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ON THE MULTIVALUED ASYMPTOTIC MARTINGALES

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INTRODUCTION. In recent years, the study of asymptotic martingales (real-valued and vector-valued) has been developed extensively by many authors, e. g. Austin, D. G., Edgar, G. A. and Ionescu Tulcea, A. [1], Bellow, A. [2], Chacon, R. V. and Sucheston, L. [4], Edgar, G. A. and Sucheston, L. [8], [9], Uhl, J. J. Jr [14]... Also multivalued martingales have been discussed by F. Hiai and H. Umegaki [10], F. Hiai [11], and later on by Coste, A. [6]. The aim of this paper is to present a theory of multivalued amarts (asymptotic martingales) considered as a simultaneous generalization of vector-valued amarts and multivalued martingales. In Section I, we introduce some preliminary notations and definitions and give several examples of multivalued amarts. In Section II, we present several convergence theorems for some suitable metric and also an almost surely convergence theorems. Theorem 2. 1 seems to be new even when restricted to the vector-valued amarts, whereas theorem 2.2. is in some sense a generalization for the case of multivalued amarts of a result in [10] (see Theorem 6.3). Let us note also that our theorem 2.4 is the multivalued version of Chacon and Sucheston's theorem (see [4, Theorem 2]) with some modifications.

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I. DEFINITIONS AND NOTATIONS

Let us collect some terminology and notations. Throughout this paper we denote by $(\Omega, \mathcal{F}, P)$ a probability space, by E a real separable Banach space with a dual E' and by 2^E the family of all nonempty subsets of E. For $X \in 2^E$, let cl X denote the closure (in the norm topology) of X. Denote

$$K(E) = \{X \in 2^E : X \text{ is bounded, closed}\}$$
$$K_c(E) = \{X \in K(E) : X \text{ is convex}\}$$
$$K_{cc}(E) = \{X \in K_c(E) : X \text{ is (norm) compact}\}$$

with the introduction of the Hausdorff distance

$$h(X, Y) = max\,\{\sup_{x \in X} d(x, Y), \sup_{y \in Y} d(y, X)\} \tag{1.1}$$

and in particular

$$|X| = h(X, \{0\}) = \sup_{x \in X} \|x\|$$

$K(E)$ can be embeded into a complete metric space. Moreover $K_c(E)$ is a closed subspace of $K(E)$ and $K_{cc}(E)$ is a closed, separable subspace of $K(E)$ (see[15]). The addition and multiplication in $K(E)$ are defined by

$$X \oplus Y = \text{cl}\,\{x + y : x \in X, y \in Y\} \quad , X, Y \in K(E)$$
$$aX = \{ax : x \in X\} \quad , a \in R, X \in K(E).$$

Let X is a bounded, nonempty subset of E. The support of E. The support function of X is the function defined on E' by

$$x' \to s(x' \mid X) = \sup\,\{\langle x', x\rangle : x \in X\}$$

DEFINITION 1.2. A sequence $(X_n : n \geqslant 1) \subset K_c(E)$ is called to converge weakly to $X \in K_c(E)$ iff $\lim_{n\to\infty} [s(x' \mid X_n) - s(x' \mid X)] = 0$ for all $x' \in E'$.

Let $X : \Omega \to 2^E$ such that $X(\omega)$ is closed subset of E for all $\omega \in \Omega$. Recall that X is (weakly) measurable with respect to б-field $\mathfrak{F}$ if $\{\omega : X(\omega) \cap U \neq \phi\} \ni \mathfrak{F}$ for every open set $U \subset E$. We consider to the following spaces of multifunctions (see [10] for related results):

$L^0(\Omega, \mathfrak{F}, P, 2^E) = L^0(\Omega, 2^E) = \{X : \Omega \to 2^E : X$ is measurable and $X(\omega)$ is closed for all $\omega \in \Omega\}$

$$L^1(\Omega, \mathfrak{F}, P, K(E)) = L^1(\Omega, K(E)) = \{X \in L^0(\Omega, 2^E) : \int_\Omega |X|\, dP < \infty\}$$
$$L^1(\Omega, \mathfrak{F}, P, K_c(E)) = L^1(\Omega, K_c(E)) = \{X \in L^1(\Omega, K(E)) : X(\omega) \in K_c(E) \text{ a.s.}\}$$
$$L^1(\Omega, \mathfrak{F}, P, K_{cc}(E)) = L^1(\Omega, K_{cc}(E)) = \{X \in L^1(\Omega, K(E)) : X(\omega) \in K_{cc}(E) \text{ a.s}\}$$

