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ON THE CLASS OF ALL PROCESS HAVING A RIESZ DECOMPOSITION

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INTRODUCTION The theory of Asymptotic Martingales (Amarts) has been developed and extentively studied in recent years by Bellow [I], Edgar and Sucheston [3], Chacon and Sucheston [2] among others. It was shown that every real, valued amart and every vector-valued uniform amart has a Riesz decomposition. Thus one problem seems to be remained is that of characterizing all processes having a Riesz decomposition. The purpose of the present is to solve this problem. Our main results are given in the section. In the section 2 we prove some convergence theorems and contruct some related examples.

I. DISCRETE PARAMETER PROCESSES HAVING A RIESZ DECOMPOSITION

Throughout the paper let B be a Banach space with some norm denoted by $\| . \|$ and L_1 (B,F) the Banach space of all B-valued Bochner integrable functions defined on some probability space (S, E, P). We shall assume that we are given an increasing sequence $F(n)$ of sub-fields of F. A process $X(n)$ is said to be adapted to F (n) if $X(n)$ in F (n)-measurable for all positive integers n. All our processes are assumed to be adapted to F (n) Unless otherwise specified all random variables will be assumed taken from L_1 $(B$, F$)$.

DEFINITION 1.1. A process $X(n)$ is said to have a Riesz decomposifion,

if $X(n) = M(n) + P(n)$ $(n \geqslant 1)$ (1. 1)

where $M(n)$ is a martingaleandP (n) a L_1-potential, i. e.

$$\lim_{n} E(\| P(n) \|) = 0 \quad (1. 2$$

By the same argument as that given in [5] the Riesz decomposition is always essentially unique.

PROPOSITION 1.1 *Every uniform amart* ([5]), *hence every quasi-martingale in* L_1 (B,F) *has a Riesz decomposition.*

PROPOSITION 1.2 *A process* $X(n)$ *is convergent in* L_1 *iff it has a Riesz decomposition (1.1) with (1.2), where* $M(n)$ *is a regular martingale.*

THEOREM 1. 1 *A process* $X(n)$ *has a Riesz decomposition iff the following condition (RD) holds*

$$(RD)\ \forall \varepsilon>0\ \exists k\geqslant 1\ \forall m\geqslant m\geqslant k\ E(\|X_n(m)-X(n)\|)\leqslant\varepsilon$$

Where $X_n(m)=E(X(m),\ (F(n))\ (m\geqslant n\geqslant 1)$

Proof. $(\Rightarrow)$ *Let* $X(n)$ be a process having a Riesz decomposition (1.1) with (1.2). Then in particular, we have

$$\forall \varepsilon>0\ \exists k\geqslant 1\ \forall m\geqslant k E(\|'P(m)\|)\leqslant\frac{\varepsilon}{2}$$

Hence $\forall \varepsilon>0\ \exists k\geqslant 1\ \forall m\geqslant n\geqslant k$, we get

$$E(\|X_n(m)-X(n)\|)$$
$$\leqslant E\|X(m)-M(m)\|)+E(\|X(n)-X(n)\|)$$
$$\leqslant(E\|P(m)\|)+E(\|P(n)\|)\leqslant\varepsilon$$

It proves condition (RD).

Proof of $(\Leftarrow)$ Suppose conversely that a process $X(n)$ satisfies condition (RD). Thus, in particular, we have

$$\forall j\geqslant 1\ \ \forall\varepsilon>0\ \exists k\geqslant j\ \forall m\geqslant k\ E(\|X_k(m)-X(k)\|)\leqslant\varepsilon$$

Therefore $\{X_j(n)\}_{n=j}^{\infty}$ is a Cauchy sequence in $L_1(B, F, (j))$ for all $j\geqslant 1$. Consequently,

$$\lim_{n\to\infty} E(\|X_j(n)-M(j)\|)=0$$

for some sequence $M(j)$. It is not hard to show that in the case the sequence $M(j)$ is a martingale. Now put

$$P(j)=X(j)-M(j)\qquad\qquad (j\geqslant 1)$$

We claim that $P(j)$ is a L_1-potential. Indeed, fix a positive number ε, by condition (RD) we have

$$\exists k\geqslant 1\ \forall m\geqslant n\geqslant k\quad E(\|X_n(m)-X(n)\|)\leqslant\frac{\varepsilon}{2}$$

On the one hand, since $\lim_{n \to \infty} E(\| X_j(n) - M(j) \|) = 0 \; (j \geqslant 1)$ then, in particular, we get

$$\forall_{j \geqslant k} \exists_{m_j} \; E(\| X(j) + m_j) - M(j) \|) \leqslant \frac{\varepsilon}{2}$$

On the other hand, since

$$E(\| P(j) \|) = E(\| X(j) - M(j) \|)$$
$$\leqslant E(\| X(j) - X_j(j) + m_j) \|) + E(\| X_j(j + m_j) - M(j) \|)$$

then finally we have $E(\| P(j) \|) \leqslant \varepsilon \; (j \geqslant k)$
In prove (1.2). The proof of theorem 1.1 is complete

DEFINITION 1.2. Let N denote the set of all positive integers. For every pair $m > n \geqslant 1$ and every subsequence $(\alpha_n) (N$ we denote by $\alpha[m, n)$ the number of elements of (α_n) contained in the interval $[m, n]$. In the following theorem if we have two sequences (α_n) and (β_n) of N then by (γ_n) we mean the superimposed sequence $(\alpha_n$ with (β_n).

