ACTA MATHEMATICA VIETNAMICA
Tom 6 N° 1

ISOPERIMETRIC INEQUALITIES FOR MULTIVARIFOLDS

ĐÀO TRỌNG THI
Hanoi University

. INTRODUCTION

In the paper [1], we have developped the theory of multivarifolds to formulate and solve the classical multidimensional Plateau problem (i. e. the Plateau problem in a homotopy class of multidimensional films). The present paper is to announce a further application of the multivarifold theory to another important question: the isoperimetric inequalities for parametrized films. Isoperimetric inequalities in various formulations were obtaine by Federer and Fleming [2], Almgren [3], Reifenberg [4] and Fomenko [5]. We emphasize that these inequalities are a key to solve the isoperimetric problems. The details of the paper will appear elsewhere.

First we recall some basic notation of the multivarifold theory which can be found in [1]. Let $\mathcal{M}$ be a Riemannian manifold and let $\mathcal{V}_k(\mathcal{M})$ be the linear space of all multivarifolds of order k on $\mathcal{M}$. For any multivarifold $V \in \mathcal{V}_k(\mathcal{M})$ let us denote its support by spt V, and its multimass and homogenous components – by $\{M_0(V), M_1(V), \ldots, M_k(V)\}$ and $\{V^{(0)}, V^{(1)}, \ldots, V^{(k)}\}$ respectively. To every multivarifold $V \in \mathcal{V}_k(\mathcal{M})$ corresponds a Radon measure $\|V\|$ on $\mathcal{M}$, which in its turn can be expressed in a canonical way as the difference of two positive Radon measures $\|V\|^+$ and $\|V\|^-$ ($\|V\| = \|V\|^+ - \|V\|^-$). Further, we denote by $\widetilde{\mathcal{R}}_k(\mathcal{M})$ and $\mathcal{R}_k(\mathcal{M})$ the subspace of the semi-rectifiable multivarifolds and the subset of the rectifiable multivarifold sin $\mathcal{V}_k(\mathcal{M})$ respectively. Consider now a continously differentiable mapping f from a Riemannian manifold $\mathcal{H}$ into a Riemannian manifold $\mathcal{M}$ and denote by $T(f): \mathcal{V}_k(\mathcal{M}) \to \mathcal{V}_k(\mathcal{M})$ the induced by f mapping. If f is only a locally Lipschitzian mapping then there does not exists the induced mapping $T(f)$ in general, but it is still defined for semi-rectifiable multivarifolds.

Finally, among the multivarifolds on $\mathcal{M}$ we have the set $\mathcal{S}\mathcal{C}_k^r(\mathcal{M})$ of the T_r-real multivarifolds and the set $J\mathcal{C}_k^r(\mathcal{M})$ of the T_r -integral multivarifolds. Such multivarifold V has a well defined boundary, wich will be denoted by $\partial_T V$. We remind that these multivarifolds have many valuabe geometric applications. Let us denote by $\rho(x, A)$ the distanee from a point $x \in \mathcal{M}$ to a subset A of $\mathcal{M}$. The main result of this paper is following :

MAIN THEOREM (the isoperimetric inequality). *Let $A \subset \mathbb{R}^n$ be a contractible in itself subset, $C \subset A$ be a compact subset of $\mathbb{R}^n$. U be neighbourhood of A in $\mathbb{R}^n$ and $\alpha : U \to A$ be a retraction, satisfying the Lipschitz's condition with the coefficient ξ on the subset $\{x\ \rho(x, C) \leqslant a\}$, where a is some positive number.*

If $V \in J\mathcal{C}_k^0(\mathbb{R}^n)$, $sptV \subset C$, $\partial_T V = 0$, $2n^{2k} C_k^n M_k(V) \leqslant a^k$ and $V = T(h_0)V_0$ where $V_0 \in \mathcal{R}_k(\mathcal{H})$, $\mathcal{H}$ is a Riemannian manifold, which can be the boundary of some Riemannian manifold and $h_0 : \mathcal{H} \to \mathbb{R}^n$ is a Lipschitzian mapping, then there exists $W \in J\mathcal{C}_{k+1}^0(\mathbb{R}^n)$, such that $SptW \subset A$, $\partial_T W = V$ and

$$\left[M_{k+1}(W)\right]^{k/k+1} \leqslant 2n^{\frac{k(+2)}{k+1}} C_k^n \xi^k M_k(V)$$

REMARK. In the case if the subset A is convex, we can choose a arbitrarily great and $\xi = 1$.

