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A CONICAL ALGORITHM FOR SOLVING A CLASS OF COMPLEMENTARITY PROBLEMS

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1. INTRODUCTION

Consider the following problem:

(P) Given a closed convex subset D of R^n and a concave function $f : R^n \to R$ satisfying $f(x) \geqslant 0$ for all $x \in D$, find a point $\bar{x} \in D$ such that $f(\bar{x}) = 0$.

Any solution of this problem must be, of course, an optimal solution of the program

$$\text{Minimize } f(x), \text{ subject to } x \in D \tag{1}$$

The converse, however, is true only if the optimal value of (1) equals 0.

Problem (P) includes as a special case the following *concave complementarity problem*:

Given a concave mapping $w : R^n \to R^n$, find $x \in R^n$ such that

$$x \geqslant 0, \quad y = w(x) \geqslant 0, \langle x,y \rangle = 0 \tag{2}$$

(when w is affine, this is the *linear complementarity problem* extensively studied during the last decade). To convert (2) into a problem (P) it suffices to set

$$D = \{x : x \geqslant 0, w(x) \geqslant 0\}, f(x) = \sum_{i=1}^{n} \min \{x_i, w_i(x)\}$$

In [2 ; 3] we have developed a method for solving the linear complementarity problem via concave programming. In the present paper this approach will be extended further in order to solve (P). The main point of this extension is in a more flexible rule for the bounding operations involved in the iterative procedure, which on one hand saves us the need to sue linear subprograms (to be solved by the simplex method), on the other hand allows the procedure to be applicable to more general situations. Thus, a variant of the new method will be absolutely «simplex free», i.e. does not use in any way the simplex method. When applied to linear programming problems, this variant yields a new iterative algorithm, quite different from the simplex one. Although at the present stage the method has not yet been tested on computers, it seems to offer some interest in view of several potential advantages: 1) it solves the problem whenever the latter is solvable; 2) it involves very simple operations; 3) it allows an easy decomposition of a large problem into many smaller subproblems.

2. DESCRIPTION OF THE ALGORITHM

Assume that an interior point of D is available at the beginning, and (using a translation if necessary) that this point is the origin O of the space. The method we propose for the solution of the problem (P) is an iterative procedure of the branch and bound type. In each step of this procedure a certain collection of cones (with vertex at O) must be examined and three basic operations must be performed:

1) *Select* a cone from this collection;

2) *Split* this cone into two smaller subcones;

3) *Fathom* each newly generated cone M by computing a number $\mu(M)$ such that $\mu(M) > 0$ indicates that $f(x) > 0$ all over $M \cap D$ and hence that M can be discarded from further examination.

We shall specify later the precise rules for these basic operations. Assuming for the moment that these rules have been defined, the algorithm can be described as follows.

Take an n simplex $T = [s^1, \ldots, s^{n+1}]$ in R^n with barycentre at 0 and for each $j = 1, \ldots, n+1$ let $M_{0,j}$ be the polyhedral convex cone generated by n halflines from 0 through s^l with $l \neq j$; let $x^{0,j}$ be the point achieving the minimum (if it exists) of $f(x)$ over the common part of D with the halfline from 0 through s^j.

Initialization. Set $x^0 = \arg\min \{f(x^{0,j}) : j = 1, \ldots, n+1\}$; $\mathcal{M}_0 = \{M_{0,1}, \ldots, M_{0,n+1}\}$. Compute $\mu(M_{0,j})$ for each $j = 1, \ldots, n+1$.

Step $k = 0,1,\ldots$ If $f(x^k) = 0$, stop: x^k is a solution of (P). Otherwise, delete all cones $M \in \mathcal{M}_k$ with $\mu(M) > 0$. Let $\mathcal{R}_k'$ be the set of remaining cones.

If $\mathcal{R}_k = \emptyset$, stop: the problem (P) has no solution. Otherwise, select a cone $M_k \in \mathcal{R}_k$ and split it into two subcones $M_{k,1}$, $M_{k,2}$. For each $j = 1,2$ compute $\mu(M_{k,j})$. These operations generate some new points of D. Let x^{k+1} be the new current best point, i.e. the point that achieves the smallest value of $f(x)$ among x^k and all newly generated points of D. Form $\mathcal{M}_{k+1} = (\mathcal{R}_k \setminus \{M_k\}) \cup \{M_{k,1}, M_{k,2}\}$ and go to the next step $k+1$.

