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STABILITY PROPERTIES OF DIFFERENTIABLE FUNCTIONS ON BANACH SPACES

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1. INTRODUCTION

Let V be a real Banach space, $\|.\|$ a norm in V, $f: V \to R$ a C^k function, $k \geqq 1$, into the real numbers R and $df(x)$ the differential of f at the point $x \in V$. Let $J^k(f)(p)$ be the *k-jet* of f at $p \in V$, thus $J^k(f)(p)$ is the set of all C^k functions $g: V \to R$, satisfying $d^i f(p) = d^i g(p)$ fon $i = 0, \ldots, k$, or equivalently, $J^k(f)(p)$ is the set of all C^k functions $g: V \to R$ with $g(p) = f(p)$ and for any $\varepsilon > 0$ there exists a neighbourhood U of p such that: $\| df(x) - dg(x) \| \leqq \varepsilon \| x - p \|^{k-1}$ f^{-1} for all $x \in U$; where $\| df(x) - dg(x) \|$ denotes the supremum norm of $df(x) - dg(x)$. The set $J^k(f)(p)$ can be extended to the set $J^\alpha(f, \omega)(p)$ as follows:

Given $\alpha, \omega \in R$, $\omega > 0$. We define:

$$J^\alpha(f, \omega)(p) = \{g: V \to R: g \in C^1, g(p) = f(p)$$

and there exists a neighborhood U of p such that

$$\| df(x) - dg(x) \| \leqq \omega \| x - p \|^{\alpha-1} \text{ for all } x \in U\}.$$

We call $J^\alpha(f, \omega)(p)$ the (α, ω)-jet of f at p. Further' we say that f has the *property* $Q(\alpha, a)$ at p with $\alpha, a \in R$, $a > 0$, if there exists a neighborhood U of p such that $\| df(x) \| \geqslant a \| x - p \|^{\alpha-1}$ for all $x \in U$. f is said to be local homeomorphic at p to g if there exists for every neighborhood O of p a neighborhood W of p and a homeomorphism $\Phi: W \approx \Phi(W)$ such that $\Phi(p) = p$, $\Phi(w) \subset O$ and $fo\Phi(x) := f(\Phi(x)) = g(x)$ for all $x \in W$.

In [10] we have proved two following theorems which relate to some local stability properties of differentiable functions on Banach spaces.

Theorem I. If $f \in C^k$, $k \geqq 2$ and f has property $Q(k, a)$ at p, then f is local homeomorphic at p to each $g \in J^k(f)(p)$.

Theorem II. If $f \in C^1$ and the mapping $x \to df(x)$ is locally Lipschitz and f has property $Q(\alpha, a)$ with $\alpha > 1$ at p, then f is local homeomorphic at p to each $g \in J^\alpha(f, \omega)(p)$ provided $\omega < \frac{a}{2^{\alpha+1}}$.

Clearly Theorem II implies Theorem I. Note that Theorem I and Theorem II extend the well-known Kuiper—Kuo Theorem (see [4], Theorem 1, and [5]), which have been proved for Hilbert spaces and R^n and only for $J^k(f)(p)$ defined on such spaces.

In this paper we shall extend the property $Q(\alpha, a)$ at p and the set $(f, \omega)(p)$ by replacing p with a compact set $G \subset V$. So, the property $Q(\alpha, a)$ of ıt G and a new set $J^{\alpha}(f, \omega)(G)$ will be defined. Then, with respect to pro- rty $Q(\alpha, a)$ at G and $J^{\alpha}(f, \omega)(G)$ a theorem like Theorem II can be proved. ›w we detail our idea: Let $f: V \to R$ be a differentiable function, G be a com- ct set in V (Thus int $G = \phi$ if V is infinite dimensional) and W^r be the neigh- rhood of G defined as follows:

$$W^r = \{x \in V : \| x - G \| < r\}$$

here $r > 0$ is a real number and

$$\| x - G \| = \min \{\| x - p \| : p \in G\}$$

(1.1) Definition. f has property $Q(\alpha, a)$ at G with $\alpha, a \in R$, $a > 0$ if there ists a neighborhood W^r of G such that:

$$\| df(x) \| \geqq a \| x - G \|^{\alpha-1}$$

all $x \in W^r$.

(1.2) Definition. Given $\alpha, \omega \in R$, $\omega > 0$, then

$J^{\alpha}(f, \omega)(G) := \{g : V \to R : g \in C^1, g(p) = f(p), p \in G$ and there exists a ghborhood W^r of G such that

$$\| df(x) - dg(x) \| \leqq \omega \| x - G \|^{\alpha-1} \text{ for all } x \in W^r\}.$$

(1.3) Definition. f is said to be homeomorphic at G to g, if there exists eoery neighborhood O of G a neighborhood W of G and a homeomorphism $W \approx \Phi(W)$ such that $\Phi(p) = p$ for $p \in G$, $\Phi(W) \subset O$ and

$$f \cdot \Phi(x) = g(x)$$

all $x \in W$.

Our main result is the following:

(1.4) Theorem. Let $f : V \to R$ be a C^1-function and suppose the mapping $\to df(x)$ is locally Lipschitz. When f has property $Q(\alpha, \text{-}a)$ with $\alpha > 1$ at G, n f is homeomorphic at G to each $g \in J^{\alpha}(f, \omega)(G)$ provided $\omega < \frac{a}{2^{\alpha+1}}$.

