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A REPRESENTATION THEOREM FOR ALMOST SURELY CONVERGENT SEQUENCES OF MULTIFUNCTIONS

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§ 1. INTRODUCTION

Let $(\Omega, \mathcal{A}, P)$ be a probability space and $\mathcal{K}$ the family of all closed bounded non-empty subsets of a Polish space $(\mathcal{B}, \rho)$. Thus $(\mathcal{K}, h)$ becomes a complete metric space with the usual Hausdorff's metric h, defined as follows

$$h(X, Y) = \max \{\sup_{y \in Y} d(X, y), \sup_{x \in X} d(Y, x)\} \quad (X, Y \in \mathcal{K}) \tag{1.1}$$

A multifunction $X : \Omega \to \mathcal{K}$ is said to be weakly $\mathcal{A}$-measurable $(X \in \mathcal{M}(\mathcal{K}, \mathcal{A}))$ if $\{\omega, X(\omega) \cap V \neq \phi\} \in \mathcal{A}$ for every open subset V of $\mathcal{B}$. A function $f : \Omega \to \mathcal{B}$ is called an $\mathcal{A}$-measurable selector of $X \in \mathcal{M}(\mathcal{K}. \mathcal{A})$, $(f \in S_X(\mathcal{A}))$ if $f(\omega) \in X(\omega)$ for all ω and $f^{-1}(V) \in \mathcal{A}$ for every open subset V of $\mathcal{B}$

In [2], Kuratowski and Ryll-Nardzewski proved the following general theorem on selectors

Theorem 1. For each $X \in \mathcal{M}(\mathcal{K}, \mathcal{A})$, the set $S_X(\mathcal{A})$ is non-empty. Castaing [1] generalised this result as follows

Theorem 2. $X \in \mathcal{M}(\mathcal{K}, \mathcal{A})$ if and only if there is a sequence $\langle f^i \rangle_{i=1}^{\infty}$ in $S_X(\mathcal{A})$ such that $X(\omega) = cl\{f^i(\omega);\ i \geqslant 1\}$ for all ω, where cl denotes the closure operator.

Using these results we have obtained, recently, the following theorem in]3].

Theorem 3. Let $X, Y \in \mathcal{M}(\mathcal{K}, \mathcal{A})$ and $\varphi : \Omega \to (0, \infty)$ an $\mathcal{A}$-measurable positive function. Then

$$\forall f \in S_x(\mathcal{A}) \exists g \in S_y(\mathcal{A})\ \rho(f(\omega), g(\omega)) \leqslant h(X(\omega), Y(\omega)) + \varphi(\omega) \forall \omega.$$

Main purpose of the note is to prove the following result.

Theorem 4. Let $\langle X_n \rangle$ be a sequence in $\mathcal{M}(\mathcal{K}, \mathcal{A})$. Then $\langle X_n \rangle$ is almost surely convergent to some element $X_\infty \in \mathcal{M}(\mathcal{K}, \mathcal{A})$

i.e.
$$\lim_{n \to \infty} h(X_n(\omega), X_\infty(\omega)) = 0, \text{ a.e.}$$

if and only if there is a countable number of sequences:

$$\langle f_1^i \rangle_{i=1}^{\infty}, \langle f_2^i \rangle_{i=1}^{\infty}, \dots \langle f_\infty^i \rangle_{i=1}^{\infty} \text{ such that}$$

1) $\langle f_n^i \rangle_{i=1}^{\infty} \subset S_{X_n}(\mathcal{A})$; $X_n(\omega) = cl(\{f_n^i(\omega),\ i \geqslant 1\}) \forall \omega, \forall n = 1, 2, \dots, \infty$

and

2) $\lim_{n\to\infty} \rho(f_n^i(\omega), f_\infty^i(\omega)) = 0$, a.e., uniformly in $i = 1, 2, \ldots$;

§ 2. PROOF OF MAIN RESULT

(Necessarity). Fix a positive integer m, Since $X_m \in \mathcal{M}(\mathcal{K}, \mathcal{A})$ then by the Castaing's representation theorem 2 there is a sequence $\langle g_m^j \rangle_{j=1}^{\infty}$ such that

$$\langle g_m^j \rangle_{j=1}^{\infty} \subset S_{x_m}(\mathcal{A}) \text{ and } X_m(\omega) = \mathrm{cl}(\{g_m^j(\omega),\ j \geqslant 1\})\ \forall \omega \tag{2.1}$$

Further, thanks to therem 3 there is a sequence $\langle p_\infty^{m,j} \rangle_{j=1}^{\infty}$ such that

$$\langle p_\infty^{m,j} \rangle_{j=1}^{\infty} \subset S_{x_\infty}(\mathcal{A}) \quad \text{and}$$

$$\rho\left(g_m^j(\omega),\ p_\infty^{m,j}(\omega)\right) \leqslant h(X_m(\omega), X_\infty)) + \frac{1}{2^m}, \text{ a.e.} \tag{2.2}$$

Analogously, for each n there is a sequence

$$\langle p_n^{m,j} \rangle_{j=1}^{\infty} \quad \text{such that} \quad \langle p_n^{m,j} \rangle_{j=1}^{\infty} \subset S_{x_n}^{\infty} \quad \text{and}$$

$$\rho\left(p^{m,j}(\omega), p_\infty^{m,j}(\omega)\right) \leqslant h(X_n(\omega)), X_\infty(\omega)) + \frac{1}{2^n}, \text{ a.e.} \tag{2.3}$$