2. SINGULARITIES OF INDUCED MAPPINGS

Because the deformations of a space $\mathbb{R}^n$ onto skeletons of its standard cubical cell decompositions, which we shall use to prove the main theorem, have singularities, we begin to consider the influence of the singularities of a mapping f upon the induced mapping $T(f)$. More exactly, we shall prove that under certain conditions the induced mapping $T(f)$ acts even then the mapping f has singularities in the supports of multivarifolds.

Let $u = \mathbb{R}^n \to \mathbb{R}$ be a nonnegative continuous function. We put:

$$O_u = \left\{x \in \mathbb{R}^n \mid u(x) > 0\right\}$$

As in [2] we say that a continuously differentiable mapping $f: O_u \to \mathbb{R}^m$ is u-admissible if it carries bounded sets into bounded sets, and $|Df(x)| < u(x)^{-1}$ for almost everywhere in O_u with respect to Lebesgue measure.

Now we have:

THEOREM 1. *Let* $u: \mathbb{R}^n \to \mathbb{R}$ *be a nonnegative continuous function,* f *and* g *be two u-admissible mappings from* O_u *into* $\mathbb{R}^m$, *and let* $h: O_u \times I \to \mathbb{R}^m$ *be the linear homotopy between* f *and* g. *If a multivarifold* $V \in \mathcal{V}_k(\mathbb{R}^n)$ *satisfies the conditions*

$$(\|V\|^+ + \|V\|^-)(\mathbb{R}^n \setminus O_u) = 0 \tag{1}$$

$$(\|V\|^+ + \|V\|^-)(u^{-j}) < \infty, \; 0 \leqslant j \leqslant k, \tag{2}$$

then we have the following statements:

1) *There exist the limits (in the weak topology)*

$$T(f)_u V = \lim_{\lambda \to +0} T(f)(V \sqcap \Lambda_u)$$

$$T(h)_u (I \times V) = \lim_{\lambda \to +0} T(h)(I \times [V \sqcap \Lambda_u])$$

where $\Lambda_u = \{x \in \mathbb{R}^n | \, u(x) > \lambda, \lambda \in \mathbb{R}$ *and* $I = [0,1] \subset \mathbb{R}$

2) *For each* i, $0 \leqslant i \leqslant k$, *there are the following estimations*

$$M_i[T(f)_u V] \leqslant \sum_{i \leqslant j \leqslant k} (\|V^{(j)}\|^+ + \|V^{(j)}\|^-)(u^{-j}), \tag{3}$$

$$M_{i+1}[T(h)_u (I \times V)] \leqslant \sum_{i \leqslant j \leqslant k} (\|V^{(j+1)}\|^+ + \|V^{(j+1)}\|^-)(u^{-j-1}) +$$

$$+ \sum_{i \leqslant j \leqslant k} (\|V^{(j)}\|^+ + \|V^{(j)}\|^-)(|f - g| u^{-j}) \tag{4}$$

$$M_0[T(h)_u (I \times V)] \leqslant \sum_{0 \leqslant j \leqslant k} (\|V^{(j)}\|^+ + \|V^{(j)}\|^-)(u^{-j}) \tag{5}$$

3) *In addition, if* V *is a semi-rectifiable multivarifold, then the multivarifolds* $T(f)_u V$ *and* $T(h)_u (I \times V)$ *are semi-rectifiable, too.*

THEOREM 2. *Let* u *and* v *be nonnegative continuous functions on* $\mathbb{R}^n$. *Suppose that* f *and* g *are u-admissible and v-admissible at the same time. Then if the multivarifold* $V \in \mathcal{V}_k(\mathbb{R}^n)$ *satisfies the conditions* (1) *and* (2) *for both* u *and* v, *we have*

$$T(f)_u V = T(f)_v V$$

$$T(h)_u (I \times V) = T(h)_v (I \times V).$$

Here h *is the linear homotopy from* f *to* g.

Accordingly, $T(f)_u V$ and $T(h)_u (I \times V)$ do not depend on the function u, so we shall denote them by $T(f) V$ and $T(h)(I \times V)$ respectively.

3. DEFORMATIONS

We describe now the deformations of $\mathbf{R}^n$ onto skeletons of its standard cubical cell decompositions and study their singularities (for the details, see [2] and [3]).