(1.5) Remark. Obviously Theorem (1.4) implies Theorem II and therefore ›orem I. The proof of Theorem (1.4) will be as similar as that one of ›orem II. However some interesting considerations and cases occur in this ·of, but not in the proof of Theorem II. For example in case the set G is a ıpact subset of V wich contains some critical point of f or G itself is a ıpact subset of the critical set $\mathbf{K} = \{p \in V : df(p) = 0\}$.

We note that if $f \in C^2$, $G = \{p\} \in \mathbf{K}$ and V is reflexive, then property , a) of f at p, as has been proved in [8], coincides with the definition of ıdegeneracy of critical point p, given by R. Palais in [6, §7], i.e., if the pping A defined by $(Ax, y) = d^2f(p)(x, y)$ is a toplinear isomorphism from ›nto its dual space V^*. In the example that follows we will see when a ction f has property $Q(\alpha, a)$ at a compact set G. We hope our result in ›orem (1.4) will find applications in some stability problems where the ›ility of objects (functions, veccor fields) is investigated around not only ngle point but a domain in both finite and infinite dimensional cases.

1.6. Example. Let us consider a C^1 function $f: V \to R$. Let G be a compact subset of V, containing no critical points of f. Then, since $\mathbf{K}$ is a closed set, G has a positive distance from $\mathbf{K}$. Hence, for small enough $0<r<1$ the set $W^r = \{x \in V : \| x - G \| < r\}$ has a positive distance from $\mathbf{K}$. Assume moreover f satisfies the Condition (C) of Palais and Smale (i. e., if for any subset S of V on which $|f|$ is bounded but $\inf_{x \in S} \| df(x) \| = 0$, then there is a critical point of f in the closure $\overline{S}$ of S), and $|f|$ is bounded on W^r then there is clearly a $\gamma > 0$ such that $\| df(x) \| > \gamma$ for all $x \in W^r$. We get $\| df(x) \| \geqq \frac{\gamma}{r} r \geqq \frac{\gamma}{r} \| x - G \|^{\alpha - 1}$ for all $x \in W^r$, if $\alpha > 1$. Hence f has property $Q(\alpha, a)$ at G with $a = \frac{\gamma}{r}$.

2. BASIC DEFINITIONS AND RESULTS

Let $B(x, r) = \{y \in V : \| y - x \| < r\}$ be the open ball with center x and radius r and $\partial B(x, r)$ be the boundary of $B(x, r)$.

Given a C^1 – function $f: V \to R$ then $df(x) \in V^*$ and, as mentioned, the critical set $\mathbf{K}$ of f is closed which implies the openness of the set $\widetilde{V} = V \setminus \mathbf{K}$. Now given an open interval $]a, b[$ in R and let $\sigma :]a, b[\to V$ be a differentiable mapping. Then $d\sigma(t) \in L(R : V)$ (the space of all linear continuous mappings from R into V). Setting $\sigma'(t) = d\sigma(t)\, 1$ we call σ' the canonical lifting of σ. Let $\mathcal{O}$ be an open subset of V. A C^k vector fielld, $l \geq 0$, on $\mathcal{O}$ is a C^k mapping from $\mathcal{O}$ into V. Let X be a C^k vector field on $\mathcal{O}$, then a solution curve of X is a C^1-mapping σ from an open interval $]a, b[$ into $\mathcal{O}$ such that $\sigma'(t) = X(\sigma(t))$ for all $t \in]a, b[$. If $0 \in]a, b[$ then $x = \sigma(0)$ is called the initial condition of σ. From now on we write σ_x to mean the solution curve of some vector field with initial condition $\sigma_x(0) = x$. σ_x is said to be a maximum solution curve of X if any other solution curve δ_x with initial condition $\delta_x(0) = x$ is a restriction of σ_x. A mapping $A : \mathcal{O} \to V$ is called locally Lipschitz if for every $x \in \mathcal{O}$ there exists a neighborhood U of x and a constant $L > 0$, such that $\| A(x_1) - A(x_2) \| \leqq L \| x_1 - x_2 \|$ for all $x_1, x_2 \in U$.

In [3] (IV, §2) the reader can find the proof of the existence of a maximum solution curve σ_x of a locally Lipschitz vector field X defined on an open set $\mathcal{O}$. Besides, one gets following results.

Denote by $J(x) =]\, t^-(x), t^+(x)\, [\subset R$ the domain of σ_x and let $D(X) = \{(t, x) \in R \times \mathcal{O} : t \in J(x)\}$ then D(X) is an open set of $R \times \mathcal{O}$, the mapping $\varphi : D(X) \to V$ defined by $\varphi(t, x) = \sigma_x(t)$ is locally Lipschitz and for $s \in J(\sigma_x(t))$ it follows

$\varphi(s + t, x) = \varphi(s, \varphi(t, x))$ and, if $y = \varphi(t, x)$ then $x = \varphi(-t, y)$. All the above results based on the theorem of local existence and uniqueness of solution curve of a locally Lipschitz vector field. A motivation of this theorem is the following:

(2. 1) Theorem. Let $X; B(x, r) \to V$ be a locally Lipschitz vector field on $B(x, r)$ satisfying $\| X(y) \| \leqq H$ for all $y \in B(x, r)$, where $H > 0$ is a constant. Then for $0 < a \leqq \frac{r}{H}$ there exists a solution curve $\sigma_x :]-a, a[\to B(x, r)$ of X, i. e., $\sigma_x(t) \in B(x, r)$ and $\sigma'_x(t) = X(\sigma_x(t))$ for all $t \in]-a, a[$.

