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SOME EXAMPLES AND THEOREMS RELATED TO THE R-N-PROPERTY IN BANACH SPACES

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1. INTRODUCTION. A Banach space X is said te have the RN Property if for every finite positive messure (S, Σ, μ) and every μ — continuous measure $m: \Sigma \to X$, of bounded variation, there is a Bochner integrable function $f: S \to X$ such that

$$m(A) = \int_A f d\mu \qquad \text{for each } A \text{ of } \Sigma$$

It was shown, at first, in [6] by Phillips that every reflexive Banach space has the RNP. Later, Rieffel [7] Davis and Phelps [1] Maynard [3] and Phelps [5] obtained many geometrie characterizations for the RNP. Recently, Huff and Morris [2], have studied the class of all closed bounded non-empty subsets of X and proved that a Banach space X has the RNP iff every closed bounded non-empty subset of X has an extreme point. In this paper we give some examples and prove some theorems related to the RNP in Banach spaces.

1. NOTATIONS, DEFINITIONS AND SOME EXAMPLES

Throughout the paper let X be a Banach space with it's continuous dual X. By $B(x, r)$ we always mean the open ball of radius $r > o$ and with center $x \in X$. Given a subset B of X we denote by $\overline{B}$; co (B) and $\overline{\text{co}}$ (B) the closure the convex hull and the closed convex hull of B, resp. Finally $\sigma - \text{co}(B)$ is defined as follows

$$\sigma - \text{co}(B) = \Big\{ \sum_{i=1}^{\infty} r_i x_i : r_i > 0 \quad \sum_{i=1}^{\infty} r_i = 1; \{x_i\} \subset B$$

$$\text{provided the series } \sum_{i=1}^{\infty} r_i x_i \text{ is convergent} \Big\}$$

DEFINITION 1.1. A subset B of X is said to be f – dentable; σ – dentable or dentable if for every $r > 0$, there is an $x \in B$ such that

$$x \notin \text{co} \ (B \setminus B(x, r)) \quad (1.1)$$
$$x \notin \sigma\text{co} \ (B \setminus B(x, r)) \text{ or} \quad (1.2)$$
$$x \notin \overline{\text{co}} \ (B \setminus B(x, r)), \quad \text{resp.} \quad (1.3)$$

If an $x \in B$ satisfies (1.1) ; (1.2) ar (1.3), resp. for all $r > 0$, then x will be called an f – denting, σ – denting or denting point of B and denoted by $x \in$ f – $D(B)$, $x \in \sigma - D(B)$ or $x \in D(B)$, resp.

DEFINITION 1. 2. Given a Banach space X let B (X) denote the class of all bounded non-empty subsets of X. We shall consider the following subclasses $F(X)$, $F_f(X)$, $F_\sigma(X)$, $F_d(X)$ and $F_{\text{ext}}(X)$ of $B(X)$

$$F(X) = \{ F \in B(X);\ F \text{ is closed}\}$$
$$F_f(X) = \{ F \in F(X);\ F \text{ is } f\text{–dentable} \}$$
$$F_\sigma(X) = \{ F \in F(X);\ F \text{ is } \sigma\text{–dentable}$$
$$E_d(X) = \{ F;\ F \text{ is dentable and } F \in F(X) \}$$
$$F_{\text{ext}}(X) = \{ F \in F(X);\ F \text{ has an extreme point} \}$$

LEMMA 1.3. *Let X be a Banach space. Then for each $B \in B(X)$ we have*

(1) $\text{co}\ (B) \subset \sigma\text{-co}\ (B) \subset \overline{\text{co}}\ (B)$ **(1.4)**

(2) $F_d(X) \subset F_\sigma(X) \subset F_f(X)$ (1.4)

(3) $\text{ext}\ (B) = \text{f} - D\ (B)$

(4) $F_{\text{ext}}(X) \subset F_f(X)$ (1.5)

Where ext (B) *denotes the set of all extreme points of* B.

Proof. Since the inclusions mentioned in (1), (2) and (4) are easily established so we give here only a proof of (3). Indeed, let B be a bounded non-empty subset of X. Suppose first that $x \in \text{ext}\ (B)$. Henc $x \notin \text{co}(B \setminus \{x\})$. Consequently, $x \notin \text{co}\ (B \setminus B(x,r))$ for every $r > 0$. It means that $x \in f - D(B)$. Suppose conversely that $x \notin \text{ext}(B)$. Equivalently, $x \in \text{co}\ (B \setminus \{x\})$. Thus there are positive numbers $r_1, \ldots, r_n$ and vectors $x_1, \ldots, x_n$ of $B \setminus \{x\}$ such that $x = \sum_{i=1}^{n} r_i x_i$.

Since $x_i \neq x$ for all $i = 1, \ldots, n$ then we have

$$r = \inf \{ \| x_i - x \| ;\ i = 1, \ldots, n\} > 0.$$

Hence $x \in \text{co}\ (B/B(x,r))$. Consequently, $x \notin f-D(B)$. It completes a proof of lemma 1.3.