Understandably, the convergence as well as the efficiency of the above Algorithm crucially depends upon the concrete rules for selecting and splitting M_k and for fathoming the newly generated cones $M_{k,1}$ and $M_{k,2}$. We proceed to describe these rules in the next sections.

3. SELECTING M_k

Let M be any cone generated during the procedure. So M is a cone with vertex at 0 and exactly n edges passing through n points $v^1,\ldots, v^n$ of some facet of the simplex $T = [s^1,\ldots, s^{n+1}]$. Let

$$\theta_j = \sup \{\theta : f(\theta v^j) > 0\} \tag{3}$$

$$\rho(M) = \min \{\theta_1,\ldots, \theta_n\} \tag{4}$$

Then, as can be easily seen, $\rho(M) > 0$ and

$$\rho(M) = \sup \{\theta : f(x) > 0 \quad \forall x \in M \cap \theta . T\} \tag{5}$$

and hence, $M' \subset M$ implies $\rho(M') \geqslant \rho(M)$.

Rule for selecting M_k:

$$M_k = \arg\min \{\rho(M) : M \in \mathcal{R}_k\} \tag{6}$$

PROPOSITION 1. *Suppose that the above rule is adopted for selecting the cone* M_k *in each step* k. *If the Algorithm generates an infinite subsequence of cones* $\{M_{k_q}\}$ *tending to a halfline* Γ *contained in* D *then the problem* (P) *has no solution.*

(We say that a sequence of cones M_{k_q} tends to a halfline Γ if any cone containing $\Gamma \setminus \{\theta\}$ in its interior contains all M_{k_q} with sufficiently large q).

Proof. It will suffice to show that, given any positive number N, we have $f(x) > 0$ for all $x \in D$ such that $\|x\| < N$. To this end, observe that, the function f being concave and bounded below by 0 on Γ, its minimum over Γ is attained at 0 (see e.g. [1], section 32). That is, $f(x) \geq f(0) > 0$ for all $x \in \Gamma$. Take $\theta > 0$ so large that the ball $\|x\| < N$ is contained in the simplex θT. Let U be a ball around a point $c \in \Gamma$, such that $x \notin \theta T$ and $f(x) > 0$ for all $x \in U$. Then for all q large enough M_{k_q} lies inside the cone generated by 0 and U. Consequently the j-th edge of M_{k_q} meets U at some point $x^{q,j}$ with $f(x^{q,j}) > 0$. Since $x^{q,j} \notin \theta T$, we must have $\rho(M_{k_q}) > \theta$ and hence, by the rule (6), $\rho(M) > \theta$ for all $M \in \mathcal{R}_{k_q}$. Now, if $x \in D$ and $\|x\| < N$, then either x belongs to some cone already deleted in a step $h \leq k_q$ or x belongs to some cone $M \in \mathcal{R}_{k_q}$: in both cases, $f(x) > 0$, because $\rho(M) > \theta$ and $x \in \theta T$. □

4. SPLITTING M_k

We shall use the same *bisection rule* as was used in [4]. More specifically, let $v^{k,1},\ldots, v^{k,n}$ be the intersections of the n edges of M_k with the boundary of the simplex T. Choose the longest side of the simplex $S_k = [v^{k,1},\ldots, v^{k,n}]$, say $[v^{k,j1}, v^{k,j2}]$, and let u^k be the midpoint of this side. Then, for each $h = 1,2$ take $M_{k,h}$ to be the cone whose set of edges obtains from that of M_k by substituting the halfline from O through u^k for the edge passing through $v^{k,jh}$.

It is immediate that $M_k = M_{k,1} \cup M_{k,2}$. In [4] we have established the following important property of this splitting method:

PROPOSITION 2. *If the Algorithm is infinte, then the diameter of the above defined simplex S_k tends to zero. Hence, any infinite decreasing subsequence of cones $\{M_{k_q}\}$ tends to a halfline emanating from O.*

(By a «decreasing sequence $\{M_{k_q}\}$ «we mean a sequence such that $M_{k_{q-1}} \supset K_{k_q}$)

We shall say that a cone M is of *first* or *second category* according to whether $M \cap D$ is bounded or not. From the previous Propositions we can draw the following

COROLLARY. *Suppose that the rule* (6) *and the bisection rule are used for selecting and splitting* M_k, *respectively. If the Algorithm generates infinitely many cones of seond category, then the problem* (P) *has no solution.*