(2.2) Definition. Let $f; \mathcal{O} \to R$ be a differentiable mapping. A vector field Z on $\mathcal{O}$ is called a pseudo-gradient vector field on $\mathcal{O}$ for F if $\| Z(x) \| \leqq 2 \| df(x) \|$ and $Zf(x) ;= df(x) Z(x) \| df(x) \geqq \|^2$ for all $x \in \mathcal{O}$

(2.3) Theorem. (Palais [7], Theorem 4.4) Let $f : V \to R$ be a C^1 – function then there exists a locally Lipschitz pseudo-gradient vector field Z for f on $\widetilde{V}$.

(2.4) Corollary. ([7], §5) Let $f : V \to R$ be a C^1 – function and suppose that the mapping $x \to df(x)$ is locally Lipschitz. So, by (2.3) there exists a locally Lipschitz pseudo-gradient field vector Z on $\widetilde{V}$. Let $X = \frac{Z}{Zf}$ with $X(x) = \frac{Z(x)}{Zf(x)}$ then X is a locally Lipschitz vector field on $\widetilde{V}$ and $df(x)X(x) = 1$. Hence, if σ_x is the solution curve of X, then f is strictly monotone increasing along σ_x i.e., $f(\sigma_x(t_1)) < f(\sigma_x(t_2))$ for $t_1 < t_2 \in J(x)$. Especially $f(\sigma_x(t)) = f(x) + t$ for $t \in J(x)$.

3. STABILITY PROPERTIES ON $J^{\alpha}(f, \omega)(G)$. THE PROOF OF THEOREM (1.4)

In this section let $f : V \to R$ be a C^1-function and suppose that the mapping $x \to df(x)$ is locally Lipschitz. Let Z be a locally Lipschitz pseudo-gradient vector field for f on $\widetilde{V}$, $X = \frac{Z}{Zf}$ and σ_x be the maximum solution curve of X with initial condition $\sigma_x(0) = x$.

First, we study some properties describing the behavior of a function $g \in J^{\alpha}(f, \omega)(G)$ and of the solution curve σ_x of the vector field X near the compact set G in the case f has property $Q(\alpha, a)$ at G. From these properties we then obtain the proof of Theorem (1.4). Throughout this section we assume $\alpha > 1$. It turns out that some propositions also hold for arbitrary α. Now it is not hard to prove the first proposition stated below:

(3. 1) Proposition. If f has properties $Q(\alpha; a)$ at G then every $g \in J^{\alpha}(f, \omega)(G)$, $\omega < a$, has property $Q(\alpha, a - \omega)$ at G.

(3. 2) Proposition. Let $g \in J^{\alpha}(f, \omega)(G)$ and W^r be the neighborhood of G such that

$$\| df(x) - dg(x) \| \leqq \omega \| x - G \|^{\alpha - 1} \text{ for all } x \in W^r \setminus G.$$

It follows that:

$$| f(x) - g(x) | < \omega \| x - G \|^{\alpha} \text{ for all } x \in W^r \setminus G.$$

Proof. Let $X \in W^r \setminus G$, then there is a point q on the boundary ∂G of G such that $\| x - G \| = \| x - q \|$.

Applying the mean value Theorem

$$f(x) - f(q) = \int_0^1 df(q + t(x - q))(x - q)\, dt \text{ for } t \in [0,1],$$

$$g(x) - g(q) = \int_0^1 dg(q + t(x - q))(x - q)\, dt.$$

Note that $f(q) = g(q)$ so we obtain by substraction:

$$f(x) - g(x) = \int_0^1 (df(z) - dg(z))(x-q)\, dt \text{ with } z = q + t(x - q).$$

Consequently:

$$|f(x) - g(x)| \leqq \int_0^1 \| df(z) - dg(z) \| \, \| x - q \| \, dt$$

$$\leqq \omega \| x - q \| \int_0^1 \| t(x - q) \|^{\alpha-1} dt = \omega \| x - q \|^{\alpha} \int_0^1 t^{\alpha-1} dt$$

$$= \omega \| x - q \|^{\alpha} \cdot \frac{1}{\alpha} < \omega \| x - q \|^{\alpha} = \omega \| x - G \|^{\alpha}.$$

Q.E.D

In the next Proposition (3.4), in the case when f has property $Q(\alpha, a)$ at G, the behavior of each $g \in J^{\alpha}(f, \omega)$ (o) on σ_x in a neighborhood W^r of G will be described. First we prove the following:

(3.3) Lemma. Suppose that f has property $Q(\alpha, a)$ at G and W^r is the neighbourhood of G such that:

$$\| df(x) \| \geqq a \| x - G \|^{\alpha-1} \quad \text{for all } x \in W^r.$$

Then the following inequality holds:

$$\| X(x) \| \leqq \frac{2}{a \| x - G \|^{\alpha-1}} \text{ for all } x \in W^r \setminus G.$$

Proof. For $x \in \widetilde{V}$ one gets:

$$\| X(x) \| = \left\| \frac{Z(x)}{df(x) Z(x)} \right\| \leqq \frac{2 \| df(x) \|}{\| df(x) \|^2} = \frac{2}{\| df(x) \|}$$

Clearly, by the property $Q(\alpha, a)$ the set $W^r \setminus G$ is a subset of $\widetilde{V}$. Hence

$$\| X(x) \| \leqq \frac{2}{\| df(x) \|} \leqq \frac{2}{a \| x - G \|^{\alpha-1}} \text{ for all } x \in W^r \setminus G.$$

Q.E.D.