with the introduction of the Hausdorff mean distance

$$H(X, Y) = \int_\Omega h(X(\omega), Y(\omega))\, dP \tag{1.2}$$

$L^1(\Omega, X(E))$ can be made into a complete metric space. Moreover, $L^1(\Omega, X_c(E)) \supset L^1(\Omega, K_{cc}(E))$ are closed subspaces of $L^1(\Omega, K(E))$. The space

$$L^1(\Omega,\mathcal{F},P,E) = L^1(\Omega,E) = \{f : \Omega \to E : \int_\Omega |f| dP < \infty\}$$

is considered to be a subspace of $L(\Omega, K_{cc}(E))$.

Let $X \in L^0(\Omega, 2^E)$. The integral of X is defined by

$$\int_\Omega XdP = \{\int_\Omega fdP : f \in S^1_X\}$$

where $S^1_X = S^1_X(\mathcal{F}) = \{ f \in L^1(\Omega,\mathcal{F},P,F) : f(\omega) \in X(\omega) \text{ a.s. } \}$

For $A \in \mathcal{C}$, $\int X = \int 1_A X$ with 1_A is the characteristic function of the set A. Apart from the distance given by (1.2), we define the weak (mean) distance as follows:

DEFINITION 1.2. For $X,Y \in L^1(\Omega,\mathcal{F},P,K_c(E))$ the number

$$H_w(X,Y) = \sup\{\int_\Omega | s(x' \mid X) - s(x' \mid Y) | dP : x' \in E', \|x'\| \leqslant 1\} \quad (1.3)$$

is called the weak distance between X and Y.

REMARKS (1) By [3, Theorem II.18] we always have

$$H_w(X,Y) \leqslant HX,Y) \qquad \text{for } X,Y \in L^1(\Omega, K_c(E))$$

(2) We also have the following inequalities for X, $Y \in L^1(\Omega, K_c(E)$

$$\sup_{A\in\mathcal{F}} h(\mathrm{cl}\int_A X, \mathrm{cl}\int_A Y) \leqslant H_w(X,Y) \leqslant 4 \sup_{A\in\mathcal{F}} h(\mathrm{cl}\int_A X, \mathrm{cl}\int_A Y) \quad (1.4)$$

Indeed, (1.4) is immediately consequence of [10, Lemma 2.2] and the inequalities

$$\sup_{A\in\mathcal{F}} |\int_A fdP| \leqslant \int_\Omega |f| dP \leqslant 4 \sup_{A\in\mathcal{F}} |\int_A fdP|$$

for scalar — valued function $f \in L^1(\Omega, \mathcal{F}, P, R)$.

We now give the definition of multivalued amarts. Without loss of generality, let us denote by $(\mathcal{F}_n : n \geqslant 1)$ an increasing sequence of $\mathcal{F}$ such that $\mathcal{F} = \sigma(\bigcup_n \mathcal{F}_n)$. A stopping time is a random variable τ taking values in $\{1,2,..., \infty\}$ such that for each $n \geqslant 1$, $(\tau = n) \in \mathcal{F}_n$. The set of all bounded stopping times is denoted by T. For $\tau \in$ T, denote $T(\tau) = \{\sigma \in T : \sigma \geqslant \tau\}$

DEFINITON 1.3. An adapted sequence $(K_n : n \geqslant 1)$ in $L^1(\Omega, K_c(E)$ (i.e., X_n is $\mathcal{F}_n$ measurable for each $n \geqslant 1$) is said to be a multivalued amart iff

(h) $\lim \mathrm{cl}\int_{\Omega} X_\tau dP = Y$ exists in $K_c(E)$. i.e., for each $\varepsilon > 0$, there exists $\tau \in T$ such that $h(\mathrm{cl}\int_{\Omega} X_\tau dP, Y) < \varepsilon$ for all $\sigma \in T(\tau)$.

