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Том 6 — № 1

ON PERIODIC SOLUTIONS OF A NEUTRAL TYPE EQUATION

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The aim of the present paper is to find sufficient conditions for the existence of a periodic solution of the nonlinear equation

$$\dot{x}(t) = f(t, x(t), x(t-h), \dot{x}(t-h)), \tag{1}$$

where h is a constant deviation.

Problems connected with periodic solutions for differential equations with a deviating argument have been considered by a number of authors, as for example, in [1], [2], [3] and [4].

We will say that conditions (A) are satisfied if:

(A1) The function $f(t, x, y, z)$ is periodic in the variable t with a period $T>0$ and it has continuous first derivatives with respect to x, y, z and is continuous for all (t, x, y, z) of the four-dimensional space.

(A2) There exists a constant $m>0$ such that the following is fulfilled:

$$\left| \int_0^t [f_x(t, \sigma_1(t), \sigma_2(t), \sigma_3(t)) + f_y(t, \sigma_1(t), \sigma_2(t), \sigma_3(t))]\, dt \right| \geqslant m \tag{2}$$

for arbitrary T-periodic functions $\sigma_1(t)$, $\sigma_2(t)$ and $\sigma_3(t)$.

(A3) The functions $f_x(t, x, y, z)$, $f_y(t, x, y, z)$ and $f_z(t, x, y, z)$ satisfy the conditions

$$\begin{gathered} |f_x(t, x, y, z)| \leqslant M, \quad |f_y(t, x, y, z)| \leqslant M \\ |f_z(t, x, y, z)| \leqslant 2M, \quad |f_x(t, x, y, z) + f_y(t, x, y, z)| \leqslant N \end{gathered} \tag{3}$$

where M, N are constants for which the following inequality holds:

$$\frac{MNT^2}{m} + \frac{2MNT}{m} + 2M < 1 \tag{4}$$

THEOREM 1. *Let conditions (A) be satisfied. Then, equation* (1) *has a T-periodic solution* $x(t)=\lim_{n\to\infty} x_n(t)$, *where*

$$x_n(t) = a_n + \varphi_n(t), \tag{5}$$

while the constant a_n *and the functions* $\varphi_n(t)$ *are determined as follows:*

$$\varphi_0(t) = 0, \quad -\infty < t < +\infty \tag{6}$$

$$\int_0^T f(t, a_0, a_0, 0)\, dt = 0 \tag{7}$$

$$\varphi_n(t) = \int_0^T f(s, a_{n-1} + \varphi_{n-1}(s), a_{n-1} + \varphi_{n-1}(s-h), \dot{\varphi}_{n-1}(s-h))\, ds \tag{8}$$

and

$$\int_0^T f(s, a_n + \varphi_n(s), a_n + \varphi_n(s-h), \dot{\varphi}_n(s-h))\, ds = 0 \tag{9}$$

Proof. From the condition (A1), (6), (8) and (9) it follows that all functions $\varphi_n(t)$ are T-periodic.

To prove that there exists a unique solution a_n of the equation (9) for every $n = 0, 1, 2,\ldots$, consider the function

$$\Gamma_n(a) = \int_0^T f(s, a + \varphi_n(s), a + \varphi_n(s-h), \dot{\varphi}_n(s-h))\, ds$$

Further, let us calculate

$$\Gamma'_n(a) = \int_0^T [f_x(s, a + \varphi_n(s), a + \varphi_n(s-h), \dot{\varphi}_n(s-h)) +$$
$$+ f_y(s, a + \varphi_n(s), a + \varphi_n(s-h), \dot{\varphi}_n(s-h))]ds$$

From condition (2) it follows that $|\Gamma'_n(a)| \geqslant m > 0$. Assume for convenience that $\Gamma'_n(a) > 0$. Then, for $a > 0$, $\Gamma_n(a)$ satisfies the inequality $\Gamma_n(a) \geqslant ma + \Gamma_n(0)$, while for $a < 0$ it satisfies the inequality $\Gamma_n(a) \leqslant ma + \Gamma_n(0)$.

Hence obviously there exists a constant $R_n > 0$, such that $\Gamma_n(R_n) > 0$, $\Gamma_n(-R_n) < 0$ for every $n = 0, 1,\ldots$ Since $\Gamma_n(a)$ is a continuous monotonic function, then there exists only one $a_n \in (-R_n, R_n)$, such that $\Gamma_n(a_n) = 0$. Thus, we have established that a_n and $\varphi_n(t)$ $(n = 0,1,\ldots)$ are determined uniquely from the relations (6), (7), (8) and (9).