THEOREM 1.2. *Let $X(n)$ be a process in $L_1(B, F)$ then the following conditions are equivalent*

(1) *$X(n)$ has a Riesz decomposition*

(2) $\forall_{\alpha > 1} \exists_{\{\alpha_k\} \subset N \alpha_k \uparrow \infty} \forall_{\{\beta_k\} \subset N \beta_k \uparrow \infty}$, if $\sum_{k=1}^{\infty} \frac{\beta[\alpha_k, \alpha_{k+1})}{\alpha^k} < \infty$

then the process $X(\gamma_k)$ is a quasi-martingale.

(3) $\exists_{\{\alpha_k\} \subset N \alpha_k \uparrow \infty} \forall_{\{\beta_k\} \subset N}$ if $\beta[\alpha_k, \alpha_{k+1}) = 1 \quad (k \geqslant 1)$

then the process $X(\gamma_k)$ is a quasi-martingale.

(4) $\exists_{\{\alpha_k\} \subset N \alpha_k \uparrow \infty} \forall_{\{\beta_k\} \subset N \beta_k \uparrow \infty}$, if $\beta[\alpha_k, \alpha_{k+1}) = 1, \quad (k \geqslant 1)$

then the process $(X \gamma_k)$ is a uniform amart.

Proof of $(1 \to 2)$. Let $X(n)$ be a process having a Riesz decomposition (1.1) with (1.2). Then, in particular, we have

$$\forall_{\alpha > 1} \exists_{\{\alpha_k\} \subset N \alpha_k \uparrow \infty} \forall_{n \geqslant \alpha_k} E(\| P(n) \|) \leqslant \frac{1}{\alpha^k} \; (1.3).$$

Suppose now that $\{\beta_1 < \beta_2 < \beta_3, < \ldots\}$ is any but fixed subsequence of

N with $\lim\limits_{k\to\infty}\beta_k = \infty$ and $\sum\limits_{k=1}^{\infty} \frac{\beta[\alpha_k, \alpha_{k+1})}{\alpha_k} < \infty$ (1.4)

Estime

$$\sum_{k=1}^{\infty} E(\| X_{\gamma_k}(\gamma_{k+1}) - X(\gamma_k) \|) = \sum_{k=1}^{\infty} \sum_{\gamma_j \in [\alpha_k, \alpha_{k+1})} E(\| X_{\gamma_j}(\gamma_{j+1}) -$$

$$- X(\gamma_j) \|) \leqslant \sum_{k=1}^{\infty} \sum_{\gamma_j \in [\alpha_k, \alpha_{k+1})} [E(\| X(\gamma_{j+1}) - M(\gamma_{j+1}) \|) + E(\|)((\gamma_j) -$$

$$- M(\gamma_j) \|)] \leqslant \sum_{k=1}^{\infty} \frac{Z\ [\beta[\alpha_k, \alpha_{k+1}) + 1]}{\alpha^k}$$

Thus since $\alpha > 1$ then by (1.3) and (1.4) we have (2).

Because the implications $(2 \to 3 \to 4)$ are clear then it remains to show that $(4 \to 1)$. Indeed.

Let $X(n)$ be a process satisfying condition (4) then, in particular in view of result of Bellow [1] we get

$$X(\alpha_k) = M(\alpha_k) + P(\alpha_k) \qquad (k \geqslant 1) \tag{1.5}$$

where $M(X_{\alpha_k})$ is a martingale and $P(\alpha_k)$ is a L_1—potential Now define

$$M(n) = E\,(M\,(\alpha_k), F(n)) \text{ for all } \alpha_{k-1} < n \leqslant \alpha_k$$

with $\alpha_0 = 0$ and $P(n) = X(n) - M(n) \qquad (n \geqslant 1)$

We claim that the process $M(n)$ and $P(n)$ satisfy condition (1.1) with (1.2). Since (1.1) is clear then we have to show only (1.2). Indeed, suppose that (1.2) does not hold, hence we can choose some positive number ε and a subsequence

$$\{\beta_k\} \text{ of } N \text{ with } \beta[\alpha_k, \alpha_{k+1}) = 1 \text{ for all } k \text{ and}$$

$$\lim_{k\to\infty} E\,(\| P\,(\beta_k) \|) \geqslant \varepsilon \tag{1.6}$$

But again, by (4) the process $X(\gamma_k)$ is a uniform amart therefore in view of [1] we get

$$X(\gamma_k) = M'(\gamma_k) + P'(\gamma_k) \qquad (k \geqslant 1) \tag{1.7}$$

Where $M'(\gamma_k)$ is a martingale and $P'(\gamma_k)$ is a uniform potential hence a L_1 —potential. Again define

$$M'(n) = E\,(M'(\gamma_k'), F(n)) \text{ for all } \| \gamma_{k-1} < n \leqslant \gamma_k \text{ with } \gamma_0 = 0$$

and $$P'(n) = X(n) - M'(n) \qquad (n \geqslant 1)$$

On the one hand, since $\{\alpha_k\} \subset \{\gamma_k$ then by (1.7) and (1.5).