An adapted sequence of random variables taking values in $K_c(E)$ is said to be of class (B) iff $\sup_T \int_{\Omega} |X_\tau| dP < \infty$

EXAMPLES

1) Multivalued martingales (see [10]). An adapted sequence $(X_n : n \geqslant 1)$ is said to be multivalued martingale if $X_m = E(X_n \mid \mathcal{F}_m)$ for all $n \geqslant m \geqslant 1$, where $E(X_n \mid \mathcal{F}_m)$ is denoted the multivalued conditional expectation of X_n relative to $\mathcal{F}_m$. By the property of conditional expectations (see [10, Theo. 5.4]), it is easy to see that $\mathrm{cl}\int_{\Omega} X_\sigma dP = \mathrm{cl}\int_{\Omega} X_\sigma dP$ for all $\tau, \sigma \in T$. Therefore, every multivalued martingale is multivalued amart.

2) Multivalued supermartingales (resp. multivalued submartingales)

An adapted sequence $(X_n : n \geqslant 1) \subset L^1(\Omega, K_{cc}(E))$ is said to be multivalued supermartingale (resp. submartingale) iff

$\int_A X_n dP \supseteq \int_A X_{n+1} dP$ (resp. $\int_A X_n dP \subseteq \int_A X_{n+1} dP$), $A \in \mathcal{F}_n$, $n \geqslant 1$.

Then, it is easy to see that $\int_A X_\tau dP \subseteq \int_A X_\sigma dP$ (resp. $\int_A X_\tau dP \subseteq \int_A X_\sigma dP$) for, $\tau, \delta \in T$ with $\tau \leqslant \delta$. Thus, if (X_n) is multivalued supermartingale, then $Y = \bigcap_n \int_{\Omega} X_n \, dP = (h) \lim \int_{\Omega} X_\tau \, dP$. If (X_n) is multivalued submartingale such that $\bigcup_n \int_{\Omega} X_n dP$ is relative norm compact set of E, then $\mathrm{cl}\,(\bigcup_n \int_{\Omega} X_n dP) = (h) \lim_T \int_{\Omega} X_\tau dP$

3) Mulitivalued quasimartingale. An adapted sequence $(X_n : n \geqslant 1)$ is said to be multivalued quasimartingale iff

$$\sum_{n=1}^{\infty} H(X_n, E(X_{n+1} \mid \mathcal{F}_n)) < \infty$$

By the same argument in the real-valued case (see [8, section I, example 3]) it will be proved that every multivalued quasimartingale is a multivalued amart.

II. CONVERGENCE THEOREMS

In this section, under « Uhl conditions » (see [13]), we shall give the convergence theorem in the weak metric and the almost surely convergence theorem for multivalued amarts in $L^1(\Omega, K_{cc}(E))$. The convergence theorems for multivalued amarts in $L^1(\Omega, K_c(E))$ will be established by using the Radon-Nikodym theorem of F. Hiai (see [11]).

We begin with a lemma, which is a fundamental statement for multivalued amarts (see [4] for the case of vector-valued amarts).

LEMMA 2.1. *Let $(X_n, \mathcal{F}_n, n \geqslant 1)$ be a multivalued amart in $L^1(\Omega, K_c(E)$ Then for each $A \in \bigcup_n \mathcal{F}_n$, the net $(\mathrm{cl}\int_A X_\tau dP : \tau \in T)$ converges to a limit $F(A)$ in $K_c(E)$ (w. r. t. the Hausdorff metric). Moreover the convergence is uniform in the following sense: For each $\varepsilon > 0$, there exists a $\tau_0 \in T$ such that*

$$\sup_{A \in \mathcal{F}_\tau} h(\mathrm{cl}\int_A X_\tau, F(A) < \varepsilon \tag{2.1}$$

for all $\tau \in T(\tau_0)$. Where, $\mathcal{F}_\tau = \{A \in \mathcal{F}: A \cap (\tau = n) \in \mathcal{F}_n$ for all $n \geqslant 1\}$

Proof. Let $\varepsilon > 0$, choose $\tau_0 \in T$ such that

$$h(\mathrm{cl}\int_\Omega X_\tau, \mathrm{cl}\int_\Omega X_\sigma) < \varepsilon$$

for $\tau, \sigma \geqslant T_{\tau_0}$). Fix $\sigma \geqslant \tau \geqslant \tau_0$ and $A \in \mathcal{F}_\tau$. Define a bounded stopping times τ_1, σ_1 as follows: $\tau_1 = \tau$ on A, $\sigma_1 = \sigma$ on A and $\tau_1 = \sigma_1 = n_0$ on $\Omega \setminus A$, where $n_0 > \max(\tau, \sigma)$. Then, from

$$X_{\tau_1} = 1_A X_\tau \oplus 1_{\Omega \setminus A} X_{n_0}$$

$$X_{\sigma_1} = 1_A X_\sigma \oplus 1_{\Omega \setminus A} X_{n_0}$$

By [10, Theorem 4.1] and [3, Theorem II . 19] we have

$$h(\mathrm{cl}\int_A X_\tau, \mathrm{cl}\int_A X_\sigma) = h(\mathrm{cl}\int_\Omega X_{\tau_1}, \mathrm{cl}\int_\Omega X_{\sigma_1}) < \varepsilon \tag{2.2}$$

