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RELATIONSHIP BETWEEN BILINEAR PROGRAMMING AND CONCAVE MINIMIZATION UNDER LINEAR CONSTRAINTS

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1. INTRODUCTION

In its general form, the Bilinear Programming Problem can be stated as follows

$$\min \{f(x, y) : x \in X, y \in Y\}, \tag{1}$$

where X, Y are given closed convex polyhedral sets in R^n, R^m respectively and $f(x, y)$ is a biaffine function of x and y.

The recent investigations by a mumber of authors ([1], [2], [3], [4]) have revealed a close relationship between this problem and the concave minimization problem as first studied by Hoàng Tụy [5]. Indeed, as will be pointed out later, the algorithm proposed by G.Gallo and A. Ülkücü in [4] for solving problem (1) is in fact the adaptation of Hoàng Tụy' s method [5] to a concave programming problem equivalent to the given bilinear programming problem.

In a converse direction, M.Altman [1] and H.Konno[7] have shown that the minimization of a concave quadratic function under linear constraints can be reduced to a bilinear programming problem. On the other hand, it is known by a result of M.Raghavachari [6] that the general zero – one integer programming is equivalent to a concave quadratic minimization problem under linear constraints. Thus, the general zero – one integer programming can also be reduced to a bilinear programming. Further, in [8] A.M.Frieze have reduced the 3-dimensional assignment problem to a bilinear programming problem and then to a special concave programming problem. It seems, however that no attempt has been made up to now to disclose the full connection between the bilinear programming problem (1) and the concave minimization problem.

It is the purpose of the paper to show that the bilinear programming problem is in fact fully equivalent to the concave piecewise linear minimization problem under linear constraints, in the sense that the former can be reduced to the later and vice versa. Hence, each algorithm for solving one problem can be used, on principle, to solve the other.

Since every concave function can be approximated with arbitrary accuracy by a concave piecewise linear function, the general concave minimization problem under linear constraints can be reduced to a bilinear programming problem.

Furthermore, we shall show that finding a feasible solution of the convex (linear) complementarity problem is equivalent to solving a concave (concave piecewise linear, resp.) minimization problem. This fact extends a result of O.L.Magasarian in [9].

2. CONCAVE PROGRAMMING FORMULATION OF THE BILINEAR PROGRAMMING PROBLEM

We shall assume below that

$$\min_{y \in Y} f(x, y) \tag{2}$$

exists for every $x \in X$ and that Y has at least one vertex (for example, these requirements are fulfiled if Y is bounded and nonempty). Then we have

Theorem 1. The problem (1) can be formulated as a concave piecewise linear minimization problem under linear constraints.

Proof. Denote by $\ddot{Y}$ the set of vertices of Y. From the theory of linear programming it follows that the optimal solution of problem (2) is attained in at least one vertex of Y, i.e., in some element of $\ddot{Y}$. The problem (1) can now be restated as

$$\min_{x\in X,\, y \in Y} f(x, y) = \min_{x\in X} \{ \min_{y\in Y} f(x, y)\}$$
$$= \min_{x\in X} \{ \min_{y\in \ddot{Y}} f(x, y)\} = \min_{x\in X} g(x),$$

where $g(x) = \min_{y\in \ddot{Y}} f(x, y) = \min_{y\in Y} f(x, y)$.

Thus, the problem (1) is equivalent to the problem

$$\min_{x\in X} g(x). \tag{3}$$

Notice that the set $\ddot{Y}$ is finite and for each $y \in \ddot{Y}$ $f(x, y)$ is an affine function of x, so $g(x)$ is a concave piecewise linear function of x, i. e., (3) is a concave piecewise linear minimization problem under linear constraints.

Remark 1. If the bilinear programming problem is stated in the form

$$\max \{ f(x, y) : x \in X,\ y \in Y \} \tag{1'}$$

then under previous conditions probrem (1') is equivalent to the problem

$$\max_{x\in X} g(x), \tag{3'}$$

where $g(x) \doteq \max_{y\in \ddot{Y}} f(x, y) = \max_{y\in Y} f(x, y)$ is a convex function of x.