Proof. Suppose that the Algorithm generates infinitely many cones of second category. Then one can select an infinite decreasing subsequence of cones M_{k_q} such that each M_{k_q} contains infiniiely many cones of second category. By Proposition 2, this subsequence of cones tends to a halfline, Γ. It is easy to see that Γ must be contained in D. Indeed, otherwise there would exist a ball U around a point $c \in \Gamma$ such that $U \cap D = \phi$. For all q large enough every halfline in M_{k_q} emanating from O would meet U. In view of the convexity of D this would imply that the intersection of D with every such halfline is a segment, hence that $M_{k_q} \cap D$ is bounded, conflicting with the fact that M_{k_q} contains cones of second category. Thus, Γ is contained in D and so, by Proposition 1, the problem (P) has no solution. ☐

5. FATHOMING THE NEWLY GENERATED CONES

By the previous Corollary, if the problem (P) is solvable, then using rule (6) and the bisection rule the Algorithm will be finite or will generate, after a certain number of steps, only cones of first category. In the latter case there exists an infinite decreasing subsequence of cones $\{M_{k_q}\}$ of first category. By Proposition 2 the intersection of this subsequence is a halfline Γ and since each set $M_{k_q} \cap D$ is unbounded, the set $\Gamma \cap D$ is a segment $[0,z^*]$ with $z^* \neq 0$ (because $0 \in \operatorname{int} D$). It turns out that using an appropriate rule for fathoming the newly generated cones in each step it is possible to guarantee that for any such subsequence M_{k_q} we have $f(z^*) = 0$, i.e. z^* is a solution.

To describe this rule, let M be a newly generated cone, with n edges meeting some facet of T at $v^1,\dots, v^n$. Consider the θ_j defined by (3).

1) If $\theta_j = +\infty$ for all $j = 1,\dots, n$, then $f(x) > 0$ for all $x \in M$. Hence, in that case we set

$$\mu(M) > 0.$$

2) In the contrary case, $\theta_j < +\infty$ for some j and since $f(x) \geqslant 0$ for all $x \in D$, the j-th edge must cut the boundary ∂D of D. So at least one point of

$M \cap \partial D$ is available, for example the intersection of ∂D with the j-th edge of M. We then take an arbitrary point $z \in M \cap \partial D$ and an arbitrary supporting hyperplane H of D at z. Let H^+ be the halfspace determined by H that contains D. Since $M \cap D \subset M \cap H^+$ we can obviously set

$$\mu(M) \begin{cases} > 0 \text{ if } \inf\{f(x): x \in M \cap H^+\} > 0 \\ = 0 \qquad \text{otherwise} \end{cases} \tag{7}$$

where the number $\inf\{f(x): x \in M \cap H^+\}$ is easy to estimate because $M \cap H^+$ is a polyhedral convex set of simple structure.

More precisely, suppose that D is given by a system of constraints of the form:

$$g_i(x) \geqslant 0 \qquad (i = 1,...,m) \tag{8}$$

where each $g_i: R^n \to R$ is a concave function. Since $0 \in \text{int } D$ we must have $g_i(0) > 0$ for all $i = 1,...,m$. If j is such that $\theta_j < +\infty$ and $\lambda_j = \sup\{\lambda > 0: g_i(\lambda v^j) \geqslant 0 \; \forall i\}$, then $z = \lambda_j v^j \in M \cap \partial D$, and we have $g_i(z) = 0$ for some i. Let t be a subgradient of $-g_i(x)$ at point z. Then $g_i(x) - g_i(z) = g_i(x) \geqslant 0$ implies $\langle t, z - x\rangle \geqslant 0$, and so the equation $\langle t, z - x\rangle = 0$ defines a supporting hyperplane H for D at z, such that $H^+ = \{x: \langle t, z - x\rangle \geqslant 0\}$. Since $0 \in \text{int } D \subset \text{int } H^+$ it follows that

$$\langle t, z\rangle > 0. \tag{9}$$

Let us distinguish two subcases:

2a) $\langle t, v^j\rangle > 0$ for all $j = 1,..., n$. Then for each j the j-th edge of M cuts H at $z^j = \zeta_j v^j$ with

$$\zeta_j = \langle t,z\rangle / \langle t,v^j\rangle.$$

So $M \cap H^+$ is a simplex with vertices $0, z^1,..., z^n$ and from the concavity of the function f we have