By the completely of $K_c(E)$ with respect to the Hausdorff metric we deduce (h) $\lim \mathrm{cl}\int_A X_\tau = F(A)$. Finally, (2.2) implies

$$\sup_{A \in \mathcal{F}_\tau} h(\mathrm{cl}\int_A X_\tau, F(A)) < \varepsilon$$

This completes the proof.

The next lemma is multivalued version of the maximal lemma (see [8]). The proof is identical to that given for Lemma 1.1 of [8].

LEMMA 2.2. (maximal lemma) *Let* $(X_n, \mathcal{F}_n, n > 1)$ *be an adapted sequence of random variables taking values in* $K_c(E)$ *such that* $\sup_T \int_\Omega |X_\tau| \, dP < \infty$. *Then for each positive number* a,

$$P(\sup_n |X_n| > a) \leqslant (1/a) \sup_T \int_\Omega |X_\tau| dP$$

THEOREM 2.1. *Let* E *be a Banach space with a separable dual. Let* $(X_n, \mathcal{F}_n : n \geqslant 1)$ *be a multivalued amart in* $L^1(\Omega, K_{cc}(E))$ *such that:*

(i) $\sup_T \int_\Omega |X_\tau| \, dP < \infty$ and.

(ii) *for a given* $\varepsilon > 0$, *there exists a norm compact convex subset* $K \subset E$ *such that for any* $\delta > 0$ *there is an index* $n_0 \geqslant 1$ *and a set* $A_0 \in \mathcal{F}_{n_0}$, $P(\Omega \setminus A_0) < \varepsilon$ *:*

$$\int_A X_n dP \subset P(A)K + \delta U$$

for all $n \geqslant n_0$ *and for all* $A \subset A_0$, $A \in \mathcal{F}_n$, U *denotes the closed unit ball of* E.

Then there exists some $X \in L^1(\Omega, K_{cc}(E))$ *such that* X_n *converges weakly a.s. to* X.

Proof. We first assume that (X_n) is a E—valued amart. As in the lemma 2.1., the E-valued set function

$$F(A) = \lim_T \int_A X_\tau dP \qquad A \in \bigcup_n \mathcal{F}_n$$

is finitely additive. By standard methods (see [5]) we can write $F = F_c + F_s$, where F_c and F_s are E-valued finitely additive set functions of bounded variation where $|F_s|$, the variation of F_s is singular with respect to P, and F_c is a countably additive set function which is P-continuous. From (ii), by the similar argument as in [13, Theorem 4] we deduce

$$F_c(A) = \int_A X dP \quad \text{for } X \in L^1(\Omega, E),\ A \in \mathcal{F}.$$

Define $Y_n = E(X \mid \mathcal{F}_n)$ and $Z_n = X_n - Y_n$. Then, Z_n is E-valued amart. On the one hand, for each $x' \in E'$, $(\langle x'; Z_n\rangle : n \geqslant 1)$ is (real-valued) amart with $\sup_n \int_\Omega |\langle x'; Z_n\rangle| \, dP < \infty$. By [1] $\langle x'; Z_n\rangle$ must converge a.s. On the other hand, it is easy to see that $\lim_{n\to\infty} \int_A \langle x'; Z_n\rangle dP = \langle x'; F_s(A)\rangle$ for $A \in \mathcal{F}$. That implies $\lim_{n\to\infty} \langle x'; Z_n\rangle = 0$ in probability. Hence, $\lim_{n\to\infty} \langle x'; Z_n\rangle = 0$ a. s. Therefore, $\lim_{n\to\infty} \langle x'; X_n\rangle = \langle x'; X\rangle$ a.s.

Now, let $(x'_i, i \geqslant 1)$ be a dense sequence of E'. By (i) and lemma 2.2 we deduce that for almost surely $\omega \in \Omega$ $\lim \langle x'; X_n \rangle = \langle x'; X \rangle$ for all $x' \in E'$.