Rieffel has pointed out in [7] that the closed unit ball of $C\,[o,1]$ is not dentable but it has an ∂-denting point. Consequently, we have $F_d(C[o,1]) \not\subseteq F_\sigma(C[o,1)$. The following example gives us further informations about inclusions (1.4).

Example 1.4. There is a closed bounded subset of c_0 which has an extreme point (then by lemma 1.3 (4), it is f-dentable) but it is not σ-dentable.

Construction. In c_0 define the closed bounded subsets F_n by $F_n = \{(x_1,\ldots,x_n,o,\ldots);\ x_i \in \{1,-1\};\ i=1,\ldots,n\}$ It is easy to check that

(1) $h\,(F_i, F_j) \geqslant 1\ (i,\ j \geqslant 1\ ;\ i \neq j)$

(2) $F_n \subset \text{co}\ (F_m)\ (m \geqslant n \geqslant 1)$

Where, by definition, $h\,(A,B) = \inf\ \{\ \| a - b \|\ ;\ a \in A,\ b \in B\}$

Now put $F = \bigcup_{n=1}^{\infty} F_n \cup \{\sum_{k=1}^{\infty} 2^{-k}.e_k\}$, where $\{e_1, e_2, \ldots, e_n, \ldots\}$ denotes the usual basis for c_0. taking into account that by (1) and (2) F is closed bounded, ext (F) $= \{\sum_{k=1}^{\infty} 2^{-k}.e_k\}$ and F is not σ-dentable. Thus we have $F_\sigma\ (c_0) \notin F_f\,(c_0)$.

It is also well-known that the closed unit ball of c has many extreme points but it is not dentable. We show now that there is a closed bounded subset of c^0 which is dentable but it has no extreme point.

Example 1.5. There is a closed bounded subset of c_0 which is dentable but it has no extreme point.

Construction. In c_0, define closed bounded subsets F_n by

$$F_n = \overline{B}\left(e_n \frac{1}{2^n}\right), \quad (n \geqslant 1)$$

It is not hard to check that

(1) ext $(F_n) = \phi\ (n \geqslant 1)$ and

(2) $h(F_i, F_j) \geqslant \frac{1}{2}\ (i,\ j \geqslant 1$ and $i \neq j)$.

Now put $F = \bigcup_{n=1}^{\infty} F_n$. Then by (1) and (2) F is closed bounded and ext $(F) = \varnothing$. But since $h(\bigcup_{n \neq k} F_n, F_k) \geqslant \frac{1}{4}$ then F is dentable.

Note that on the one hand this example shows that the classes $F_{ext}(X)$ and $F_d(X)$, in general, are uncomparable. On the other hand it shows that $F_{ext}(c_o) \notin F_f(c_o)$.

COROLLARY 1. 6. *The Banach space c_0 has no the RNP.*

Proof. See ([2])

2. SOME THEOREMS RELATED TO THE R-N-PROPERTY

REMARK 2. 1. In [2], Huff and Morris have shown that in a Banach space which has no the RNP we can contruct a sequence $\{F_n\}$ of finite bounded subsets contained in the closed unit ball of X satisfying the following conditions

(1) There is a positive number r such that

$$h(F_i, F_j) \geqslant r \qquad (i, j \geqslant 1 \text{ and } i \neq j) \text{ and}$$

(2) $F_n \subset (\text{co}(F_m) \qquad (m \geqslant n \geqslant 1)$.

Now put $F = \bigcup_{n=1}^{\infty} F_n$. Then by [2] F is closed bounded and $\text{ext}(F) = \phi$. We note that in the case F is not f-dentable. This remark gives us the following result.

THEOREM 2.2. *Let X be a Banach space; Then the following conditions are equivalent*

(1) *X has the RNP.*

(2) *Every closed bounded subset of X is ∂-dentable.*

(3) *Every closed bounded subset of X is f-dentable.*

DEFINITION 2. 3 A Banach space X is said to have the f-Non-empty Intersection Property (f-NIP), similarly, σ-NIP or NIP, if for every subclass $\{F_t ; t \in T\}$ of $F(X)$ with $\text{co}(F_t) = \text{co}(F_t)$, similarly, with $\sigma\text{-co}(F_t) = \sigma\text{-co}(F_t')$ or $\overline{\text{co}}(F_t) = \overline{\text{co}}(F_{t'})$ for all t, t' of T we have

$$\bigcap_{t \in T} F_t \neq \phi$$

THEOREM 2. 4. *Let X be a Banach space. Then the following conditions are equivalent*

(1) *X has the RNP.*
(2) *X has the NIP.*
(3) *X has the σ-NIP.*
(4) *X has the f-NIP.*