$f(x) \geqslant \min\{f(0), f(z^1), ..., f(z^n)\}$ for all $x \in M \cap H^+$. Noting that $f(0) > 0$, we can set

$$\mu(M) \begin{cases} > 0 \qquad \text{if } f(z^j) \text{ for all } j; \\ = 0 \qquad \text{otherwise} \end{cases} \tag{10}$$

2b) $\langle t, v^j\rangle \leqslant 0$ for at least some j. Then the j-th edge of M lies in $M \cap H^+$ which is thus unbounded. In this case we set

$$\mu(M) = 0.$$

PROPOSITION 3. *Suppose that the above rule is used for fathoming the newly generated cones in each step. If the Algorithm generates an infinite decreasing subsequence of cones M_{k_q} such that $\left(\bigcap_{q=1}^{\infty} M_{k_q}\right) \cap D = [0, z^*]$ with $z^* \neq 0$, then $f(z^*) = 0$, i.e. z^* is a solution. Furthermore, if for each M we denote by $z(M)$ the point $z \in M \cap \partial D$ constructed for M as indicated above, then $z(M_{k_q}) \to z^*$.*

Proof. Let v^* (v^{*q}, resp.) be the point where the halfline from O through z^* ($z(M_{k_q})$, resp.) cuts the boundary of T, and let $z^* = \xi^* v^*$, $z(M_{k_q}) = \xi_{*q} v^{*q}$. Clearly the sequence M_{k_q} tends to the halfline Γ from O through z^*, so that $v^{*q} \to v^*$. If φ is the gauge of the convexe set D then $\varphi(\xi^* v^*) = 1$, hence $\xi^* = 1/\varphi(v^*)$ and similarly, $\xi_{*q} = 1/\varphi(v^{*q})$. But φ being continuous, it follows that $\xi_{*q} \to \xi^*$ and hence, $z(M_{k_q}) \to z^*$. Now observe that, since $M_{k_q} \in \mathcal{R}_{k_q}$, we have $\mu(M_{k_q}) = 0$. Denote by $v^{q,j}$, t^q the vectors v^j, t constructed for $M = M_{k_q}$. By taking a subsequence if necessary we may assume that $t^q/\|t^q\| \to t^*$, $v^{q,j} \to v^{*j}$ as $q \to +\infty$. Since $\langle t^q, z(M_{k_q}) - x\rangle \geq 0$ for all $x \in D$, it follows that $\langle t^*, z^* - x\rangle \geq 0$ for all $x \in D$. In view of the fact $0 \in \text{int } D$ this implies $\langle t^*, z^*\rangle > 0$. Hence, $\langle t^*, v^*\rangle > 0$, so that for all q large enough $\langle t^q, v^{q,j}\rangle > 0$ for all $j = 1, \dots, n$. Therefore, $\mu(M_{k_q})$ is computed according to (10) where $z^j = z^{q,j} = \xi_{q,j} v^{q,j}$ with

$$\xi_{q,j} = \langle t^q, z(M_{k_q})\rangle / \langle t^q, v^{q,j}\rangle.$$

Clearly, $\xi_{q,j} \to \langle t^*, z^*\rangle / \langle t^*, v^*\rangle = \xi^*$, hence $z^{q,j} \to z^*$. But from (10) and the relation $\mu(M_{k_q}) = 0$ it follows that for each q large enough there is $j = j(q)$ such that $f(z^{q,j}) \leq 0$. Consequently, $f(z^*) \leq 0$, which implies $f(z^*) = 0$, because $f(x) > 0$ for all $x \in D$. □

REMARK 1. In the above method for computing $\mu(M)$ there is a large freedom in the choice of the point $z = z(M) \in M \cap \partial D$ and the supporting hyperplane H.

For example, if $\theta_j < +\infty$ for all $j = 1, \dots, n$, so that every j-the dge of M cuts ∂D at some z^j, then one can choose any of these z^j as z. Often it is more efficient to take z to be the point where ∂D cuts the halfline from O through the barycentre $v = (v^1 + ... + v^n)/n$ of the simplex $[v^1, ..., v^n]$. Once z has been chosen, there are in general several possible choices for the supporting hyperplane H. Of course, using this flexibility but without spending too much effort, one could try to get $\mu(M) > 0$ whenever possible and thus quickly discard M if this cone is irrelevant.

6. CONVERGENCE OF THE ALGORITHM

From the previous results we can now derive the following convergence theorems.

Suppose that the above rules for selecting and splitting M_k and for fathoming the newly generated cones are applied in the Algorithm described in section 2.