In the general case, when (X_n) is $K_{cc}(E)$-valued amart, then by [10, Theo. 4. 5] and by [11, Coro. 5.4] (X_n) can be considered as an amart taking values in a Banach space, hence by [6], the arguments of the above part can be used for this case Q.E.D.

THEOREM 2.2. *Let $(X_n, \mathfrak{F}_n, n \geqslant 1)$ be a multivalued amart in $L^1(\Omega, K_{cc}(E))$ such that*

(i) *$(|X_n| : n \geqslant 1)$ is uniformly integrable, i.e.,*

$$\lim_{a \uparrow \infty} \sup_n \int_{\{|X_n| > a\}} |X_n| \, dP = 0 \quad \text{and,}$$

(ii) *for a given $\varepsilon > 0$, there exists a norm compact convex subset $K \subset E$ such that for any $\delta > 0$ there is an index $n_0 \geqslant 1$ and a set $A_0 \in \mathfrak{F}_{n_0}$, $P(\Omega \setminus A_0) < \varepsilon$:*

$$\int_A X_n \, dP \subset P(A)K + \delta U$$

for all $n \geqslant n_0$ and for all $A \subset A_0$, $A \in \mathfrak{F}_n$.

Then there exists some $X \in L^1(\Omega, K_{cc}(E))$ such that $\lim_{n \to \infty} H_w(X_n, X) = 0$

Conversely, if $\lim_{n \to \infty} H_w(X_n, X) = 0$, then (ii) is satisfied.

Proof. The « if » part is a consequence of the theorem of Uhl (see [14, Coro. 3]). Indeed, regarding X_n as a vector-valued amart the conditions (i) and (ii) guarantee that the limit measure (in the sense of Lemma 2.1) has Radon-Nikodym derivative contained in $L^1(\Omega, K_{cc}(E))$. That implies

$$\lim_{n \to \infty} H_w(X_n, X) = 0 \text{ for some } X \in L^1(\Omega, K_{cc}(E)).$$

Conversely, if $\lim_{n \to \infty} H_w(X_n, X) = 0$, then by [13, Propo. 1] and by Lemma 2.1, so the condition (ii) holds. Q.E.D.

REMARK. If (X_n) is $K_{cc}(E)$—valued martingale, then the following three assertions are equivalent.

(a) (X_n) is « regularly », i.e., $X_n = E(X \mid \mathfrak{F}_n), n \geqslant 1$, where $X \in L^1(\Omega, K_{cc}(E))$.

(b) $\lim_{n \to \infty} H(X_n, X) = 0$

(c) $\lim_{n \to \infty} H_w(X_n, X) = 0$.

Proof. (a) $\Rightarrow$ (b) is the content of the theorem 6.1 of [10].

(b) $\Rightarrow$ (c) is triviality.

(c) $\Rightarrow$ (a) From the inequality (1.4), we have

$\int_A X_n dP = \int_A X dP \quad A \in \mathfrak{F}_n$. Using the Proposition 6.1 of [11], we deduce $X_n = E(X \mid \mathfrak{F}_n)$ for all $n \geqslant 1$.

For multivalued amarts in $L^1(\Omega, K_c(E))$, we present the following:

THEOREM 2.3. *Let E be a Banach space with the Radon-Nikodym property and let $(X_n, \mathfrak{F}_n)$ be a $K_c(E)$— valued multivalued amart such that*

(i) $\bigcup_n \int_\Omega X_n dP$ *is relatively norm compact in E and,*

(ii) $(|X_n|, n \geqslant 1)$ *is unformly integrable.*

Then, X_n converges in the weak distance to some $X \in L^1(\Omega, K_c(E))$.

Proof. By (i) and [11, Theorem 1.3], for each n, $F_n(A) = \mathrm{cl} \int_A X_n dP$ is the measure with values in $K_c(E)$. Then, by (ii)

$F(A) = (h) \lim_T \mathrm{cl} \int_A X_\tau dP$ is $K_c(E)$-valued measure, P-continuous and of bounded variation. By [11, Theorem 4.3, and Corollary 4.4] F has a generalized Radon-Nikodym derivative contained in $L^1(\Omega, K_c(E))$, i. e., $F(A) = \mathrm{cl} \int_A X dP$ for $X \in L^1(\Omega, K_c(E))$. Moreover, (ii) guarantes that $F(\Omega)$ is relatively norm compact in E, this fact implies that $\bigcup_{A \in \cup \mathfrak{F}_n} F(A)$ is relatively compact w.r.t. the Hausdorff topology. Regarding F as a vector — valued measure and using Hoffmann-Jorgensen's theorem [12, Theorem 9] we deduce $\overline{\lim_{n\to\infty}} H_w(X_n, X) = 0$

REMARKS 1, If (X_n) is mutivalued martingale, then the condition (i) of the theorem can be writen as follows: $\int_\Omega X_1 \, dP$ is relatively norm compact in E.