(3.4) Proposition. Suppose that f has property $Q(\alpha, a)$ at G. Take $g \in J^{\alpha}(f, \omega)$, (G) with some $\omega < \frac{a}{2^{\alpha+1}}$. Then there exists a neighborhood W^r of G such that:

(1) g is strictly monotone increasing along δ_x in $W^r \setminus G$ for each $x \in W^r \setminus G$, i.e., for $t_1 < t_2 \in J(x)$ with $\sigma_x(t_1), \sigma_x(t_2) \in W^r \setminus G$ the following inequality holds: $g(\sigma_x(t_1)) < g(\delta_x(t_2))$.

(2) If $\sigma_x(t) \in W^r \setminus G$ with $t > 0$ then $g(x) + t\left(1 - \frac{2\omega}{a}\right) \leqslant g(\sigma_x(t)) \leqslant g(x) + t\left(1 + \frac{2\omega}{a}\right)$.

(3) If $\sigma_x(t) \in W^r \setminus G$ with $t < 0$ then $g(x) + t\left(1 + \frac{2\omega}{a}\right) \leqslant g(\sigma_x(t)) \leqq g(x) + t\left(1 - \frac{2\omega}{a}\right)$.

Proof. First, by using the assumptions we can find a neighborhood W^r of G, such that:

$$\| df(x) \| \geqq a \| x - G \|^{\alpha-1}$$

and

$$\| df(x) - dg(x) \| \leqq \omega \| x - G \|^{\alpha-1}$$

for all $x \in W^r \setminus G$. We shall indicate that W^r is the neighborhood we are looking for. Indeed, for $x \in W^r \setminus G$ and $\sigma_x(t) \in W^r \setminus G$ with $t \in J(x)$ the following equality holds

$$\begin{aligned} \frac{d}{dt} g(\sigma_x(t)) &= \frac{d}{dt} f(\sigma_x(t)) + \left[\frac{d}{dt} g(\sigma_x(t)) - \frac{d}{dt} f(\sigma_x(t))\right] \\ &= df(\sigma_x(t))\, X(\sigma_x(t)) + [...] \\ &= 1 + [...] \end{aligned}$$

where

$$\begin{aligned} [...] &= dg(\sigma_x(t)) X(\sigma_x(t)) - df(\sigma_x(t))\, X(\sigma_x(t)) \\ &= (dg(y) - df(y))\, X(y), \quad \text{with } y = \sigma_x(t) \end{aligned}$$

Since

$$\begin{aligned} |[...]| &\leqq \| dg(y) - df(y) \| \; \| X(y) \| \\ &\leqq \omega \| y - G \|^{\alpha-1} \cdot \frac{2}{a \| y - G \|^{\alpha-1}} \\ &= \frac{2\omega}{a} < 1 \end{aligned}$$

Hence $$\frac{d}{dt} g(\sigma_x(t)) > 0$$

(1) now follows from:

$$g(\sigma_x(t_2)) - g(\sigma_x(t_1)) = \int_{t_2}^{t_2} \frac{d}{dt} g(\sigma_x(t))\, dt$$

To prove (2) and (3) one deduces from $|[...]| < \frac{2\omega}{a}$ i. e., $\left| \frac{d}{dt} g(\sigma_x(t)) - 1 \right| < \frac{2\omega}{a}$ the following inequality:

$$1 - \frac{2\omega}{a} \leqq \frac{d}{dt} g(\sigma_x(t)) \leqq 1 + \frac{2\omega}{a}.$$

Thus, we get for $t > 0$:

$$g(\sigma_x(t)) = g(x) + \int_0^t \frac{d}{ds} g(\sigma_x(s))\, ds$$

$$\leqq g(x) + t\left(1 + \frac{2\omega}{a}\right)$$

and

$$g(\sigma_x(t)) \geqq g(x) + t\left(1 - \frac{2\omega}{a}\right)$$

This proves (2). (3) can be proved in the same way. Q.E.D.

In that follows our investigating object is the neighborhood W^r determined as above by a fixed ω with $0 < \omega < \frac{a}{2^{\alpha+1}}$. Clearly, by the compactness of G such a W^r can be chosen that W^r is contained in an arbitrary given neighborhood $\mathcal{O}$ of G.

(3.5) Proposition Let $W = W^{\frac{r}{2}} = \{x \in V : \| x - G \| < \frac{r}{2}\}$

Then $W \subset W^r$ and for each $x \in W \setminus G$ the point $\sigma_x(t)$ belongs to $W^r \setminus G$ if $t \in J(x) =]-i(x), i(x)[$ where $i(x) = \omega \| x - G \|^{\alpha}$.

Proof. Let us consider a point $x \in W \setminus G$. Then there is a point $p \in \partial G$ such that x belongs to the ball $B\left(p, \frac{r}{2}\right)$. Setting $r(x) = \frac{1}{2} \| x - G \|$ then

$r(x) = \frac{1}{2} \| x - q \|$ for some $q \in \partial G$. Now for $y \in B(x, r(x))$:

$$\begin{aligned} \| y - G \| &= \| y - q' \| \qquad \text{(for some } q' \in \partial G) \\ &\geqq \| x - q' \| - \| y - x \| \\ &> \| x - q \| - r(x) \\ &= 2r(x) - r(x) = r(x) \end{aligned}$$

and

$$\begin{aligned} \| y - G \| &\leqq \| y - p \| \\ &\leqq \| y - x \| + \| x - p \| \\ &< \frac{1}{2} \| x - q \| + \| x - p \| \\ &< \frac{1}{2} \| x - p \| + \| x - p \| \\ &= \frac{3}{2} \| x - p \| < \frac{3}{2} \cdot \frac{r}{2} = \frac{3}{4} r \end{aligned}$$

Thus $B(x, r(x)) \subset W^r \setminus G$.

