A FAMILY OF ANALYTIC FUNCTIONS WHOSE MEMBERS HAVE EQUAL NUMBER OF ZEROS

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ABSTRACT: Sufficient conditions are given for $pf(z)+qg(z)$ to have (in a region) the same number of zeros for every nonnegative real number p and q not both zero, where $f(z)$ as well as $g(z)$ is an analytic function.

In what follows every function is complex valued and is defined on a subset of complex numbers.

THEOREM. *Let $f(z)$ and $g(z)$ be functions continuous on a compact set F and analytic in the interior of F such that for every nonnegative real number p and q not both zero,*

(1) $\qquad pf(z)+qg(z) \neq 0$ *for every z on the boundary of F*

Then, for every nonnegative real number p and q not both zero $pf(z)+qg(z)$ has the same number of zeros (counting multiplicities) in the interior of F.

Proof. First, let us observe that (1) implies

(2) $\qquad f(z) \neq 0 \neq g(z) \qquad$ *for every z on the boundary of F*

Next we show that

(3) *if $p>0$ and $q>0$ then in the interior of F,*

$pf(z)+qg(z)$ and $f(z)$ and $g(z)$ have the same number of zeros.

To this end, starting with $pf(z)+qg(z)$, let us consider the sequence $pf(z)+\dfrac{n}{1+n}qg(z)$ of functions, where $n=0, 1, 2, 3, \ldots$ From (1), in view of Hurwitz's theorem [1], it follows that for some $n_0 > 1$

$$pf(z) + qg(z) \text{ and } pf(z) + \frac{n_0}{1+n_0} qg(z)$$

have the same number of zeros in the interior of F.

Again, let us consider the sequence $pf(z) + \frac{n}{1+n} \cdot \frac{n_0}{1+n_0} qg(z)$ of functions, where $n = 0, 1, 2, 3, \ldots$ Again, from (1), in view of Hurwitz's theorem, it follows that for some $n_1 > 0$

$$pf(z) + qg(z) \text{ and } pf(z) + \frac{n_0}{1+n_0} qg(z) \text{ and } pf(z) + \frac{n_1}{1+n_1} \cdot \frac{n_0}{1+n_0} qg(z)$$

have the same number of zeros in the interior of F.

Continuing in the above manner, we see that for $k = 0, 1, 2, 3, \ldots$

(4) $pf(z) + qg(z)$ *and* $pf(z) + \frac{n_k}{1+n_k} \cdots\cdots \frac{n_2}{1+n_2} \cdot \frac{n_1}{1+n_1} \cdot \frac{n_0}{1+n_0} qg(z).$

have the same number of zeros in the interior of F. Clearly, for $k = 0, 1, 2, 3, \ldots$ the products of real numbers appearing in (4) form a strictly decreasing sequence of positive terms which converges to a real number, say, $r_\omega \geq 0$. Let us consider the corresponding limit function:

(5) $pf(z) + r_\omega g(z)$

Since $r_\omega \geqslant 0$, from (1), (2), (4), (5), in view of Hurwitz's theorem, it follows that:

(6) $pf(z) + qg(z)$ *and* $pf(z) + r_\omega g(z)$ *with* $p > 0$

have the same number of zeros in the interior of F.

Now, if in (6) it is the case that $r_\omega = 0$ then $pf(z) + qg(z)$ and $f(z)$ have the same number of zeros in the interior of F. If however, $r_\omega \neq 0$, then we continue the process starting with $pf(z) + r_\omega g(z)$ instead of $pf(z) + qg(z)$. Since there is no nondenumerable strictly decreasing (or increasing) sequence of real numbers (because otherwise there would be nondenumerably many rational numbers) we see that the above process must terminate at a denumerable limit ordinal u with corresponding $r_u = 0$. But then $pf(z) + r_u g(z) = pf(z)$ which implies that $pf(z) + qg(z)$ and $f(z)$ have the same number of zeros in the interior of F (with $p > 0$).

To complete the proof of (3), it is enough to show that if $p > 0$ then in the interior of F it is the case that $pf(z) + qg(z)$ and $g(z)$ have the same number of zeros. To this end we again start with $pf(z)+qg(z)$ but this time we conside the sequence $\frac{n}{1+n} pf(z) + qg(z)$ of functions (instead of the sequence $pf(z) + \frac{n}{1+n} qg(z)$ and proceed as in the above.

Hence, (3) is proved.

But then the conclusion of the Theorem follows readily from (3). Indeed, from (3) it follows (choosing $p=q=1$) that in the interior of F it is the case that $f(z)+g(z)$ and $f(z)$ and $g(z)$ and $pf(z)+qg(z)$ have the same number of zeros for every real number $p>0$ and $p>0$. Thus, $pf(z)+qg(z)$ has the same number of zeros even if one of p and q (but not both) is equal to zero. Hence, $pf(z)+qg(z)$, in the interior of F, has the same number of zeros for every nonnegative real number p and q.

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REFERENCE

1. Saks. S. and Zygmund, A., *Analytic Functions*, Elsevier Pub. Co., New York (1971), 158.