we get $M(\alpha_k) = M'(\alpha_k) \qquad (k \geqslant 1)$, hence

$M(n) = M'(n) \qquad (n \geqslant 1)$. It implies that

$P(n) = P'(n) \qquad (n \geqslant 1)$.

On the other hand, since $\{\beta_k\} \subset \{\gamma_k\}$ then

$$\lim_{k\to\infty} E\,(\| P(\beta_k) \|) = \lim_{k\to\infty} E\,(\| P'(\beta_k) \|) = 0$$

which contradicts (1.6). The proof of Theorem 1. 2 is thus complete.

2. THE CONVERGENCE THEOREMS AND SOME EXAMPLES

The following convergence theorems for processes having a Riesz decomposition can be easily established from and at the same time can be regarded as some extensions of results in [6] and [3] given by Uhl and Chatterji, resp.

THEOREM 2. 1. *A process $X(n)$ having a Riesz decomposition is convergent in L_1 iff the following conditions hold*

(1) *$X(n)$ is uniformly integrable and L_1 — bounded*

(2) *For every positive number ε there is a convex compact subset K of B such that*

$$\forall_{\partial > 0}\ \exists_{n_0}\ \exists_{A_0 \in F(n_0)}\ P(A_0) > 1 - \varepsilon \ \forall_{n > 0}\ \forall_{A \in F(n)}$$

if $A \subset A_0$ then $\int_A X(n)\, dP \in P(A)\, K + \partial U$

where U denotes the unit ball of B.

THEOREM 2.2. *A Banach space B has the RN property w. r. t. (S, F, P) iff every uniformly integrable and L_1—bounded process which has a Riesz decomposition is convergent in L_1'.*

Note that is was shown in [1] that every quasi-martingale is a uniform amart. The following example shows that the converse implication is not true.

Example 1. There is a uniform amart in L_1 (l_2, [0,1]) which fails to be a quasi-martingale.

Construction. Let P be a Lebesgue measure on Borel sets of $[0,1]$ and (e_n) the usual basis for l_2. Define

$$A_k^j = \left[2^{-k}(j-1), 2^{-k}.j\right] \qquad \text{for } k \geqslant 1,\ 1 \leqslant j \leqslant 2^k$$

and $$X(k, t) = \frac{1}{k} \sum_{j=1}^{2^k} 1|_{A_k^j}(t)\ e_{n_{k-1}+j} \quad (k \geqslant 1);\ t \in [0,1]$$

where $n_k = \sum_{j=1}^{k} 2^j$ with $n_0 = 0$ and $1|_A(.)$ denotes the characteristic function of a Borel set A.

Note that, if $k \neq k'$ then for all $t \in [0,1]$ we have $X(k, t) \perp X(k', t)$, *hence*

$$\| X(k, t) - X(k', t) \| \geqslant \| X(k', t) \| \qquad (t \in [0,1])$$

It follows that

$$\sum_{k=1}^{\infty} E(\| X_k(k+1) - X(k) \|) \geqslant \sum_{k=1}^{\infty} E(\| X(k) \|) \geqslant$$

$$\geqslant \sum_{k \geqslant 2} \frac{1}{k} = \infty$$

Therefore $X(n)$ is not a quasi-martingale. But since $E(\| X(\tau) \|) \leqslant \frac{1}{k}$ for all **bounded** stopping times $\tau \geqslant k$ then $X(n)$ is a uniform amart (see [1]).

Example 2. There is a real, valued process which has a Riesz decomposition but it is not a uniform amart.

Construction. Let P and A_n^k be defined as in preceeding example. Put

$$Y_n^k(f) = \frac{1}{4n} \sum_{i=1}^{2^n} \alpha^i 1|_{A_n^i}(f) \qquad (n \geqslant 1),\ 1 \leqslant k \leqslant 2^n$$

where $\alpha_i = i$ for $i \neq k$ and $\alpha_k = 4^n$. Define moreover $Y_n^k \geqslant Y_{n'}^{k'}$ iff either $n > n'$ or $n = n'$ and $k > k'$ and let the resulting sequence renumbered with the order be denoted by $X(n)$. It is not hard to check that $X(n)$ is a L^1-potential but $\lim E(X(\tau))$ does not exist hence, by definition, it fails to be an amart where T denotes the set of all bounded stopping times with the usual order.

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