But in view of (2.2) one can suppose that for each m

$$p_m^{m,j}(\omega) = g_m^j(\omega)\ \forall \omega\ \forall j = 1, 2, \ldots \tag{2.4}$$

Now, since $X_\infty \in \mathcal{M}(\mathcal{K}, \mathcal{A})$ then again by theorem 2 there is a sequence $\langle q_\infty^k \rangle_{k=1}^{\infty}$ such that

$$\langle q_\infty^k \rangle_{k=1}^{\infty} \subset S_{X_\infty}(\mathcal{A}) \text{ and } X_\infty(\omega) = \mathrm{cl}(\{q_\infty^k(\omega),\ k \geqslant 1\})\ \forall \omega \tag{2.5}$$

Further, again by theorem 3, for each n there is a sequence $\langle q_n^k \rangle_{k=1}^{\infty}$ such that $\langle q_n^k \rangle_{k=1}^{\infty} \subset S_{x_k}(\mathcal{A})$ and

$$\rho(q_n^k(\omega), q_\infty^k(\omega)) \leqslant h(X_n(\omega), X_\infty(\omega)) + \frac{1}{2^n}, \text{ a.e.} \tag{2.6}$$

Finally, for each $n = 1, 2, \ldots$, we put

$$\langle f_n^i \rangle_{i=1}^{\infty} = \langle p_n^{m,j} \rangle_{m,j=1}^{\infty} \cup \langle q_n^k \rangle_{k=1}^{\infty}$$

then it is easy to check that by (2,1), (2.4) and (2.5) we get first assertion 1. Further by (2.2), (2.3), (2.6) we obtain that for all $i = 1, 2, \ldots$

$$\rho(f_n^i(\omega), \rho_\infty^i(\omega) \leqslant h(X_n(\omega), X_\infty(\omega)) + \frac{1}{2^n}, \text{ a. e.}$$

Thus by the almost sure convergence of the sequence $\langle X_n \rangle$, the second assertion 2 is satisfied. It completes the proof of necessarity. Since sufficiency can be established easily from conditions (1) (2) and definition (1.1) then our main result is obtained.

In particular, if **B** is a separable Banach space then according to [3] we an also consider a sequence $\langle X_n \rangle$ of integrably bounded multi-functions, i. e. or each n

$$\int_\Omega \sup \{ \| x \|, x \in X_n(\omega)\} \, dP < \infty$$

Therefore our main result implies the following corollary 5 which gives s a representation for L_1-convergent sequences of integrable bounded multi- unctions.

Corollary 5. Let $\langle X_n \rangle$ be a sequence of integrably bounded multifuncti- ns contained in $\mathcal{M}(\mathcal{K}, \mathcal{A})$. Then $\langle X_n \rangle$ is L_1-convergent to some integrably ounded multifunction $X_\infty \in \mathcal{M}(\mathcal{K}, \mathcal{A})$

e.
$$\lim_{n \to \infty} \int_\Omega h(X_n(\omega), X_\infty(\omega)) \, dP = 0$$

' and only if there is a countable number of sequences

$$\langle f_1^i \rangle_{i=1}^\infty, \langle f_2^i \rangle_{i=1}^\infty, \ldots, \langle f_\infty^i \rangle_{i=1}^\infty \text{ such that}$$

1) $\langle f_n^i \rangle_{i=1}^\infty \subset S_{X_n}(\mathcal{A}), X_n(\omega) = cl(\{f_n^i(\omega), \ i \geqslant 1\}) \forall \omega \forall n = 1, 2, \ldots, \infty$ and

2) $\lim_n \int_\Omega h(f_n^i(\omega), f_\infty^i(\omega)) dP = 0$, uniformly in $i = 1, 2, \ldots$

§ 3. QUESTION

Given an increasing sequence $\langle \mathcal{A}_n \rangle$ of sub σ-fields of $\mathcal{A}$ and a sequence $\langle X_n \rangle$ of $\mathcal{M}(\mathcal{K}, \mathcal{A})$ adapted to $\langle \mathcal{A}_n \rangle$, i.e. each $X_n \in \mathcal{M}(\mathcal{K}, \mathcal{A}_n)$. Suppose that $\langle X_n \rangle$ s almost surely convergent to some $X_\infty \in \mathcal{M}(\mathcal{K}, \mathcal{A})$. Our question is, whether here is countable number of sequences

$$\langle f_1^i \rangle_{i=1}^\infty, \langle f_2^i \rangle_{i=1}^\infty, \ldots, \langle f_\infty^i \rangle_{i=1}^\infty \text{ such that}$$

1) $\langle f_n^i \rangle_{i=1}^\infty \subset S_{X_n}(\mathcal{A}_n), X_n(\omega) = cl(\{f_n^i(\omega), \ i \geqslant 1\}) \ \forall \omega \ \forall n = 1, 2, \ldots, \infty$ and

2) $\lim_n \rho(f_n^i(\omega), f_\infty^i(\omega)) = 0$, a.e. for each $i = 1, 2, \ldots$

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