2. If E' is separable and (X_n) is multivalued martingale such that $\lim_{n\to\infty} H_w(X_n, X) = 0$ for $X \in L^1(\Omega, K_c(E))$. Then $X_n = F(X \mid \mathfrak{F}_n)$ for alln.

The following is multivalued version of a theorem of Chacon and Sucheston (see [4, Theo. 2]).

THEOREM 2.4. *Let E be a Banach space with the Radon-Nikodym property and with a separable dual. Let $(X_n : n \geqslant 1)$ be a $K_c(E)$ — valued amart such that*

(i) *for each* $n \geq 1$, $\int_\Omega X_n \, dP$ *is relatively weakly compact*, $F(\Omega)$ *is weakly compact and*

(ii) $\sup_T \int_\Omega |X_\tau| \, dP < \infty$

Then there exists a $X \in L^1(\Omega, K_c(E))$ *such that* X_n *converges to X weakly a. s.*

Proof. The proof is similar to that of theorem 2 in [4].

The following corollary is the multivalued version of the theorem of Bellow (see [2]).

COROLLARY 2. 5. *The following assertions are equivalent for a given Banach space E.*

(a) *E is finite-dimensional.*

(b) *Every multivalued amart* (X_n) *in* $L^1(\Omega, K_c(E))$ *such that* $\sup_T \int_\Omega |X_\tau| \, dP < \infty$ *converges a.s. in the Hausdorff topology.*

Proof. (b) $\Rightarrow$ (a) is immediately consequence of the theorem of Bellow [2] because every E-valued amart is $K_c(E)$-valued amart. (a) $\Rightarrow$ (b) is consequence of the theorem 2.4. Indeed, by theorem 2.4, X_n converges weakly a. s. to $X \in L^1(\Omega, K_c(E))$. For any $\sigma > 0$ choose $x'_1, \dots, x'_k$ with $\| x'_j \| \leqslant 1$ $(1 \leqslant J \leqslant k)$ such that for each $x' \in E'$, $\| x' \| \leqslant 1$ there is a x'_j satisfying $\| x' - x'_j \| < \sigma$.

Then from the inequality

$$|s(x' \mid X_n) - s(x' \mid X)| \leqslant |s(x' \mid X_n) - s(x'_j \mid X_n)| +$$
$$|s(x'_j \mid X_n) - s(x'_j \mid X)| + |s(x'_j \mid X) - s(x' \mid X)|$$

we have

$$h(X_n, X) \leqslant \max_{1 \leqslant j \leqslant k} \{ |s(x'_j \mid X_n) - s(x'_j \mid X)| \} + \sigma(|X_n| + |X|)$$

That implies

$$\lim_{n \to \infty} h(X_n, X) = 0 \quad \text{a. s.}$$ Q.E.D.

REMARK. The theorem 6.3. of [11] is special case of the Corollary 2.5. Indeed, if (X_n) is $K_c(E)$-valued supermartingale or submartingale (see example 2, section I), then (X_n) is also $K_c(E)$-valued amart and $\sup_n \int_\Omega |X_n| \, dP < \infty$

implies $\sup_T \int_\Omega |X_\tau| \, dP < \infty$. Moreover, if $(|X_n| : n \geqslant 1)$ is uniformly integrable, then $H(X_n, X) \to 0$.

Finally, we give the Riesz decomposition for multivalued amarts.

THEOREM 2.6. *Let E be a Banach space with a separable dual and with the Radon-Nikodym property. Let (X_n) be a $K_c(E)$ valued amart such that for each $n \geqslant 1$, $\int_\Omega X_n dP$ is relatively weakly compact, $F(\Omega)$ is weakly compact and such that $\lim_n \inf \int_\Omega |X_n| dP < \infty$. Then*

(i) *There exists a unique $K_c(E)$-valued martingale (Y_n) such that $\lim_T H_w(X_\tau, Y_\tau) = 0$.*

(ii) *if $\sup_T \int_\Omega |X\tau| \, dP < \infty$, then for a.s. $w \in \Omega$ $\lim_{n\to\infty} [s(x'|X_n) - s(x'|Y_n)] = 0 \ \forall x' \in E'$*

Proof. The proof is similar to that of theorem 1 in [9].