Let $\mathbf{Z}^n \subset \mathbf{R}^n$ be the latice of the integral points in $\mathbf{R}^n$, and let $\mathbf{Z}^n_j \subset \mathbf{R}^n$ be the subset of all points, having j even and $n-j$ odd coordinates. To each point $\xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbf{Z}^n_j$ correspond the j — dimensional cube

$\xi' = \{ x = (x_1, x_2, \ldots, x_n) \in \mathbf{R}^n \mid |x_i - \xi_i| < 1$ if ξ_i is even and $x_i = \xi_i$ if ξ_i is odd$\}$.

and the n-j-dimensional cube

$\xi'' = \{ x = (x_1, x_2, \ldots, x_n) \in \mathbf{R}^n \mid |x_i - \xi_i| > 1$ if ξ_i is odd and $x_i = \xi_i$ if ξ_i is even$\}$.

Then,

$$C' = \bigcup_{0 \leqslant j \leqslant n} \bigcup_{\xi \in \mathbf{Z}^n_j} \xi' \quad \text{and} \quad C'' = \bigcup_{0 \leqslant j \leqslant n} \bigcup_{\xi \in \mathbf{Z}^n_j} \xi''$$

are the dual cubical cell complexes with the k-dimensional skeletons

$$C'_k = \bigcup_{0 \leqslant j \leqslant k} \bigcup_{\xi \in \mathbf{Z}^n_j} \xi' \quad \text{and} \quad C''_k = \bigcup_{n-k \leqslant j \leqslant n} \bigcup_{\xi \in \mathbf{Z}^n_j} \xi''$$

Consider the mappings

$$P_k = C'_{k+1} \setminus C''_{n-k-1} \to C'_k \; (k = 0, \; 1, \ldots, n-1)$$

defined as follows: the restriction of P_k on C'_k is identity mapping and the restriction of P_k on each $k+1$ - dimensional cell (except the center) is a central projection of this cell onto its boundary.

Suppose that $s : [-1, 1] \to [-1, 1]$ is a monotone increasing odd smooth function such that $s'(0) = s(1) = 1$ and all derivaties $s^{(i)}(1) = 0$ $(1 \leqslant i < +\infty)$. We define then the mapping $\vec{s} : \mathbf{R}^n \to \mathbf{R}^n$ by

$$\vec{s}(x) = z + \left(s(y_1), s(y_2), \ldots, s(y_n) \right),$$

if $x = z + y$ with $z \in \mathbf{Z}^n_n$, $y \in [-1, 1]^n$. $\vec{s}$ is clearly a smooth mapping. We put now

$$P_k^s = \vec{s} \circ P_k,\ \sigma_k^s = P_k^s \circ P_{k+1}^s \circ \dots \circ P_{n-1}^s = \mathbb{R}^n \setminus C''_{n-k-1} \to C'_k$$

We also consider $\sigma_n^s = P_n^s$ to be the identity mapping of the space $\mathbb{R}^n$.

It is easy to verify that a function s can be chosen so that P_k^s arbitrarily closely approximates P_k and σ_k^s is a smooth mapping, U_k^s – admissible with some nonnegative continuous function U_k^s.

Denote by $l\,(i, i-1)$ the linear homotopy between σ_i^s and σ_{i-1}^s, i.e

$$l\,(i,i-1)\,(x, t) = t\ \sigma_i^s(x) + (1-t)\ \sigma_{i-1}^s(x),\ 0 \leqslant t \leqslant 1.$$

Then we have a new homotopy between σ_i^s and σ_{i-1}^s, given by the formula

$$h^s(i, i-1)\,(x, t) = l\,(i, i-1)\left(x, \frac{1+s\,(2t-1)}{2}\right),\ \ 0 \leqslant t \leqslant 1.$$

We can show that the homotopy $h^s(p, q)$ $(p > q)$ from σ_p^s to σ_q^s, constructed from the $h^s(p, p-1), \dots, h^s(q+1, q)$, is a smooth homotopy. In particular, $h^s(n, k)$ is a smooth homotopy, deforming $\mathbb{R}^n \setminus C''_{n-k-1}$ into C'_k for each $h = 0, 1, \dots, n-1$.

3. ISOPERIMETRIC INEQUALITIES FOR MULTIVARIFOLDS

By means of the theorems 1 and 2 and the smooth deformations given above, we can obtain the following two lemmas (the analogous results for currents were given by Federer and Fleming [2]).