THEOREM 1. *If the problem* (P) *is solvable, then either the Algorithm terminates after finitely many steps, yielding a solution, or it generates a sequence* $\{x^k\} \subset D$ *such that* $f(x^k) \searrow 0$. *In the latter case the sequence* $z(M_k)$ *has at least one cluster point and every such cluster point is a solution.*

Proof. If the problem is solvable, then, as seen at the beginning of the previous section, the sequence M_k, whenever infinite, contains an infinite decreasing subsequence M_{k_q} such that $(\bigcap_{q=1}^{\infty} M_{k_q}) \cap D$ is a line segment $[0,z^*]$ with $z^* \neq 0$. Then by Proposition 3, $z(M_{k_q}) \to z^*$ and z^* is a solution. Furthermore, we have from the definition of x^k : $f(x^{k_q}) \leqslant f(z(M_{k_q}))$, hence $f(x^{k_q}) \searrow 0$. Since the sequence $f(x^k)$ is nonincreasing, we conclude $f(x^k) \searrow 0$.

THEOREM 2. *For any two given positive numbers* ε, N, *the Algorithm either finds after finitely many steps an* ε*-approximate solution*, i.e. *a point* $x^k \in D$ *such that* $f(x^k) < \varepsilon$, *or establishes after finitely many steps that the problem has no solution in the ball* $\|x\| \leqslant N$.

Proof. If the problem is solvable, the first alternative holds by the previous Theorem. If the problem is unsolvable, the sequence M_k whenever infinite, contains a decreasing subsequence M_{k_q} tending to a halfline Γ. We must have $\Gamma \subset D$ (otherwise, $\Gamma \cap D$ would be a line segment and the problem would have nosolution by Proposition 3). Therefore, as seen in the proof of Proposition 1, for any given $\theta > 0$ there is k_q such that $\rho(M) > \theta$ for all $M \in \mathcal{R}_{k_q}$. In particular, there is k_q such that for all $M \in \mathcal{R}_{k_q}$ the ball $\| x \| < N$ is contained in the simplex $\rho(M).T$: then, at this step k_q we are assured that $f(x) > 0$ for all $x \in D$ satisfying $\| x \| < N$. □

7. CASE WHERE NO INTERIOR POINT OF D IS AVAILABLE

The above Algorithm supposes that an interior point of D is known. In some practical cases such a point is actually readily available or easy to find. There are, however, circumstances where the search for an interior point of D may be hard. Therefore, the question of how to overcome the difficulty if no interior point of D is available is worth discussing.

Fortunately, at least for the important special case of the concave complementarity problem (2), this difficulty can be easily overcome at the cost of an extra dimension.

Indeed, it is readily seen that the problem (2) is equivalent to the following one:

Find $(x, x_{n+1}) \in R^n \times R$ such that

$$x_i \geqslant 0 \ (i = 1, \ldots, n+1), \ w_i(x) + x_{n+1} \geqslant 0 \ (i = 1, \ldots, n) \tag{11}$$

$$\sum_{i=1}^{n} \min \{x_i , w_i(x) + x_{n+1} \} + x_{n+1} = 0 \tag{12}$$

This is obviously a special problem (P). Given any $x > 0$, we can always choose an x_{n+1} so that (x, x_{n+1}) satisfies all conditions (11) as strict inequalities, i.e. so that (x, x_{n+1}) is an interior point of the constraint set defined by (11). Therefore, the above Algorithm can be applied, starting from this interior point.

If we know a feasible point, then to avoid an extra dimension one can also proceed as follows.

Observe first that all the previous results remain valid if instead of assuming the function f to be nonnegative on the constraint set D we assume only that f is bounded below on this set. Now, let us slightly relax the constraints (8) and consider instead of D the set D_ε defined by system

$$g_i(x) + \varepsilon \geqslant 0 \ (i = 1, \ldots, m).$$

where ε is a small positive number. Clearly, any point of D is an interior point of D_ε, so if the function f is still bounded below on D_ε (as it is the case with the function f in the concave complementarity problem (2), then, starting from a point of D, we can apply the previous Algorithm to find a zero of f in D_ε : this will provide an approximate solution to the original problem.

8. CASE WHERE THE CONSTRAINTS ARE LINEAR

A special case deserving detailed consideration is that in which the constraints are linear, i.e. have the form

$$g_i(x) = \langle a^i, x\rangle + \beta_i \geq 0 \ (i = 1,\dots, m) \tag{13}$$

It turns out that in this case the method can be applied even if the origin O is only a boundary point of D.

