ACTA MATHEMATICA VIETNAMICA
Tom 6, No 2 (1981)

MEASURABLE RELATIONS WITH CLOSED BALL VALUES IN BANACH SPACES

HÀ LÊ ANH and ĐINH QUANG LƯU
Institute of Mathematics
Hanoi

INTRODUCTION. The theory of multi-functions and vector-valued amarts has been developed and extentively studied in recent years by Kuratowski and Ryll-Nardzewski [8], Himmelbeg [7] Castaing and Valadier [3] Hiai and Umegaki [6] B.K. Đam and N.D.Tien [5], Bellow [2] D.Q.Luu ([9], [10]) among others. The main purpose of this paper is to consider the class of all multi-functions with closed ball values in a separable Banach space. It is shown that one can embed the class of all integrably bounded multi-functions with closed ball values into some Banach space. These results lead to consider various processes of multi-functions with closed ball values. In Section I we recall some necessary notations of multi-functions. Main results are proved in Section 2. In Section 3 we give some applications of main results to the study of some classes of amarts with closed ball values in Banach spaces.

I. Notations. Throughout the paper let B denote a Banach space with some norm denoted by $\|\cdot\|$ and $L_1(B, \Sigma)$ the Banach space of all B-valued Bochner integrable functions defined on some probability space (Ω, Σ, P). By K we mean the class of all closed bounded non-empty subsets of B with the usual Hausdorff's metric h (. , .). A function $F : \Omega \to K$ is called measurable if every set $\{\omega : F(\omega) \cap V \neq \phi\}$ is measurable, where V denotes an open subset of B. If this occurs then we write $F \in \mu(K, \Sigma)$ with

$$S_F(\Sigma_1) = \{f \in \mu(B, \Sigma_1) ; f(\omega) \in F(\omega), \text{ a. e}\}$$

where Σ_1 is a subσ-field of Σ. Such a function is called integrably bounded, if the real-valued function $\omega \mapsto h(F(\omega), \{o\})$ is integrable. If this occurs then we write $F \in L_1(F, \Sigma)$. In the case the integral of F over $A \in \Sigma$ is defined as follows

$$\int_A F dP = \{\int_A f dP;\ f \in S_F(\Sigma)\}$$

Moreover, if $F_1, F_2 \in L_1(K, \Sigma)$ then

$$H(F_1, F_2) = \int_\Omega h(F_1(\omega), F_2(\omega))\, dP$$

defines some complete metric in $L_1(K, \Sigma)$.

Remark If we denote

$K_c = \{A \in K,\ A \text{ is eonvex}\}$

$K_{cc} = \{A \in K,\ A \text{ is convex compact}\}$ and

$K_s = \{A \in K,\ A \text{ is a closed ball}\}$ then

in the case, where B is an infinite dimentional Banach space we have $K_{cc} \neq K_s$.

2. Measurability and Integrability of Relations With Closed Ball Values

Before giving main results of the paper we note that every closed ball of B can be written only in a unique way $x + rU$, where $x \in B$, $r > o$ and U denotes the unit ball of B. Thus every multifunction F with closed ball values can be written only in an essentially one way, i. e.

$$^*F(\omega) = f(\omega) + r(\omega)\, U, \text{ a. e.}$$

where $f: \Omega \to B$ and $r: \Omega \to [0, \infty)$. The natural question arises whether f and rare measurable if F is measurable? and integrable if F is measurable? We shall give some positive answers to these questions.

LEMMA 2.1. Let B be a Banach space (it not to be separable) and let $|A| = h(A, \{0\})$ for all $A \in K$. Then $(K, |\cdot|)$ can be regarded as a closed convex cone of $B \times R$ with the norm $\|(x \cdot r)\| = \|x\| + |r|$. More precisely, there is a linear (w.r.t. the positive scalars) one-to-one isometric embedding I from K into $B \times R$.

Proof. Let B be a Banach space. Define

$I(x + rU) = (x, r)$. It is easy to check that the operator I satisfies all the required mentioned above conditions

COROLLARY 2.2. A Banach space B has the R-N- Property iff K has.