From Lemma (3.3) we obtain:

$$\| X(y) \| \leqq \frac{2}{a \| y - G \|^{\alpha - 1}} \text{ for all } y \in B(x, r(x)), \text{ Consequently}$$

$$\| X(y) \| \leqq \frac{2}{a (r(x))^{\alpha - 1}} =: H(x) \text{ for all } y \in B(x, r(x))$$

It then follows from Theorem (2.1) that there exists a solution curve $\sigma_x :]-a(x), a(x)[\to B(x, r(x))$ with $a(x) = \frac{r(x)}{H(x)} = \frac{a}{2}(r(x))^\alpha$. So $\sigma_x(t)$ belongs to $B(x, r(x))$ for all $t \in]-a(x), a(x)[$. But $i(x) = \omega \| x - G \|^\alpha = \omega(2r(x))^\alpha = \omega 2^\alpha (r(x))^\alpha < \frac{a}{2}(r(x))^\alpha = a(x)$. This means that $I(x) \subset]-a(x), a(x)[$ what implies $\sigma_x(t) \in W^r/G$ for all $t \in I(x)$. Q.E.D.

3.6. Proposition. For each $x \in W \setminus G$ there exists one and only one $t(x) \in I(x)$ such that $g(x) = f(\sigma_x(t(x)) = f(\varphi(t(x), x))$. By setting $t(p) = 0$ for $p \in G$ we then have a centinuous functions t from W into R.

Proof. First, assume that $f(x) < g(x)$ or $g(x) - f(x) > 0$. It follows:

$$g(x) = f(x) + g(x) - f(x)$$
$$= f(x) + |g(x) - f(x)|$$

Apply (3.2) we obtain:

$$g(x) < f(x) + \omega \| x - G \|^\alpha$$
$$= f(x) + i(x) = f(\varphi(i(x), x))$$

Thus the following inequality holds:

$$f(x) = f(\varphi(0, x)) < g(x) < f(\varphi(i(x), x))$$

Observe that f is strictly monotone increasing along $\varphi(., x)$ we conclude that there is only one $t(x)$ with $0 < t(x) < i(x)$ such that

$$f(\varphi(t(x), x)) = g(x)$$

If $f(x) > g(x)$ then we can write $-f(x) < -g(x)$. As above a $t(x)$ with $-i(x) < t(x) < 0$ can be found such that:

$$f(\varphi(t(x), x)) = g(x)$$

If $f(x) = g(x)$ then by setting $t(x) = 0$,

$$f(\varphi(0, x)) = f(x) = g(x).$$

Now we prove the continuity of the function t at every $x \in W$.

Two following cases are to be investigated:

(1) $x \in W \setminus G$

(2) $x = p \in \partial G$

(the case $x = p \in \text{int } G$ is evident by the definition of t).

to (1)

Given a sequence $x_i \to x$. We can obviously assume that $x_i \in W \setminus G$. We have to show that for any $\varepsilon > 0$ there exists a i_1 so that $|t(x_i) - t(x)| < \varepsilon$ for $i \geq i_1$. First $\varphi(t(x),) \in W^r \setminus G$. Since $W^r \setminus G$ is open and φ is uniformly continuous there exist $t_1, t_2 \in I(x)$ with $t_1 < t(x) < t_2$, $|t_1 - t_2| < \varepsilon$ and an open neighborhood U of x such that $\varphi(t, y) \in W^r \setminus G$ for $t \in [t_1, t_2]$ and $y \in U$. Consequently, there exists an i_0 such that $\varphi(t, x_i) \in W^r \setminus G$ for all $t \in [t_1, t_2]$ and $i \geqslant i_0$ Setting $a = f(\varphi(t_1, x))$, $b = f(\varphi(t_2, x))$. $a_i = f(\varphi(t_1\, x_i))$, $b_i = f(\varphi(t_2, x_i))$

This means that $]-s(y), s(y)[\subset I(z)$ and therefore $t_1 \in I(z)$. It follows:

$$y = \sigma_z(-t_1) \in B(z, r(z)) \text{ (see(3.5)) where}$$

$$r(z) = \frac{1}{2} \| z - G \| = \frac{1}{2} \frac{r'}{3} = \frac{r'}{6}.$$

Apply the triangle inequality and $\varepsilon < \frac{r'}{6}$ we come to the following contradiction:

$$\frac{r'}{\varepsilon} = \| z - G \| \leqq \| z - p \| \leqq \| z - y \| + \| y - p \| \leqq \frac{r'}{6} + \varepsilon < \frac{r'}{\varepsilon}$$

which proves (1).

to (2). Suppose that $t^+(y) \leqq s(y)$ but there exists an $\varepsilon_1 > 0$ such that $\|\sigma_y(t) - G\| > \varepsilon_1$ for some $y \in B(p, \varepsilon)\setminus G$, all $0 \leqq t < t^+(y)$. Choore s such that $0 < s < t^+(y)$ and $t^+(y) - s < \omega \varepsilon_1^\alpha$.