EXAMPLE. Let $([0,1), \mathcal{F}, P)$ be the Lebesgue measure space. Let for each $n \geqslant 1$ $f_n : [0,1) \to [0,1)$ be a $\mathcal{F}$-measurable function such that $\forall \omega \in [0,1)$ $\lim_n \sup f_n(\omega) > 0$ and $\sum_{n=1}^{\infty} (\int_0^1 f_n dP)^2 < \infty$. Let E be an l_2 space with the usual basis (e_n).

Define the multivalued function $X : [0,1) \to 2^{l_2}$ as follows:

$$X(\omega) = \overline{co} \{f_n(\omega) e_n : n \geqslant 1\}$$

Clearly

$X \in L^1(\Omega, K_c(l_2))$ but for all $w \in [0.1)$ $X(\omega \notin K_{cc}(l_2)$ and $\int_A X dP \in K_{cc}(l_2)$ for all $A \in \mathcal{F}$.

Let $\mathcal{F}_n$ be a δ-field generated by $\{[(k-1)2^{-n}, k_1^{-n}, k = 1, \ldots, 2^n\}$

We define $X_n = E(X|\mathcal{F}_n)$ for all $n \geqslant 1$.

Then it is easy to see that $H(X_n, X) \nrightarrow 0$, but by theorem 2.3.

$H_w(X_n, X) \to 0$. Moreover $h(X_n, X) \nrightarrow 0$ for all $\omega \in [0,1)$, but by theorem 2.4 X_n converges weakly a.s. to X.

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REFERENCES

1. Austin, D.G., Edgar, G.A. and Ionescu Tulcea, A.: *Pointwise convergence in terms of expectations*. Z. Wahrscheinlichkeitstheorie verw. Gebiete, 30, 17—26 (1974)

2. Bellow, A.: *On vecto-valued asymptotic martingales*. Proc. Nat, Acad. Sci, U.S.A., 73, 1798—1799 (1976).

3. Castaing, C. and Valadier, M.: *Convex Analysis and measurable multifunctions* Lecture Note N⁰580. Springer Verlag (1977).

4 Chacon, R.V. and Sucheston, L.: *On convergence of vectorvalued asymptotic martingales.* Z. Wahrscheinlichkeitstheorie verw. Gebiete, 33, 55—59 (1975).

5. Chatterji, S.D.: *Martingale convergence and the Radon-Nikodym theorem in Banch spaces.* Math. Scand. 22, 21—41 (1968)-

6. Coste, A.: *Sur les martingales multivoques.* C. R. Acad. Sc. Paris. T. 290, Series A—B, N⁰ 20 (2 juin 1980),

7. Dunfor, N. and Schwarts, J.T.: *Linearr operators.* Part I; General theory. Interscience publi. New York, London (1958).

8. Edgar. G.A. and Sucheston, L.: *Amarts: a class of asymptotic martingales A. discrete parameter.* J. Multivariate Anal., 6, 193—221 (1976).

9. Edgar. G.A. and Sucheston, L.: *The Riesz decomposition for vector-valued amarts.* Z. Wahrscheinlichkeitstheorie verw. Gebiet 36, 85—92 (1976b),

10. F. Hiai and H. Umegaki: *Integrals, conditional expectations and martingales of multivalued functions.* J. Multivariate Anal. 7, 149—182 (1977)

11. F. Hiai: *Radon-Nikodym theorems for set-valued measures.* J. Multivariate Anal., 8, 96–118 (1978).

12. J. Hoffmann-Jrgensen: *Vector measures.* Math. Scand., 28. 5–32 (1971)

13, Uhl. J.J.Jr: *Applications of Radon-Nikodym theorems to martingale convergence.* Tran. Amer. Math. Soc., 145, 271—285 (1969).

14. Uhl, J.J.Jr; *Pettis mean convergence of vector-valued asymtotic martingales.* Z. Wahrscheinlichkeitstheorie verw. Gebiete, 37, 291–295 (1977).

15. Kuratowski, K. *Topololy,* Vol. I. Academic Press, New York-London-Warszawa (1966) (Transl. from French)