Suppose first that O is a *nondegenerate extreme point of* D, i.e. an extreme point incident to exactly n edges of D. Then we can take the initial set $\mathcal{M}_0$ to consist of a single cone, viz. the cone M_0 with vertex at O generated by the n edges of D incident to O. Indeed, for this it suffices to choose first n vertices $s^1,\dots, s^n$ of the simplex $T = [s^1,\dots, s^{n+1}]$ to be the extreme points of D adjacent to O (if some edge of D incident to O is unbounded, the corresponding s^j may be taken to be any point distinct from O on this edge). Then the cones $M_{O,j}$ (see section 2) with $j = 1,\dots, n$ are immediately discarded, because $M_{O,j} \cap D = \{O\}$. So the only cone that must be explored is $M_0 = M_{O,\, n+1}$.

With $\mathcal{M}_0$ chosen in that way it is easy to verify that all the operations described in section 4 for fathoming a cone make sense. In more details let M be the cone to be fathomed and assume that $\theta_j < \infty$ for at least one j (see section 4, case 2). Then a point $z = z(M) \in M \cap \partial D$ is available which is distinct from O. In section 4 the inequality (9) was secured by the hypothesis $O \in \operatorname{int} D$. In the present case this is secured by the following

LEMMA 1. *Assume that* $[O, z]$ *with* $z \neq 0$ *is the intersection of the polyhedral set (13) with a halfline emanating from* O. *Then there is an* $i \in \{1,\dots, m\}$ *for which* $g_i(z) = 0 < g_i(0)$, *i.e.* $\langle a^i, z\rangle + \beta_i = 0$, *while* $\beta_i > 0$.

Proof. Denote $I = \{i : \langle a^i, z\rangle + \beta_i = 0\}$. Since $0 \in D$, we must have $\beta_i \geq 0$ $(i = 1,\dots, m)$. If $\beta_i = 0$ for all $i \in I$, then for some $\theta > 1$ we would have

$\langle a^i, \theta z\rangle + \beta_i = 0$ ($i \in I$), while $\langle a^i, \theta z\rangle + \beta_i \geqslant 0$ ($i \notin I$) (since $\langle a^i, z\rangle + \beta_i > 0$ for $i \notin I$). This would mean $\theta z \in D$, contradicting the hypothesis. □.

Thus, for some i we have $\langle a^i, z\rangle + \beta_i = 0, \beta_i > 0$. Then, taking $t = -a^i$ (which is the only subgradient of $-g_i(x) = -\langle a^i, x\rangle - \beta_i$ at point z), we are assured that $\langle t, z\rangle = \beta_i > 0$, i. e. (9) holds, and hence, we can proceed further just in the same way as we did in section 4.

Of course we still have to show that Proposition 3 remains valid. But upon close scrutiny it is easily seen that the basic points in the proof of this proposition are the following:

1) Whenever $\left(\bigcap_{q=1}^{\infty} M_{k_q}\right) \cap D = [0, z^*]$ with $z^* \neq 0$, then $z(M_{k_q}) \to z^*$;

2) any cluster point t^* of the sequence $t^q / \| t^q \|$ satisfies $\langle t^*, v^*\rangle > 0$ (the notations being the same as in the proof of Proposition 3)

In section 4 both these points are secured by the hypothesis $0 \in \operatorname{int} D$. In the present case, these points follow from the linearity of the constraints. In fact we have the following

LEMMA 2. *Let $v^* \neq 0$ be a point such that the intersection of the halfline from O through v^* with the polyhedral set D defined by (13) is a line segment $[0, z^*]$ with $z^* \neq 0$. For every neighbourhood U of z^* there is a neighbourhood V of v^* such that whenever $v \in V$ the intersection of the halfline from O through v with D is either $\{0\}$ or a line segment $[0, x]$ with $x \in U$.*

(A proof this Proposition can be found in [3])

Since $z(M_{k_q}) \neq 0$ (which can be easily seen from the fact that M_{k_q} is a subcone of M_0), it follows from the above Lemma that $z(M_{k_q}) \to z^*$. Furthermore, if $t^q / \| t^q \| \to t^*$, then, since $t^q = -a^i$ for some $i = i(q)$, we may assume $i(q) = i^*$ (constant) for all q, hence $\langle t^q, z(M_{k_q})\rangle = \beta_{i^*} > 0$, so that $\langle t^*, z^*\rangle = \beta_{i^*} > 0$. This proves points 1) and 2) and there by Proposition 3 and Theorems 1 and 2 for the case where O is a nondegenerate extreme point of D (if O is a degenerate extreme point of D then a slight perturbation will make it nondegenerate).