THEOREM 2.3. *Let B be a separable Banach space and* $F: \Omega \to K_s$ *i.e.* $F(\omega) = f(\omega) + r(\omega)U$, *a. e. for some* $f: \Omega \to B$ *and* $r: \Omega \to [0, \infty[$. *Then F is a measurable iff both f and r are measurable.*

Proof. The sufficiency has been proved in ([3], theorem III. 41). The major part of the proof contains in showing the necessarity of condition. Indeed, let $F: \Omega \to K_s$, i. e. $F(\omega) = f(\omega) + r(\omega) U$, a. e. Suppose that F is measurable. Then in view of [3] there is a sequence (f_n) of $S_F(\Sigma)$ such that

$$F(\omega) = cl \{f_n(\omega).\ n \geqslant 1\}, \text{ a. e.}$$

Hence $r(\omega) - \frac{1}{2} \sup \left\{ \| f_n(\omega) - f_m(\omega) \| ;\ m, n \geqslant 1 \right\}$, a.e. It implies that $r(.)$ is measurable. Now let $x \in B$ then it is clear that $F(\omega) - x = (f(\omega) - x) + r(\omega) U$.

Therefore in view of lemma 2. I. we have

$$| F(\omega) - x | = \| f(\omega) - x \| + r(\omega) \qquad (\omega \in \Omega)$$

But since F is measurable then again by [3] the function $\omega \mapsto | F(\omega) - x |$ is also measurable for all $x \in B$. It follows thus the function $\omega \to \| f(\omega) - x \|$ is mearurable for all $x \in B$. In other words, $f(.)$ is itself measurable. It completes our proof of Theorem 2.3.

COROLLARY 2.4. A function $F: \Omega \to K_s$ is measurable iff there is a sequence of simple functions $F_n: \Omega \to K_s$ such that

$$\lim_{n\to\infty} h(F_n(\omega), F(\omega)) = 0, \text{ a. e.}$$

Moreover, by the same technique given in [11] we can choose the sequence (F_n) so that

$$\| f_n(\omega) \| \leqslant \| f(\omega) \| \text{ and } r_n(\omega) \leqslant r(\omega), \text{ a. e.}$$

for all n, where $F(\omega) = f(\omega) + r(\omega) U$ and

$$F_n(\omega) = f_n(\omega) + r_n(\omega) U, \text{ a.e.} \qquad (n \geqslant 1)$$

COROLLARY 2.5. Let $F(\omega) = f(\omega) + r(\omega)U$, a.e. then F is integrably bounded iff the both functions f and r are integrable.

THEOREM 2.6. *Let B be a separable Banach space. Then there is a linear (w. r. t. the positive bounded functions) one-to-one isometric embedding J from* $L_1(K_s, \Sigma)$ *into* $L_1(R, \Sigma) \times L_1(B, \Sigma)$. *Moreover,*

(1) $cl \int_\Omega F dP$ *is equal to the Bochner integral taken as a function in* $L_1(R, \Sigma) \times L_1(B, \Sigma)$.

(2) For every $F \in L_1(K_s, \Sigma)$ *and every subσ-field* $\Sigma_1 \subset \Sigma$ *we have* $E(F_1 \Sigma_1) \in L_1(K_s, \Sigma_1)$ *and the conditional expectation* $E(F, \Sigma_1)$ *defined in the sense of [6] is equal to the conditional expectation taken as a function in* $L_1(R, \Sigma) \times L(B, \Sigma)$.

Proof. It follows from Corollary 2.5. in Theorem 4 . I(2), and Theorem 53(I, 2) in [6].

APPLICATIONS.

Definition 3.1. A sequence (F_n) is said to be adapted to an increasing sequence (Σ_n) of subσ-fields of Σ if F_n is Σ_n-measurable for all n. Unless otherwise mentioned all our sequences are assumed adapted $(\Sigma_n.)$

Definition 3.2. A sequence (F_n) is said to be a martingale, quasi-martingale, uniform amart or a L_1-amart in $L_1(K_s, \Sigma)$ if the following conditions hold, resp.

$$F_n = E(F_{n+1}, \Sigma_n) \qquad (n \geqslant 1) \tag{3.1}$$

$$\sum_{n=1}^{\infty} H\,[E(F_{n+1}, \Sigma_n), F_n] < \omega \tag{3.2}$$

$$\forall \varepsilon > 0 \ \exists k \geqslant 1 \ \forall \sigma \geqslant \tau \geqslant k \ H[E(F_\sigma, \Sigma_\tau), F_\tau] \leqslant \varepsilon \tag{3.3}$$

$$\forall \varepsilon > 0 \ \exists k \geqslant 1 \ \forall m \geqslant n \geqslant k \ \ H[E(F_m, \Sigma_n), F_n] \leqslant \varepsilon \tag{3.4}$$

where σ, τ denote the bounded stopping times with the order defined by $\sigma(\omega) \geqslant \tau(\omega)$, a. e. iff $\sigma \geqslant \tau$.