3. BILINEAR PROGRAMMING FORMULATION OF THE CONCAVE PIECEWISE LINEAR MINIMIZATION PROBLEM UNDER LINEAR CONSTRAINTS

Consider now the concave minimization problem

$$\min_{x \in X} g(x). \tag{4}$$

ıere, as before, X is a given closed convex polyhedral set in R^n and $g(x)$ is ːonvave piecewise linear function defined as the pointwise minimum of m ıffine functions

$$l_1(x), \ldots, l_m(x):$$
$$g(x) = \min_{i=1,\ldots,m} l_i(x).$$

Theorem 2. *The problem (4) can be formulaled as a biltnear programming ɔroblem.*

Proof. We construct a biaffine function $f(x, y)$ of variables $x \in R^n$ and $y \in R^m$ as follows

$$f(x, y) = \sum_{i=1}^{m} y_i l_i(x). \tag{5}$$

Let us define

$$Y = \{ y \in R^m : \sum_{i=1}^{m} y_i = 1, \ y \geqslant 0 \}, \tag{6}$$

Then Y has exactly m vertices $y^1, \ldots, y^m$, with y^i being the i-th unit, vector of R^m and from (5) we have $f(x, y^i) = l_i(x)$, $i = 1, \ldots, m$. Consequently-

$$\min_{y \in Y} f(x, y) = \min_{i=1,\ldots,m} f(x, y^i)$$
$$= \min_{i=1,\ldots,m} l_i(x) = g(x)$$

for every $x \in R^n$.

Thus, the problem (4) can be restated as

$$\min_{x \in X} g(x) = \min_{x \in X} \{ \min_{y \in Y} f(x, y) \}$$
$$= \min \{ f(x, y) : x \in X, \ y \in Y \}. \tag{7}$$

Since $f(x, y)$ is obviously a biaffine function and Y is the standard $(m-1)$ - simplex in R^m, (7) is a bilinear programming problem.

4. CONCAVE PROGRAMMING FORMULATION OF THE COMPLEMENTARITY PROBLEM

Consider now the following complementarity problem: Find an n – vec tor x satisfying

$$x \in D, \ f(x) \geqslant 0, \ g(x) \geqslant 0, \ \langle f(x), g(x) \rangle = 0 \tag{8}$$

where D is a given convex set in R^n, $f(x) = (f_1(x), \ldots, f_m(x))$, $g(x) = (g_1(x), \ldots, g_m(x))$ are given vector functions and $\langle f, g \rangle$ denotes the scalar product of f and g. Let

$$l(x) = \sum_{i=1}^{m} \min(f_i(x), g_i(x)),$$

We have the following.

Theorem 3. Let $f_i(x)$, $g_i(x)$, $i = 1, \ldots, m$, be concave functions on D. Then (8) is equivalent to the problem

$$\min \{ l(x) : x \in D,\ f(x) \geqslant 0,\ g(x) \geqslant 0 \} \tag{9}$$

in the sense: $\tilde{x} \in R^n$ is a solution of (8) if and only if $\tilde{x}$ is an optimal solution of (9) with $l(\tilde{x}) = 0$.

Note that the objective function of problem (9) is concave and its set of feasibles solutions is convex because of the assumed concavity of f_i and g_i. Hence, (9) is actually a concave programming problem.

Proof. Let $\tilde{x}$ be a solution of (8). Then it is obvious that $\tilde{x}$ is a feasible solution of (9). Since $f(\tilde{x}) \geqslant 0$, $g(\tilde{x}) \geqslant 0$ and $\langle f(\tilde{x}), g(\tilde{x}) \rangle = 0$ we have min $(f_i(\tilde{x}), g_i(\tilde{x})) = 0$ for $i = 1, \ldots, m$, i.e., $l(\tilde{x}) = 0$. But $l(x) \geqslant 0$ for every feasible solution x of (9). Consequently, $\tilde{x}$ is an optimal solution of (9).

Conversely, let $\tilde{x}$ be an optimal solution of (9) with $l(\tilde{x}) = 0$. Since $f_i(\tilde{x}) \geqslant 0$, $g_i(\tilde{x}) \geqslant 0$ we have min $(f_i(\tilde{x}), g_i(\tilde{x})) \geqslant 0$, $i = 1, \ldots, m$, hence min $(f_i(\tilde{x}), g_i(\tilde{x})) = 0$, $i = 1, \ldots, m$. This implies $\langle f(\tilde{x}), g(\tilde{x}) \rangle = 0$, which completes the proof.

A particular case of problem (8) is the linear complementarity problem: Find an n-vector x, an n-vector y, a p-vector z satisfying

$$\left.\begin{aligned} Ax + By + Cz &= b, \\ \langle x, y \rangle &= 0, \\ x, y, z &\geqslant 0, \end{aligned}\right\} \tag{10}$$

where A, B are m by n matrices, C is an m by p matrix, b is an m-vector and $\langle x, y \rangle$ denotes, as usuall, the scalar product of vectors x and y.

As specialized to this problem, Theorem 3 yields.