9. ALTERNATIVE VARIANT FOR THE CASE OF LINEAR CONSTRAINTS

Besides the above basic algorithm, one can also indicate several other variants for the case where D is a polyhedral set given by (13).

I. Suppose that O is a *boundary but not an extreme point* of D. Then in carrying out the fathoming operation as indicated above it may occur that, in the case where $\theta_j < +\infty$ for some j (case 2 in section 4), no point $z \in M \cap \partial D$ distinct from O is available and so we cannot determine the hyperplane H by the previous method. To circumvent this difficulty, we can proceed as follows,

Computation of $\mu(M)$,

1) If $\theta_j = +\infty$ for all $j = 1,\ldots, n$, then, as before, set $\mu(M) > 0$,

2) Otherwise, consider two subcases:

2a) If there is some $i = 1,\ldots, m$ such that for all $j = 1,\ldots, n$: $\langle a^i, v^j \rangle < 0$ and $f(\zeta_j v^j) > 0$, where $\zeta_j = -\beta_i / \langle a^i, v^j \rangle$, set $\mu(M) > 0$;

2b) Otherwise, set $\mu(M) = 0$.

It is easy to see that $\mu(M) > 0$ actually implies $f(x) > 0$ for all $x \in M \cap D$. Indeed, in case 2a) the hyperplane $H = \{ x : \langle a^i, x \rangle + \beta_i = 0\}$ cuts the edges of the cone M at $z_j = \zeta_j v^j$ $(j = 1,\ldots,n)$, so the set $M \cap D$ is contained in the simplex $[0, z^1,\ldots, z^n]$. Since $f(0) > 0$, if we have $f(z^j) > 0$ for all j then from the concavity of f it follows that $f(x) > 0$ for all $x \in M \cap D$.

PROPOSITION 4. *The constraints being* (13), *suppose that* $0 \in D$ *and the above rule for computing* $\mu(M)$ *is applied. If the Algorithm generates an infinite decreasing subsequence of cones* M_{k_q} *such that* $\left(\bigcap_{q=1}^{\infty} M_{k_q} \right) \cap D = [0, z^*]$ *then* $f(z^*) = 0$, i.e. z^* *is a solution.*

Proof. Let $z^* = \zeta^* v^*$, where v^* denotes the intersection of the halfline $\Gamma = \bigcap_{q=1}^{\infty} M_{k_q}$ with the boundary of T. If $z^* = 0$ then $\theta v^* \notin D$ for all $\theta > 0$, hence there is some i such that $\beta_i = 0$, $\langle a^i, v^* \rangle < 0$; if $z^* \neq 0$ there is by Lemma 1 some i such that $\langle a^i, z^* \rangle + \beta_i = 0$ and $\beta_i > 0$, hence $\langle a^i, \zeta^* v^* \rangle = -\beta_i < 0$. Thus, in either case there is some i such that $\langle a^i, v^* \rangle < 0$,

$\zeta^* = -\beta_i / \langle a^i, v^* \rangle$. Since $v^{q,j} \to v^*$, we must have, when q is large enough, $\langle a^i, v^{q,j} \rangle < 0$ for all j. The equality $\mu(M_{k_q}) = 0$ then implies the existence of an index $j = j(q)$ such that $f(\zeta_{q,j} v^{q,j}) < 0$, where $\zeta_{q,j} = -\beta_i / \langle a^i, v^{q,j} \rangle$. Obviously, $\zeta_{q,j} \to \zeta^*$, $\zeta_{q,j} v^{q,j} \to \zeta^* v^* = z^*$. Hence, $f(z^*) = 0$, because f is nonnegative on D. □

From this Proposition, we deduce:

THEOREM 3. *The constraints being (13), suppose that $O \in D$ and the above rule for computing $\mu(M)$ is applied. If the problem (P) is solvable, then the Algorithm either finds a solution after a finite number of steps or generates an infinite decreasing subsequence of cones M_{k_q} such that $\left(\bigcap_{q=1}^{\infty} M_{k_q}\right) \cap D = [0, z^*]$, where z^* is a solution.*

II. The previous method for computing $\mu(M)$ is simple enough, but does not guarantee that $f(x^k) \searrow 0$ when the problem is solvable. Although we actually have $z^{q,j} = \zeta_{q,j} v^{q,j} \to z^*$ whenever $\left(\bigcap_{q=1}^{\infty} M_{k_q}\right) \cap D = [0, z^*]$, we are not sure that $z^{q,j} \in D$, and so a subsequencce $z(M_{kq}) \in D$ with $z(M_{kq}) \to z^*$ may fail to exist. Another method which guarantees the latter result even in the case where 0 is only a boundary point of D is the following.

