tau \in T) \qquad (4.5)$$

Hence $X(k) = E(X, E(k)), \forall k \in N$ *iff*

$$X(\eta) = E(X, E(\eta)) \; \forall \eta \in T$$

In other words $\langle X(n) \rangle$ *in a regular martingale iff* $\langle X(\tau) \rangle$ *is.*

THEOREM 4.5. (See /3/, theor.6)

Let B be a separable B-space. Then B has the RN-property iff for every uniformly integrable and L_1*-bounded martingale* $\langle X(n) \rangle$ *in* $L_1(K_c, E)$ *there is a (unique) function* $X \in L_1(K_c, E)$

s.t $$X(\tau) = E(X, E(\tau)) \qquad (\tau \in T) \qquad (4.6)$$

Proof. It follows from Corollary 3.5 and Proposition 4.4.

THEOREM 4.6. (See [13], Theor. 2)

Let $\langle X(n) \rangle$ be a martingale in $L_1(K_c E,)$ with properties (1) and (2), given in Corollary 3.6, then there is a (unique) function $X \in L_1(K_c, E)$ *s.t.*(4.6) holds.

Proof. It follows immediately from Corollary 3.6 and Proposition 4.4.

At the end we note that, if $\langle X(n) \rangle$ is a sequence in $L_1(K_c, E)$, and $F(n)$ is asmallest sub σ—field w.r.t. which every $X(k)$ is measurable for $k = 1, 2, \ldots, n$. Then $F = \sigma - (VF(n))$ is always separable. Moreover, if $\langle X(n) \rangle$ is a (regular) martingale *w.r.t.* $\langle E(n) \rangle$, then $\langle X(n) \rangle$ is also a (regular) martingale w.r.t. $\langle F(n) \rangle$. Thus, Corollary 2.6 and 3.4. can be applied to $\langle X(n) \rangle$ w.r.t. $\langle F(n) \rangle$.

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