Put $z = \sigma_y(s)$. then $i(z) = \omega \| z - G \|^\alpha > \omega \varepsilon_1^\alpha$. Since $s \in]-s(y), s(y)[$ and $z \notin G$ hence $z \in W^{\frac{r'}{3}} \setminus G$ (see above) such that $\sigma_z(t)$ is, by (3.5), defined for $t \in I(z) =]-i(z), i(z)[$. But $i(z) > \omega_1^\alpha$ thus $\omega\varepsilon_1^\alpha \in I(z)$. It follows that $\sigma_y(t)$ is defined for $t = s + \omega\varepsilon_1^\alpha$ since $z = \sigma_y(s)$. This is a contradiction to $t^+(y) < \omega\varepsilon_1^\alpha + s$. Hence there exists a sequence $q_i \in \partial G$ and a sequence $t_i \to t^+(y)$ such that $\| \sigma_y(t_i) - q_i \| \to 0$ as $i \to \infty$. Now, by the compactness of G one can even assume that $q_i \to q \in \partial G$ as $i \to \infty$. Then q is a limit point of $\sigma_y(t)$ for $t \to t^+(y)$ since $\| \sigma_y(t_i) - q \| \leqq \| \sigma_y(t_i) - q_i \| + \| q_i - q \|$.

to (3). The proof is the same as by (2) *to* (4) *and* (5). (4) follows immediately from the fact that f is strictly monotone increasing along σ_y (see [8, Lemma 6.9]) and (5) from (4). Q.E.D.

(3.8) Proposition. Let $0 < r' < r$ and $\lambda = 1 - \frac{2\omega}{a}$. With respect to this λ and r' let $B(p, \varepsilon)$ be the ε-ball chosen as in proposition (3.7) for some $p \in \partial G$. Then for each $y \in B(p, \varepsilon)$ there exist two points z_1 and z_2 on the closure of σ_y such that the path connecting z_1, z_2 lies in $W^{r'}$ and the following inequality holds:

$$g(z_1) < f(y) < g(z_2)$$

Proof. (a) Suppose that $f(y) < g(y)$. We have by (3.2) $g(y) - \omega \| y - G \|^\alpha < f(y)$. If $-s(y) \leqslant t^-(y) < t^+(y) \leqslant s(y)$ then we can put $z_2 = y$, $z_1 = q'$, q' being the limit point of $\sigma_y(t)$ as $t \to t^-(y)$. Clearly, we have

$$g(z_1) = g(q') = f(q') < f(y) < g(y) = g(z_2).$$

If $t^-(y) < -s(y) < t^+(y) \leqslant s(y)$, then put $z_2 = y$ and $z_1 = \sigma_y(-s(y))$. By the Proof of (1) by (3.7) we have seen that

$$\sigma_y(t) \in W^{\frac{r'}{3}}$$

for $t \in]-s(y), s(y)[\cap]t^-(y), t^+(y)[$, what implies $z_1 \in W^{r'}$. Apply now Proposition (3.4) and the definition of s(y) we get:

$$g(\sigma_y(-s(y))) \leqslant g(y) - s(y)\left(1 - \frac{2\omega}{a}\right)$$
$$= g(y) - \omega \| y - G \|^{\alpha} < f(y)$$

Thus $$g(z_1) < f(y) < g(z_2).$$

If $$t^-(y) < -s(y) < s(y) < t^+(y)$$
$$(\text{or } -s(y) < t^-(y) < s(y) < t^+(y)),$$

then let $z_2 = y$, $z_1 = \sigma_y(-s(y))$ (or $z_2 = y$, $z_1 = q'$) and we still have, by an argument analogous to the above,

$$g(z_1) < f(y) < g(z_2)$$

b) Suppose that $f(y) > g(\hat{y})$ This case can be investigated in the similar way as by (a). Notice that by choosing z_1 and z_2 as above the path connecting z_1 and z_2 obviously lies in $W^{r'}$.

c) Suppose that $f(y) = g(y)$. This case is evident since g is strictly monotone increasing along σ_y in W^r. The proof of (3.8) is complete. Q.E.D.

Now the proof of Theorem (1.4) follows directly from the following:

(3.9) Proposition. Define a mapping $\Phi : W = W^{\frac{r}{2}} \to V$ by setting

$$\Phi(x) = \begin{cases} \varphi(t(x), x) & \text{for } x \in W \setminus G \\ p & \text{for } x = p \in G \end{cases}$$

Then:

(1) Φ is continuous on W, $\Phi(W) \subset W^r$ and

$$f\,\Phi(x) = g(x) \text{ for all } x \in W$$

(2) Φ is an injective mapping from W onto $\Phi(W)$

(3) Φ is an open mapping from W onto $\Phi(W)$, thus Φ is a homeomorphism from W onto $\Phi(W)$.

Proof. *to (1)* Evidently $\Phi(W) \subset W^r$ and $f\,\Phi(x) = g(x)$ for all $x \in W$.

If $x \in W \setminus G$ then Φ is continuous at x since t and φ are continuous at x. If $x = p \in \partial G$ then let us consider a sequence $x_i \to p$. We shall show that $\Phi(x_i) \to p$. Obviously, one can assume that all x_i belong to $W \setminus G$. Therefore all $\varphi(t(x_i), x_i)$ belong to $B(x_i, r(x_i))$ where $r(x_i) = \frac{1}{2} \| x_i - G \|$. So our statement follows from.

$$\| \Phi(x_i) - p \| = \| \varphi(t(x_i), x_i) - p \|$$
$$\leqq \| \varphi(t(x_i), x_i) - x_i \| + \| x_i - p \|$$
$$\leqq \frac{1}{2} \| x_i - G \| + \| x_i - p \|$$
$$\leqq \frac{1}{2} \| x_i - p \| + \| x_i - p \| = \frac{3}{2} \| x_i - p \|$$

then $a < g(x) < b$ and $a_i < b_i$. When x_i tends to x, so $a_i \to a$, $b_i \to b$ and $g(x_i) \to g(x)$. Therefore, there is an $i_1 \geqq i_0$ such that

$$a_i < g(x_i) < b_i \text{ for } i \geqq i_1.$$

This is nothing but

$$f(\varphi(t_1, x_i)) < f(\varphi(t(x_i), x_i)) < f(\varphi(t_1, x_i))$$

for $i \geqq i_1$. It follows that

$$t_1 < t(x_i) < t_2 \text{ or } |t(x_i) - t(x)| < t_2 - t_1 < \varepsilon$$

in so far as $i \geqq i_1$.