Consider the hyperplane K passing through all $y^j = \theta_j v^j$ with $\theta_j < +\infty$ and parallel to each v^j with $\theta_j = +\infty$, and let H be the supporting hyperplane of $M \cap D$ that is parallel to K. Then the point $z = z(M_{k_q})$ where H touches $M \cap D$ is determined by solving the auxiliary linear subprogram:

Maximize $\sum_{j \in J} p_j / \theta_j$, subject to $Bp \in D$, $p \geqslant 0$, (14)

where $J = \{j : \theta_j < \infty\}$, B is the $n \times n$-matrix with columns $v^1, \ldots, v_n$ and p is a column vector with components $p_1, \ldots, p_n$. Set $\mu(M) > 0$ if z lies on the side of K containing 0 (i.e. if the optimal value of (14) is $\leqslant 1$), $\mu(M) = 0$ otherwise.

This method, which has been used in our earlier work [3] (the interested reader is referred to this work for all the relevant details), yields in general a better value for $\mu(M)$ than the previous method. Furthermore, it provides in any event a point $z = z(M) \in M \cap \partial D$ such that $z(M_{k_q}) \to z^*$ for any sequence

M_{k_q} as mentioned in Theorem 3. Hence, under the conditions stated in this Theorem, $f(x^k) \searrow 0$ whenever the problem is solvable. However, these advantages are achieved at the expense of a great deal of computational efforts (preliminary experiments with this kind of algorithms have shown that most of the computational time is expected to be spent solving the auxiliary sub-programs).

III. Suppose that the polyhedral set D contains no line (as is the case with the concave complementarity problem). In this case the Algorithm can be made finite by the following procedure. In each step k of the current x^k is not an extreme point of D, one can find a vector $u \neq 0$ such that $x^k \pm u$ still lies in D (u is a nonzero vector of the linear manifold determined by the constraints that are binding for x^k). Let us replace x^k by the point that achieves the minimum of f over the lines segment (or halfline) $D \cap \{x^k + \theta u: -\infty < \theta < +\infty\}$. Repeating this operation as many times as necessary we shall ultimately reach an extreme point $\bar{x}^k$ of D such that $f(\bar{x}^k) \leqslant f(x^k)$. Thus, by performing if necessary some additional computations we may assume that the current best feasible point x^k in each step is an extreme point of D. In the case of solvability, since $f(x^k) \searrow 0$ and the number of extreme points is finite, the Algorithm will necessarily terminate after finitely many steps with an extreme point x^k such that $f(x^k) = 0$.

10. CONCLUDING REMARKS

In conclusion, let us mention some particular features of the method developed above:

1) The method solves the problem (P) (in particular, the concave complementarity problem (2)) whenever the problem is solvable, but is not able to decide in a finite number of steps whether or not the problem is solvable. In a finite number of steps the method can only decide whether or not the problem has a solution in a bounded region (as large as we please, but given before hand).

2) The basic operation throughout the computational process is the search for the minimum of a concave function over a halfline (or line segment), which is an easy one-dimensional search. Round-off errors accumulation is not expected to be a major problem for this method, because the calculations in each step do not explicitly depend on the results of the previous step.

3) In constrast with many methods of nonlinear programming, this method does not necessarily involves solving linear subproblems. In the special case of the concave (in particular, linear) complementarity problem it is even not necessary to know a feasible point in order to start the algorithm. Thus, by applying this method to linear programming problems we shall get a new iterative algorithm quite different from the simplex method.

4) At any step k, the problem can be easily decomposed into smaller subproblems, since for each cone M in $\mathcal{R}_k$ the subproblem of finding a zero of f in $M \cap D$ can be solved autonomously. This should allow an efficient use of several parallel computers for solving the problem.

We hope to be able to report computational experience with the method presented above in a subsequent paper.

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