Now we will prove that the sequence $x_n(t)$ converges uniformly on the segment $[0, T]$.

Introduce the notations

$$\psi_n = \varphi_n(t) - \varphi_{n-1}(t),\ b_n = a_n - a_{n-1},$$
$$\rho_n = \| \psi_n(t) \| = \max_t | \psi_n(t) | + \max_t | \dot{\psi}_n(t) | \tag{10}$$

Estimate $| \Psi_n(t) |$

$$\psi_n(t) = \int_0^t [f_x(s, \sigma_n(s), \tau_n(s), \theta_n(s)) (\psi_{n-1}(s) + b_{n-1}) +$$
$$+ f_y(s, \sigma_n(s), \tau_n(s), \theta_n(s))(\psi_{n-1}(s-h) + b_{n-1}) + \tag{11}$$
$$+ f_2(s, \partial_n(s), \tau_n(s), \theta_n(s)) \dot{\psi}_{n-1}(s-h)]ds = \int_0^t [f_x(s, \tau_n(s), \tau_n(s), \theta_n(s)]\varphi_{n-1}(s) +$$
$$+ f_y(s, \sigma_n(s), \tau_n(s), \theta_n(s)) \psi_{n-1}(s-h)] ds + b_{n-1} \int_0^t [f_x(s, \sigma_n(s), \tau_n(s), \theta_n(s)) +$$
$$+ f_y(s, \partial_n(s), \tau_n(s), \theta_n(s))] ds + \int_0^t f_x(s, \sigma_n(s), \tau_n(s), \theta_n(s)) \dot{\psi}_{n-1}(s-h) ds$$

On the other hand, from (9) we have

$$b_{n-1} \int_0^T [f_x(s, \sigma_n(s), \tau_n(s), \theta_n(s)) + f_y(s, \sigma_n(s), \tau_n(s), \theta_n(s))]ds =$$
$$= -\int_0^T [f_x(s, \sigma_n(s), \tau_n(s), \theta_n(s)) \psi_{n-1}(s) +$$
$$+ f_y(s, \sigma_n(s), \tau_n(s), \theta_n(s)) \psi_{n-1}(s-h))]ds - \tag{12}$$
$$- \int_0^T f_z(s, \sigma_n(s), \tau_n(s), \theta_n(s)) \dot{\psi}_{n-1}(s-h) ds,$$

hence we can write (11) in the form

$$| \psi_n(t) | = \left| \left[1 - \frac{\int_0^t [f_x(s, \sigma_n, \tau_n, \theta_n) + f_y(s, \sigma_n, \tau_n, \theta_n)]ds}{\int_0^T [f_x(s\ \sigma_n, \tau_n, \theta_n) + f_y(s, \sigma_n, \tau_n, \theta_n)]\, ds} \right] \right| \times$$

$$\times \int_0^t [f_x(s,\sigma_n,\tau_n,\theta_n)\psi_{n-1}(s) + f_y(s,\sigma_n,\tau_n,\theta_n)\psi_{n-1}(s-h) +$$

$$+ f_z(s,\sigma_n,\tau_n,\theta_n)\dot{\psi}_{n-1}(s-h)]\,ds - \frac{\int_0^t [f_x(s,\sigma_n,\tau_n,\theta_n) + f_y(s,\sigma_n,\tau_n,\theta_n)]ds}{\int_0^T [f_x(s,\sigma_n,\tau_n,\theta_n) + f_y(s,\sigma_n,\tau_n,\theta_n)]ds} \times$$

$$\times \int_t^T [f_x(s,\sigma_n,\tau_n,\theta_n)\psi_{n-1}(s) + f_y(s,\delta_n,\tau_n,\theta_n)\psi_{n-1}(s-h) +$$

$$+ f_z(s,\sigma_n,\tau_n,\theta_n)\dot{\psi}_{n-1}(s-h)]ds \Bigg| \leqslant \frac{2MN}{m}[(T-t)\,t + t(T-t)]\,\|\psi_{n-1}(t)\|$$

Since the maximum of the function $(T-t)t$ on the segment $[0,T]$ is equal to $T^2/4$, we obtain

$$|\psi_n(t)| \leqslant \frac{MNT^2}{m}\,\|\psi_{n-1}(t)\| \tag{13}$$

Estimate $|\dot{\psi}_n(t)|$. From (8) we get

$$\dot{\psi}_n(t) = f_x(t,\sigma_n,\tau_n,\theta_n)\,\psi_{n-1}(t) + f_y(t,\sigma_n,\tau_n,\theta_n)\,\psi_{n-1}(t-h) +$$

$$+ f_z(t,\sigma_n,\tau_n,\theta_n)\,\dot{\psi}_{n-1}(t-h) + b_{n-1}[f_x(t,\sigma_n,\tau_n,\theta_n) + f_y(t,\sigma_n,\tau_n,\theta_n)]$$