to (2)

Let $x_i \to p$, $x_i \in W \setminus G$. Notice that by Lemma (3.5) we have shown that $t(x_i) \in I(x_i) =]-i(x_i), i(x_i)$ [where $i(x_i) = \omega \| x_i - G \|^{\alpha}$. Hence it follows that $t(x_i) \to 0 = t(p)$ if $x_i \to p$ and this completes the proof of (3.6) Q.E.D.

(3.7) Proposition There exists for every $p \in \delta G$ and $0 < r' < r$ an $\varepsilon > 0$ such that for each $y \in B(p, \varepsilon) \setminus G$ and for $s(y) = \frac{1}{\lambda} \omega \| y - G \|^{\alpha}$ with $0 < \lambda < 1$ the following properties hold:

(1) $\sigma_y(t) \in W^{r'}$ for all $t \in]-s(y), s(y)[\cap]t^-(y), t^+(y)[$

(2) If $t^+(y) \leqq s(y)$ then $\sigma_y(t)$ has a limit point $q \in \partial G$ as $t \to t^+(y)$.

(3) If $t^-(y) \geqq -s(y)$ then $\sigma_y(t)$ has a limit point $q' \in \partial G$ as $t \to t^-(y)$.

(4) If $-s(y) \leqq t^-(y) < t^+(y) \leqq s(y)$ and q, q' are limit points of $\sigma_y(t)$ as $t \to t^+(y)$ and $t \to t^-(y)$ respectively, then $q \neq q'$.

(5) If $B = \{p\}$ then from $t^+(y) \leqq s(y)$ it follows that $t^-(y) < -s(y)$ and, if $-s(y) \leqq t^-(y)$ then $t^+(y) > s(y)$.

Proof. to (1).

Let $p \in \partial G$. We chose $\varepsilon > 0$ such that $\varepsilon < \frac{r'}{6}$ and $\beta < \frac{r'}{3} \lambda^{\frac{1}{\alpha}}$ where $\beta = \sup\{\| z - G \| : z \in B(p, \varepsilon)\}$.

For all $y \in B(p, \varepsilon) \setminus G$ we have

$$s(y) = \frac{1}{\lambda} \omega \| y - G \|^{\alpha} < \frac{1}{\lambda} \omega \beta^{\alpha} < \frac{1}{\lambda} \omega \left(\frac{r'}{3} \lambda^{\frac{1}{\alpha}}\right)^{\alpha} = \omega \left(\frac{r'}{3}\right)^{\alpha}.$$

Now the proof will be finished if $\sigma_y(t) \in W^{\frac{r'}{3}}$ with $W^{\frac{r'}{3}} = \left\{x \in V : \| x - G \| < \frac{r'}{3}\right\}$, for each $y \in B(p, \varepsilon) \setminus G$ and $t \in]-s(y), s(y)[\cap]t^-(y), t^+(y)[$. Assume that there is a $t_1 \in]-s(y), s(y)[\cap]t^-(y), t^+(y)[$ such that if $z = \sigma_y(t_1)$ then

$$z \in \partial W^{\frac{r'}{3}} = \left\{ x : \| x - G \| = \frac{r'}{3} \right\}.$$

Then

$$\| z - G \| = \frac{r'}{3} \text{ and } i(z) = \omega \| z - G \|^{\alpha} = \omega \left(\frac{r'}{3}\right)^{\alpha}.$$

Note that Φ is trivially continuous at $x = p \in \text{int } G$.

to (2). Given $x_1, x_2 \in W$. We shall show that $\Phi(x_1) = \Phi(x_2)$ implies $x_1 = x_2$. Clearly we can restrict ourselves to the case $x_1, x_2 \in W \setminus G$. First $g(x_1) = f \cdot \Phi(x_1) = f \cdot \Phi(x_2) = g(x_2)$. On the other hand $y = \Phi(x_1) = \varphi(t(x_1), x_1) = \Phi(x_2) = \varphi(t(x_2), x_2)$. Hence $x_1 = \varphi(-t(x_1), y)$ and $x_2 = \varphi(-t(x_2), y)$ lie on the solution curve σ_y starting at y, which implies $x_1 = x_2$ since g is strictly monotone increasing along σ_y in W^r and $g(x_1) = g(x_2)$.

to (3). We shall show that $\Phi(M)$ is open for each open set $M \subseteq W$.

Given $z = \Phi(x) \in \Phi(M)$, $x \in M$. One has to show that there exists an ε-ball $B(z, \varepsilon)$ contained in $\Phi(M)$ i. e., to each $y' \in B(z, \varepsilon)$ there exists a $x' \in M$ such that $y' = \Phi(x')$. By the definition of Φ, such a $B(z, \varepsilon)$ obviously exists when $x = p \in \text{int } G$. It remains two following cases to investigate:

(A) $z = \Phi(x)$ with $x \in M \setminus G$.