Corollary. $(\tilde{x}, \tilde{y}, \tilde{z})$ is a solution of (10) if and only if $(\tilde{x}, \tilde{y}, \tilde{z})$ is an optimal solution of the following problem

$$\min \{ l(x, y, z) = \sum_{i=1}^{m} \min(x_i, y_i) : Ax + By + Cz = b,\ x, y, z \geqslant 0 \}, \tag{1}$$

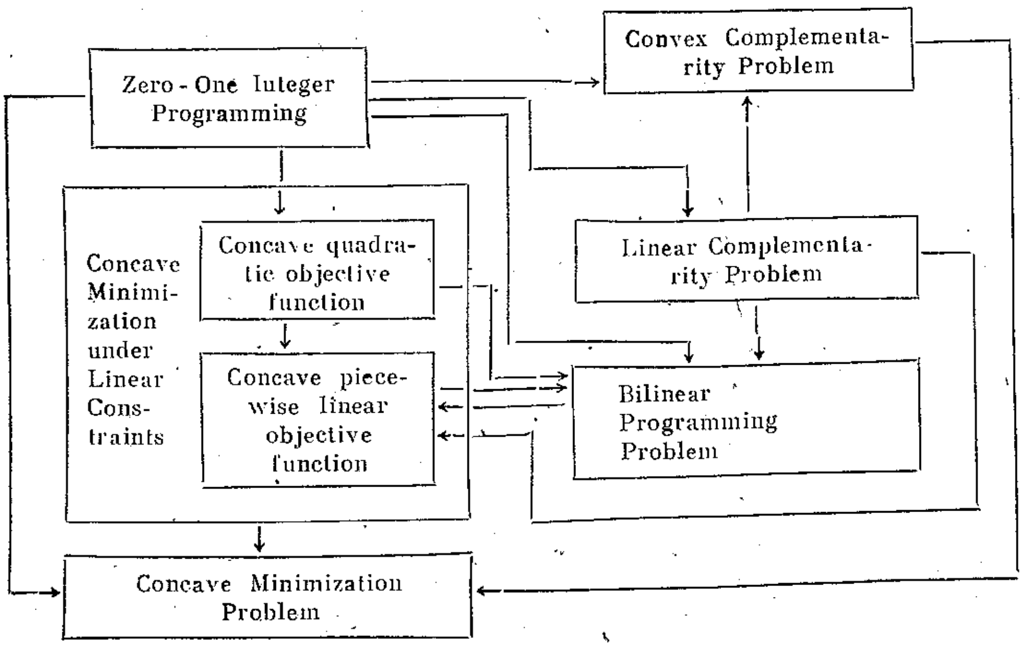
with $l(\tilde{x}, \tilde{y}, \tilde{z}) = 0$.

Clearly the objective function of (11) is a concave piecewise linear function.

If $A = M$, $B = -E$ (E is a unit matrix), $C = 0$ and $b = -q$ the problem (10) becomes: Find $x \in R^n$ satisfying $Mx + q \geqslant 0$, $x \geqslant 0$, $\langle x, Mx+q \rangle = 0$. This spesial problem has been considered by O.L. Mangasarian in [9]. Theorem 3 extends Lemma 1 is [9] of this author.

Theorem 3 shows that instead of finding a feasible solution of the complementarity problem (8), we can solve a corresponding concave minimization problem.

The connection among all the problem discused above are summarized in the following scheme



5. REMARKS ON A ALGORITHM OF GALLO AND ÜLKÜCÜ FOR BILINEAR PROGRAMMING

We have seen in the previous section that the bilinear programming problem is equivalent to the concave piecewise linear minimization problem under linear constraints.

In the light of this fact, we now show that the algorithm proposed by Gallo and Ülkücü in [4] for bilinear programming can be considered as a simple adaptation of Hoàng-Tuy's method [5] to a concave programming problem equivalent to the given bilinear progamming problem.

Let us first recall briefly the essential features of Gallo and Ülkücü's algorithm as presented in [4].

The blinear progamming problem has been considered in [4] in the following form:

$$\max\{f(x, y) : x \in X, y \in Y\} \tag{12}$$

with

$$f(x, y) = c^T x + x^T Q^T y + d^T y,$$
$$X = \{x \in R^n : Ax \leqslant a, x \geqslant 0\},$$
$$Y = \{y \in R^{n'} : B^T y \leqslant b, y \geqslant 0\},$$

where A is an m by n matrix, B^T an m' by n' matrix, Q^T an n by n' matrix; c, d, a, b are n, n', m, m' - vectors respectively. X and Y are assumed to be bounded and nonempty.