Using the same technique given in [9] by the auther we can prove the following theorem

THEOREM 3.1. (see [2] and [9]).

(1) every quasi-martingale in $L_1(K_s, \Sigma)$ is a uniform amart

(2) A sequence (F_n) is a uniform amart is $L_1(K_s, \Sigma)$ iff there is a unique martingale (M_n) in $L_1(K_s, \Sigma)$ such that

$$\lim_{T} H(F_\tau, M_\tau) = 0$$

where T denotes the set of all bounded stopping times with the usual order.

(3) A sequence (F_n) is a $L_1 - amart$ in L_1 (K_s, Σ) iff there is a unique martingale(M_n) in $L_1(K_s, \Sigma)$ such that

$$\lim_{n \to \infty} H(F_n, M_n) = 0$$

Hence every uniform amart in $L_1(K_s, \Sigma)$ is a L_1 – amart.

Using main results given in previous section we get the following result.

THEOREM 3.2. *Let F_n, $F_n \in L_1(K_s, \Sigma)$ with $F_n = f_n + r_n U$ and $F = f + rU$ for some f_n; $f \in L_1$ (B, Σ) and r_n, $r \in L_1(R, \Sigma)$ with $r_n \geqslant 0$, $r \geqslant 0$ for all n and almost $\omega \in \Omega$ then*

(1) $\lim_n h(F_n(\omega), F(\omega)) = 0$. a.e.iff

$\lim_n \| f_n(\omega) - f(\omega) \| - 0$ and $\lim_n \| r_n(\omega) - r(\omega) | = 0$, a.e.

(2) $\lim_n H(F_n, F) = 0$ iff

$\lim_n E(\| f_n - f \|) = 0$ and $\lim_n F(| r_n - r |) = 0$.

(3) The sequence (F_n) is a martingale (quasi-martingale, uniform amart, L_1—amart) in $L_1(K_s, \Sigma)$ iff the sesuences (f_n) and (r_n) are martingales (quasi-martingales, uniform amarts, L_1-amarts) in $L_1(B, \Sigma)$ and $L_1(A, \Sigma)$, resp.

Note that it has been shown in [9] that every and only L_1-amart in $L_1(B, \Sigma)$ has a Riesz decomposition in $L_1(B, \Sigma)$ In general, even for quasi-martingales with convex compact values in a Banach space we have no chance to expect a Riesz decomposition. But for L_1-amarts in $L_1(K_s, \Sigma)$ we have the following statement.

THEOREM 3.3. *A L_1-amart (F_n) in $L_1(K_s, \Sigma)$ with $F_n = f_n + r_n U$ $(n \geqslant 1)$ has a Riesz decomposition iff the L_1-amart (r_n) has a Riesz decomposition wiih a positive potential.*

Proof ($\Rightarrow$) Let (F_n) be a L_1-amart in $L_1(K_s, \Sigma)$ with $F_n = f_n + r_n U$ $(n \geqslant 1)$. Suppose that

$$F_n = M_n + P_n \quad (n \geqslant 1)$$

Where (M_n) is a martingale and (P_n) is a L_1—potential. i.e.

$$\lim_n E(\| P_n \|) = 0 \tag{3.5}$$

Since M_n and P_n belong to $L_1(K_s, \Sigma)$ then $M_n = g_n + \alpha_n U$ and $P_n = p_n + \beta_n U$. a.e. and for all n. Hence

$$\begin{aligned} F_n &= (g_n + \alpha_n U) + (p_n + \beta_n U)(n > 1) \\ &= (g_n + p_n) + (\alpha_n + \beta_n) U \\ &= f_n + r_n U \text{, a.e. } (n \geqslant 1). \end{aligned}$$

Therefore by Throrem 3,2 (3) and (3.5) the sequence (r_n) has a Riesz decomposition $r_n = \alpha_n + \beta_n$ $(n \geqslant 1)$ with the positive potential (β_n)