LEMMA 1. *Suppose* $V \in S\mathcal{C}_k^r(\mathbb{R}^n)$ *and* $\varepsilon > 0$. *Then there exist multivarifolds* $P \in S\mathcal{C}_k^r(\mathbb{R}^n)$, $Q \in S\mathcal{C}_k^r(\mathbb{R}^n)$ *and* $S \in S\mathcal{C}_{k+1}^r(\mathbb{R}^n)$, *satisfying the following conditions :*

1) $V = P + Q + \partial_T S$

2) $(\mathrm{spt}P) \cup (\mathrm{spt}S) \subset \{x \mid S\,(x, \mathrm{spt}\ V) \leqslant 2n\varepsilon\}$

$(\mathrm{spt}\ \partial_T P) \cup (\mathrm{spt}Q) \subset \{x \mid S_*(x, \mathrm{spt}\ \partial_T V) \leqslant 2n\varepsilon\}$

3) $M_k(P) \leqslant 2n^k \cdot [C_k^n M_k(V) + \varepsilon C_{k-1}^n M_{k-1}(\partial_T V)]$

$$M_{k-1}(\partial_T P) \leqslant 2n^{k-1} \left[C^n_{k-1} M_{k-1} (\partial_T V) \right]$$

$$M_k (Q) \leqslant 6n^k C^n_{k-1} M_{k-1} (\partial_T V)$$

$$M_{k+1}(S) \leqslant 4n^{k+1} \varepsilon\ C^n_k M_k(V)$$

4) *In addition, if* $V \in J\mathcal{C}^r_k(\mathbb{R}^n)$, *then* $P, Q \in J\mathcal{C}^r_k(\mathbb{R}^n)$ *and* $S \in J\mathcal{C}^r_{k+1}(\mathbb{R}^n)$ *and the homogeneous component* $P^{(k)}$ *of the degree* k *of* P *is a polyhedral kdimensional chain consisting of cubes in* μ_ε (C') *with integral coefficients, where* μ_ε *denotes the homothetic transformation of the space* $\mathbf{R}^n$ *carrying a point* x *tot he point* εx.

LEMMA 2. *Let* A *and* B *be two subsets of* $\mathbb{R}^n$ *such that* $B \subset A$ *and* A *is deformable to* B. *Suppose that* U *and* O *are neighbourhoods of* A *and* B *in* R^n, C *is a compact subset of* A, $a > 0$, $b > 0$, $\alpha = U \to A$ *is a retraction, satisfying the Lipschit's condition with the coefficient* ξ *on the subset* $\{x \mid S(x_1 C) \leqslant a\}$, $\beta : O \to B$ *is a retraction, satisfying the Lipschit's condition with the coefficient* η *on the subset* $\{x \mid S(x, B \cap C) \leqslant b\}$.

$$\text{If } V \in \tau\mathcal{C}^0_k(\mathbb{R}^n),\ \text{spt } V \subset C,\ \text{spt } \partial_T V \subset B$$

and $3n\varepsilon \leqslant \inf \{ b, (\eta + 2)^{-1}a\}$, *where* $\varepsilon > 0$ *is defined by the equation*

$$2n^k [C^n_k M_k(V) + \varepsilon\, C^n_{k-1} M_{k-1} (\partial_T V)] = 2^k \varepsilon^k$$

then there exists multivarifold $W \in \tau\mathcal{C}^0_{k+1}(\mathbb{R}^n)$ *such that* spt $W \subset A$, spt $(V - \partial_T W) \subset B$ *and*

$$M_{k+1}(W) \leqslant \xi^{k+1} (\eta + 2)^k\, [4n^{k+1} C^n_k (\eta + 2) + 3n (\eta+1)]\varepsilon\, M_k(V)$$

$$M_k (\partial_T W) \leqslant M_k(V) + 6n^k C^n_{k-1} (\eta+2)^k\, \varepsilon\, M_{k-1} (\partial_T V)$$

By means of the results given above we can prove the main theorem formulated in the introduction.

Received on June 1981

REFERENCES

1. Đao Trong Thi (Đao Chong Thi) – *Multivarifolds and classical multidimensional Plateau Problems*, Izv. Akad. Nauk. SSSR Ser. Mat., 44, No 5, 1980 (in Russian).
2. Federer H., Fleming WH – *Normal and integral currents*, Ann. Math., 72, No 3, 1960, 458–520.
3. Almgreen F.J. – *Existence and regularity almost everywhere of solutions to elliptic variational problem among surfaces of varying topological type and singularity structure*, Ann. Math., 87, No 2, 1968, 321–391.
4. Reifenberg E.R. – *Solution of the Plateau problem for m-dimensional surfaces of varying topological type*. Acta Math., 104, No I, 1960, 1–92.
5. Fomenko A.T. – *Multidimensional Plateau problems on Riemannian manifolds and extraordinary homology and cohomology theories I*, Trudy sem. Vektor. Tenzor Anal., 19, 1974, 3–176 (in Russian).