(B) $z = \Phi(x)$ with $x = p \in \partial G$, thus $z = p \in M \cap \partial G$.

to (*A*). Suppose that $t(x) \geqq 0$ (the case $t(x) < 0$ can be proved in the similar way). We have:

$$x = \Phi^{-1}(z) = \varphi(-t(x), z) \in M \setminus G$$

Evidently there exists a $\delta > 0$ and an $\varepsilon > 0$ such that $\varphi(t, y') \in M \setminus G$ for all $t \in I =]-t(x) - \delta, t(x) + \delta[$ and $y' \in B(z, \varepsilon)$. Choose $t_1, t_2 \in I, t_1 < -t(x) < t_2$ and denote $x_1 = \varphi(t_1, z)$, $x_2 = \varphi(t_2, z)$, $x'_1 = \varphi(t_1, y')$, $x'_2 = \varphi(t_2, y')$ for some $y' \in B(z, \varepsilon)$. By fixing δ and reducing ε one gets $x'_1 \to x_1, x'_2 \to x_2$, which deduces $g(x'_1) \to g(x_1)$, $g(x'_2) \to g(x_2)$. On the other hand, it follows that $f(y') \to f(z) = g(x)$ and since $g(x_1) < g(x) = f(z) < g(x_2)$ there exists an ε such that $g(x'_1) < f(y') < g(x'_2)$ for all $y' \in B(z, \varepsilon)$ Consequently, there is a $x' = \varphi(t', y')$ with $t_1 < t' < t_2$, satisfying $g(x') = f(y')$. Together with $g(x') = f(t(x'), x')$ one concludes that $y' = \varphi(t(x'), x') = \Phi(x')$, what implies $B(z, \varepsilon) \subset \Phi(M)$.

to (*B*) Choose $r' > 0$ such that $B(p, r') \subset M$. Clearly we have $B(p, r') \subset W^{r'}$. By Proposition (3.8) there is a $B(p, \varepsilon)$ such that for every $y \in B(p, \varepsilon) \setminus G$ there exists two points z_1, z_2 on the closure of σ_y, satisfying $g(z_1) < f(y) < g(z_2)$ where the path connecting z_1 and z_2 lies in $W^{r'}$. Consequently there is a x on this path such that $g(x) = f(y)$. As before we obtain $y = \varphi(t(x), x) = \Phi(x)$ with $x \in W^{r'}$. Then, since $f = g$ on G and f is strictly monotone along σ_y, it follows that $x \notin G$, thus $x \in W^{r'} \setminus G$. As seen by (3.5) we have $\|x - y\| \leqq \frac{1}{2}\|x - G\|$, thus $\|x - y\| \leqq \frac{1}{2} r'$.

Using the triangle inequality we obtain: $\|x - p\| \leqq \|x - y\| + \|y - p\| < \frac{r'}{2} + \varepsilon$.

It follows $x \in B(p, r')$ if we choose $B(p, \varepsilon)$ as above with $\varepsilon < \frac{r'}{2}$. Then $x \in M$ and this means that $B(p, \varepsilon) \setminus G \subset \Phi(M)$. Hence $B(p, \varepsilon) \subset \Phi(M)$. The proof of (3. 9) is complete Q. E. D.

To conclude this section we set the following:

(3.10) Problem. The property $Q(\alpha, a)$, $\alpha > 1$, of a function f near a point p, as has been proved by J. Bochnak and S. Lojasewicz in [2], is a necessary codition for f to be local homeomorphic at p to each $g \in J^k(f)(p)$ defined on R^n. However, for the infinite dimensional cases it has not been known, whether f would have property $Q(\alpha, a)$ at p when f is local homeomophic at p to each $g \in J^\alpha(f, \omega)$ (p) for some $\omega > 0$. In the same way we can ask the question relative to the property $Q(\omega, a)$ at a compact set G and to the set $J^\alpha(f, \omega)$ (G), defined on R^n or on an infinite dimensional Banach space. This is whether f has property $Q(\alpha, a)$ at G, if f is local homeomorphic at G to each $g \in J^\alpha(f, \omega)$ (G) for some $\omega > 0$.

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REFERENCES

1. A. Bielecki, *Une remark sur la methode de Banach – Caccipoli – Tichonow dans la théorie des equations differentielles ordinaires,* Bull. Acad.Pol. 4 (1956) 261 – 264.

2. J.Bochnak, S. Lojasiewicz, *A converse to Kuiper – Kuo Theorem.,* Proc. of Liverpool Singularities Symposium I., Springer Lecture Notes.

3. S. Lang, *Introduction to Differentiable Manifolds,* Inc. New York 1962.

4. N.H. Kuiper, *C^1-equivalence of functions near isolated critical points.* Ann. of Math. Studies 69. Princeton Univ. Press., Princeton N.J. (1972). 199–218.

5. T. C. Kuo, J. C. Lu, *A complete determination of C^0-sufficiency in J^r (2,1).* Invent. Math. 8 (1969) 226 – 235.

6. R. Palais, *Morse Theory on Hilbert manifolds.* Topology 2 (1963), 269 – 340.

7. R. Palais, *Lusternik – Schnirelman Theory on Banach manifolds,* Topology 5 (1966) 115 – 132.

[8] N. A. Tú, *Locale topologische Eigenschaften der auf Banachmannigfaltigkeiten definierten differenzierbaren Funktionen, Math. Nachr. 86 (1978), 283 – 289.*

9. N. A. Tú, *Locale topologische Eigenschaften der auf Finslermannigfaltigkeiten definierten differenzierbaren.*

Funktionen und ihre Anwendung in der Morse – Theorie Dissertation, Greifswald 1978.

10. N. A. Tú. *Locale topologische Eigenschaften der anf Banachmannigfaltigkeiten definierten differenzierbaren Funktionen* (Fortset -zung), to appear.