Note that from (12) one can easily obtain the inequality

$$|b_{n-1}| \leqslant \frac{2MT}{m}\,\rho n^{-1} \tag{14}$$

whence

$$|\dot{\psi}_n(t)| \leqslant \left(2M + \frac{2MNT}{m}\right)\rho_{n-1} \tag{15}$$

From (13) and (15) follows the inequality

$$\rho_n \leqslant \left(\frac{MNT^2}{m} + \frac{2MNT}{m} + 2M\right)\rho_{n-1} \tag{16}$$

From (4), (14) and (16) it follows that the sequence a_n is convergent.

Set

$$a = \lim_{n\to\infty} a_n, \quad \varphi(t) = \lim_{n\to\infty} \varphi_n(t), \quad x(t) = \lim_{n\to\infty} (a_n + \varphi_n(t))$$

One can see from (8) that $x(t)$ is a T-periodic solution of equation (1). Thus, the theorem is proved.

REMARK 1. The theorem holds if condition (6) is replaced by: $\varphi_0(t) = \overline{\varphi}(t)$, where $\overline{\mathscr{C}}(t)$ is an arbitrary T-periodic and continuously-differentiable for $t \in (-\infty, +\infty)$ function.

REMARK 2. If $T = h$, the conditions (A) can be weakened, as follows: Conditions (3) and (4) are replaced by

$$|f_x(t, x, y, z) + f_y(t, x, y, z)| \leqslant N, \; |f_z(t, x, y, z)| \leqslant N \tag{3'}$$

$$\frac{\mathcal{N}^2 h}{m} + \frac{(\mathcal{N} h^2)}{2m} + N < 1 \tag{4'}$$

REMARK 3. Every T-periodic solution of the equation (1) can be considered as the limit of the sequence of the type (5), where

$$a_0 = x(0), \; \varphi_0(t) = \int_0^t f(s, x(s), x(s-h), \dot{x}(s-h))ds$$

while a_n and $\varphi_n(t)$ for $n \geqslant 1$ are determined by the relations (8) and (9).

Indeed, in this case we obtain

$$\varphi_n(t) = \varphi_0(t), \; a_n = a_0 \; (n = 1, 2, \ldots,) \; x_n(t) = x(t)$$

and the assumption is obvious.

THEOREM 4. *Let the conditions (A) be satisfied. Then the T-periodic solution of equation (1) is unique.*

Proof. Let $x(t)$ be a T - periodic solution of equation (1), defined by Theorem 2, i. e.,

$$x(t) = \lim_{n \to \infty} x_n(t)$$

where $x_n(t)$ are obtained from (5), (6), (7), (8) and (9).

By $\omega(t)$ denote an arbitrary T — periodic solution of equation (1). Determine the sequence $\omega_n(t) = \tilde{a}_n + \tilde{\varphi}_n(t)$ according to Remark 3.

Consider the difference $\tilde{\psi}_n(t) = \tilde{\varphi}_n(t) - \varphi_n(t)$. It is easy to obtain the estimation

$$|\tilde{\psi}_n(t)| \leqslant \frac{MTN^2}{m} \|\tilde{\psi}_{n-1}(t)\|$$

$$|\tilde{\psi}_n(t)| \leqslant \left(\frac{2MTN}{m} + 2M\right) \|\tilde{\psi}_{n-1}(t)\|$$

whence

$$\|\tilde{\psi}_n(t)\| \leqslant \left(\frac{M\,NT^2}{m} + \frac{2MNT}{m} + 2M\right)^n \|\tilde{\psi}_0(t)\|, \tag{17}$$

$$|\tilde{a}_n - a_n| \leqslant \frac{2MT}{m} \|\tilde{\psi}_n(t)\| \tag{18}$$

From (17) and (18) and using condition (4), we get

$$\left\|\tilde{\psi}_n(t)\right\| \underset{n\to\infty}{\longrightarrow} 0, \quad \left|\tilde{a}_n - a_n\right| \underset{n\to\infty}{\to} 0$$

i. e.,

$$\tilde{\varphi}(t) = \varphi(t), \ \tilde{a} = a, \ x(t) = \omega(t)$$

Thus, the theorem is proved.

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