Proof of ($\Leftarrow$). Suppose conversely that (r_n) has a Riesz decomposition $r_n = (r_n - \gamma_n) + \gamma_n \geqslant 0$. Since $r_n \geqslant 0$ $(n \geqslant 1)$ then by construction of the Riesz decomposition we have $(r_n - \gamma_n) \geqslant 0$, a.e. $(n \geqslant 1)$. Hence, if (f_n) has a Riesz decomposition

$$f_n = k_n + h_n \ (n \geqslant 1) \text{ then}$$

$$F_n = f_n + r_n U = (k_n + h_n) + [(r_n - \gamma_n) + \gamma_n]U$$

$$= [k_n + (r_n - \gamma_n)U] + (h_n + \gamma_n U)$$

Thus, if we put $M_n = k_n + (r_n - \gamma_n)U$ and $p_n = h_n + \gamma_n U$ $(n \geqslant 1)$ then(F_n) has a Riesz decomposition

$$F_n = M_n + P_n \ (n \geqslant 1).$$

Where(M_n) is a martingale and (P_n) is a L_1 — potential which proves Theorem 3.3.

COROLLARY 3. 4. (see [4], [1] and [10]).

Let B be a separable Banach space then the following conditions are equivalent

(I) B has the RN Property w.r.t. (Ω, Σ, P)

(2) Every martingale F_n in L_1 (K_s, Σ) which is L_1 — bounded is convergence almost surely w.r.t. the metric h (...).

(3) Every L_1-amarti n L_1 (K_s, Σ) which is uniformly integrable and L_1 — bounded is convergent in L_1 w. r. t. the metric $H(.,.)$

Proof. It follows from Theorem 3. 2(3), Theorem 2.2. in [0] and the well-known result of Chatterji in [4].

COROLLARY 3. 5. see [12], also [9].

A L_1 -amart in L_1 (K_s, Σ) is convergent in L_1 — norm iff the following conditions hold

(1) F_n is uniformly integrable and L_1 — bounded.

(2) $\forall\, \alpha > 1$ $\exists$ a convex compact subset C of B such that
$\forall\, \beta < 0$ $\exists n_o \exists A_o \in \Sigma n_o$ $P(A_o) > s - \alpha$ $\forall\, n > n_o$ $\forall A \in \Sigma_n$ if $A \subset A_o$ then $\int_A F_n dP \subset P(A)\, C + \beta U$ Note that (2) and (3) in Corollary 3.4. fail to be true even for martingales with convex compact values in Hilbert spaces (see [6]).

Finally the authors wish to express their thanks to Dr. N.D. Tien, and Mr. B.K. Dam and Mr. N.X. Tan for some useful discussions.

Received April 15, 1981

REFERENCES

[1] A. Costé *Sur les martingales multivoques uniformément intégrables*, C. R. Acad. Sci. Paris sér. A 290 (1980).

[2] A. Bellow, Uniform Amarts : *A class of asymptotic martingales for which strong almost sure convergence obtains.*

Z. Warsch. Verw. Gebiete 38 (1978) 278 — 290.

[3] C. Castaing, M. Valadier. *Convex An analysis and Measurable Multi-functions. Lectures Notes in Math.* 580 (1977).

[4] Chatterji S.D. *Martingale conuergence and the R-N-Theorems in B-spaces. Math. Scand.* 22 (1968) 21 — 41.

[5] B.K. Dam, N. D. Tien *On the multi-valued asymptotic martingales Act. Math. Viet* no 1 (1981).

[6] F. Hiai, H, Umegaki. *Integrals, Conditional Expectations and Martingales of Multi-functions.* J, Multi. Anal. 7 (1977) 149 — 182

[7] Himmelberg C. J. *Measurable Relations.*

Fund. Math. LXXXVII (1975) 54 — 72.

[8] Kuratowski, Ryll-Nardzewski C. *General theorem on selectors Bull. Polon Acad. Sci. Sér. Sci. Math. Astronom. Phys.* 13 (1965) 317 — 322.

[9] D. Q. Luu *On the class of all processes having a Riesz decomposition.* Acta Math. Viet. no 1 (1981).

[10] Đ. Q. Lưu *Representations and regularity of multi-valued martingales.* Act. Math. Viet, no 2 (1982) to appear.

[11] Neveu J. *Martingales à temps discrete.*

Paris Masson (1972).

[12] J. J. Uhl Jr. *Applications of R-N-theorems to martingale convergence.* **Trans. Amer. Math. Soc. vol. 145 (1969).**