Using the Duality theory it can be easily shown that (12) is equivalent to the problem

$$\max_{\substack{Ax \leqslant 0 \\ x \geqslant 0}} \left\{ c^T x + \min_{\substack{Bu \geqslant d + Qx \\ u \geqslant 0}} b^T u \right\}. \tag{13}$$

Define $P = \{(x, u) : x \in X, u \geqslant 0, Bu \geqslant d + Qx\}$

and $\Lambda = \{(x, u) \in P : b^T u \leqslant b^T w, \forall (x, w) \in P\}$.

let $V = \{x^1, x^2, ..., x^k\}$ be a finite set of points in X. Define

$$z(V) = \max\{c^T x^i + b^T u^i : x^i \in V, (x^i, u^i) \in \Lambda\}$$

and

$$S(V) = \{x : \exists \bar{u} \geqslant 0, B\bar{u} \geqslant d + Qx, c^T x + b^T \bar{u} \leqslant z(V)\}.$$

As seen previously (see, remark 1)

Remark 2. (12) is equivalent to the problem $\max_{x \in X} g(x)$

with

$$g(x) = \max_{y \in Y} f(x, y) = \max_{y \in Y} \{c^T x + (d + Qx)^T y\}$$
$$= c^T x + \min_{\substack{Bu \geqslant d + Qx \\ u \geqslant 0}} b^T u, \tag{14}$$

i.e., $g(x)$ is just the objective function of (13).

Remark 3. From the definition of z(V) Λ, and from (14) we have

$$z(V) = \max\{c^T x^i + b^T u^i : x^i \in V, (x^i, u^i) \in \Lambda\}$$
$$= \max\{c^T x^i + \min b^T u : x^i \in V, Bu \geqslant d + Qx^i, u \geqslant 0\}$$
$$= \max\{g(x^i) : x^i \in V\},$$

i.e., z(V) is the best value of the objective function (14) at the considered points.

Remark 4. From (14) and the definition of S(V) we have for every $x \in S(V)$, $g(x) \leqslant c^T x + b^T \bar{u} \leqslant z(V)$, i.e., the value of the objective function $g(x)$ at every point of S(V) is not greater than the best value reached at that time.

Gallo and Ülkücü's algorithm.

Step 0. Pick in X a non-degenerate vertex x^o, that is, a vertex with exactly n neighboring vertices $x^1, x^2, ..., x^n$. For any vertex x^i of X, call v^i the unit vector defining the halfline emanating from x^o and containing x^i. Let $V = \{x^o, ..., x^n\}$, $\omega = \{v^1 ..., v^n\}$ and $\Delta = \{\omega\}$. Go to step 1.

Step 1. If $\Delta = \phi$, the algorithm terminates, and z (V) is the optimum value of the objective function. Otherwise, select a set $\omega = \{v^{i_1}, v^{i_2}, ..., v^{i_n}\}$ in Δ and go to step 2.

Step 2. By solving a linear program for each $j = 1, ..., n$, compute

$\bar{\theta}_{ij} = \max\{\theta_{ij} : x^o + \theta_{ij} v^{ij} \in S(V)\}$, and go to step 3.

Step 3. Find $(\lambda^*_{i_1}, ..., \lambda^*_{i_n})$, an optimal extreme solution of the linear program

$$\text{maximize} \sum_{j=1}^{n} (1/\bar{\theta}_{ij})\, \lambda_{ij}$$

subject to
$$x^o + \sum_{j=1}^{n} \lambda_{ij}\, v^{ij} \in X$$

and go to step 4.

Step 4. (a) If $\sum_{j=1}^{n} (1/\bar{\theta}_{i_j})\, \lambda^*_{ij} \leqslant 1$, delete the set ω from Δ, and go to step 1.

(b) Otherwise let $x^q = x^o + \sum_{j=1}^{n} \lambda^*_{ij}\, v^{ij}$. Include x^q in V and update z(V). For all j with $\lambda^*_{ij} > 0$, generate a set substituting v^q in ω for v^{ij}, then replace ω in Δ by these new sets. Go to step 1.

From remark 4 we see that step 2 consists just in constructing for each $j = 1, ..., n$ the farthest possible point on the ray from x^o through v^{ij} in which the value of the objective function $g(x)$ is still inferior or equal to the current best value z(V). The step 3 consists in finding the point of X that lies on the opposite side of x^o with respect to the hyperplane through the n points just constructed in step 2, and as far as possible from this hyperplane.

Thus, the steps in Gallo and Ülkücü algorithm for solving the bilinear programming (12), are exactly the same as in Hoàng-Tuy's algorithm for solving the concave minimization problem equivalent to the given bilinear programming problem. The difference is only that at step 4b:

The new sets are generated in Gallo and Ülkücü's algorithm only for those j with $\lambda^*_{ij} > 0$, when as in Tuy's original algorithm they were generated for every j with $\lambda^*_{ij} \neq 0$.

Finally it is worth noticing that no proof of finiteness has been given for either algorithm.

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