Proof. (1 → 2) Suppose that a Banach space X has the RNP and a subclass $\{F_t ; t \in T\}$ satisfies conditions that $F_t \in F(X)$ for all $t \in T$ and $\overline{co}(F_t) = \overline{co}(F_{t'})$ for all $t, t' \in T$. Put $K = \overline{co}(F_t)$ for some $t \in T$. It is clear that K does not depend on the choise of $t \in T$. Since $D(K) \neq \phi$ (see [5]) then it is sufficient to show that $D(K) \subset F_t$ for all $t \in T$. Indeed, suppose that there is an $x \in D(K)$ such that $x \notin F_{t_0}$ for some $t_0 \in T$. Hence

$$r = \inf \left\{ \| x - y \| ; \quad y \in F_{t_0} \right\} > 0. \text{ Thus}$$

$$x \in K = \overline{co}\left(F_{t_0}\right) \subset \overline{co}(K \setminus B(x, r)). \text{ It contradicts the assumption}$$

that $x \in D(K)$.

Since the implications (2→3→4) are clear then it remains to prove (4→1) Suppose conversely that a Banach space X does not haoe the RNP. Take F and r from remark 2.1 and define

$F_x = F \setminus B(x, r)$ for all $x \in F$. Then by (1) and (2) in remark 2.1 we have $co(F_x) = co(F_{x'})$ for all $x, x' \in F$ but

$$\bigcap_{x \in F} F_x = \phi. \text{ It contradicts (4).}$$

DEFINITION 2.5. On the algebraic tensor product $E \otimes F$ of two Banach spaces E and F we shall consider only one normtopology, that is the ε—topology (inductive, least cross-norm topplogy) defined as follows

$$\| z \|_\varepsilon = \sup \left\{ \left| \sum_{r=1}^{n} < X_r, f > < Y_r, g > \right| ; f \in S(E^*), g \in S(F^*) \right\}$$

for $z = \sum_{r=1}^{n} x_r \otimes y_r \in E \otimes F$, where $S(X)$ denotes the unit ball of a Banach space X.

Note that the value of $\| z \|_\varepsilon$ is independent of the special representation of z (see [4]). We denote the ε-completion of $E \check{\otimes} F$ by $E \check{\otimes}$. It was shown that the Banach tensor product $E \check{\otimes} F$ of two Banach spaces E and F with it's ε-norm is a Banach space.

The natural question is whether $E \check{\otimes} F$ has the RNP if we suppose that each of two Banach spaces E and F has the RNP. The following theorem has been suggestad by professor Ryll-Nardzewski.

THEOREM 2.6. *The Banach tensor product $l_p \check{\otimes} l_q$ fails to have the RNP, where $\frac{1}{p} + \frac{1}{q} = 1$ and $p,q > 0$.*

Proof. At first we show that $l_p \check{\otimes} l_q$ contains the space of all compact operators from l_p to l_p. Indeed,

let $z = \sum_{r=1}^{n} x_r \otimes y_r \in l_p \otimes l_q$, by definition 2 . 5, we have

$$\| z \|_\varepsilon = \sup \left\{ \left| \sum_{r=1}^{n} < X_\gamma , Y >< Y_\gamma , X >; X \in S(l_p); Y \in S(l_q) \right\} \right.$$

On the one hand we can consider z as a finite dimentionel operator from l_p to l_q, by

$$z(x) = \sum_{r=1}^{n} < y_r , x > x_r \text{ , for each } x \in l_p$$

Estime

$$\| z \| = \sup \{ \| z(x) \|_p ; x \in s\ l_p) \}$$

$$= \sup \left\{ \| \sum_{r=1}^{n} < y_r , x > x_r : x \in S(l_p) \right\}$$

$$= \sup \left\{ \left| \sum_{r=1}^{n} < y_\gamma , x >< x_r , y > \right| ; x \in S(l_p); y \in s(l_q) = \right\} = \| z \|_\varepsilon$$

Hence $l_p \check{\otimes} l_q$ contains all finite dimentional operators from l_p to l_p. Thus it contains the space of all compact eperators from l_p to l_p. Therefore in view of corollary 1.6 it is sufficient to show that the space of all compact operators from l_p to l_p contains some subspace isometrie to c_o. Indeed, let $A = (a_n) \in c_o$ and $\| A \|_{c_o} = \sup_n \{ | a_n | \} < \infty$. We can consider A as an operator from l_p to l_p by

$A(x) = (a_n x_n) \in l_p$, where $x = (x_n) \in l_p$. It is easily seen that

$$\| A \| = \sup_n \{ | a_n | \} = \| A \|_{c_o}$$

We show now that A is a compact operator. To see this let define $A_n : l_p \to l_p$ by

$$A_n(x) = (a_1 x_1 , a_2 x_2 ,..., a_n x_n , 0,...) \in l_p .$$

Since a_n tends to zero so A_n tends to A in the operator norm. Finally note that all operators A_n are finite dimentional then A must be compact. It completes the proof of the theorem 2.6.

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