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SOME FIXED POINT THEOREMS FOR MULTIVALUED MAPPINGS

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1. INTRODUCTION

In recent years the Kakutani's fixed point theorem for multivalued mappings which is very useful in optimal theory, complementary problem and mathematical programming e.t.c. has been extended by various authors, H. Scarf [11]. B.C. Eaves [4] and O.H. Merrill [9] and H. Tui [13], have proved it by constructive method. K. Fan extended it to case of topological linear locally convex Hausdorff spaces.

In this paper, we shall be concerned with the continuity of fixed points of multivalued mappings. Let $\{F_v, v \in I\}$ be a system of multivalued mappings having the fixed points $\{x_v, v \in I\}$. The question arises as to what happens if $\{F_v, v \in I\}$ converges to a multivalued mapping F in some sense. With some necessity conditions we shall show that every limit point of a net of fixed points $\{x_v, v \in I\}$ of $\{F_v, v \in I\}$ will be a fixed point of the multivalued mapping F. Furthermore, we shall apply this result to obtain some fixed point theorem on topological linear locally convex Hausdorff spaces and on metric spaces Some results obtained here are more general than that of K. Fan [5], C.I. Himmelberg [7]. W.G. Dotson [2], [3], L.F. Guseman and B, C. Peters [6], L.A. Talman [12].

2. NOTATIONS AND DEFINITIONS

Let X be a topological linear space or a metric space, K be a non-empty subset of X. We shall denote by 2^K the family of all nonempty subsets of K and $\mathcal{B}(K)$ the family of all nonempty bounded closed subsets of K. $\overline{K}$ will denote the closure of K, and I stands for anindex set with a partial ordering.

In the sequel, we shall consider the following multivalued mappings:

$$F_v : K \to 2^X, \ v \in I$$

$$F : K \to 2^X.$$

Definition 1. A multivalued mapping F is called *closed* if for any net $\{x_v\} \subset K$, $x_v \to x$ and $\{y_v\} \subset X$, $y_v \in F(x_v)$, $y_v \to y$ it implies $y \in F(x)$.

Definition 2. A family of multivalued mappings $\{F_\nu, \nu \in I\}$ is calle *convergent to the multivalued mapping F in the sense* (*), in symbols $F_\nu \xrightarrow{(*)} F$ if for any net $\{x_\nu\} \subset K$, $x_\nu \to x$ for any net $\{y_\nu\} \subset X$, $y_\nu \in F_\nu(x_\nu)$ there exists a net $\{z_\nu\} \subset X$, $z_\nu \in F(x_\nu)$ such that $z_\nu - y_\nu \to 0$ if X is a topological linear space and $d(z_\nu, y_\nu) \to 0$ in the case where (X, d) is a metric space.

Let (X, d) be a metric space, we shall denote by H(B, A) the Hausdorff distance between two arbitrary sets A and B from $\mathcal{B}(K)$. It is defined by:

$$H(A, B) = \max \{\sup_{x \in A} d(x, B), \sup_{y \in B} d(A, y)\}.$$

It is a common knowledge that $(\mathcal{B}(K), H)$ is also a metric space.

Definition 3. A family $\{F_\nu, \nu \in I\}$ is called convergent uniformly to F, in the topology determined by the Hausdorff distance, if:

$$\lim_{\nu} \sup_{x \in K} H(F_\nu(x), F(x)) = 0.$$

Definition 4. Let (X, d) be a metric space. $\psi: X \to X$ is called contraction if there exists a constant $a \in (0, 1)$ such that:

$$d(\psi(x), \psi(y)) \leqslant a\, d(x, y), \text{ for all } x, y \in X.$$

Definition 5. A nonempty subset K of the metric space (X, d) is said to have ψ — contraction structure if there exists a sequence of contraction mappings $\psi = \{\psi_n\}_{n=0}^{\infty}$ converging uniformly to the identity mapping id on K, i.e. for any n there exists $a_n \in [0, 1)$ such that:

$$d(\psi_n(x), \psi_n(y)) \leqslant a_n\, d(x, y), \quad \text{for all } x, y \in X$$

and

$$\lim_{n \to +\infty} \sup_{x \in K} d(\psi_n(x), x) = 0.$$

Definition 6. A multivalued mapping F from a metric space X into itself is called nonexpansive if

$$H(F(x), F(y)) \leqslant d(x, y), \text{ for all } x, y \in X.$$

Definition 7. A subset K of the linear space X is said to be starshaped at a point x_0 if

$$\alpha x_0 + (1 - \alpha)x \in K, \text{ for all } x \in K \text{ and } \alpha \in [0, 1].$$

Definition 8. A subset K of a topological linear locally convex space X is called *almost convex* if for any neighborhood V of the origin and any finite subset $\{v_1, \ldots, v_n\} \subset K$ there exists a finite subset $\{z_1, \ldots, z_n\} \subset K$ such that:

co $\{z_1, \ldots, z_n\} \subset K$ and $v_i - z_i \in V$, for all $i = 1, 2, \ldots, n$; where co denotes the convex hull.

3. THE MAIN RESULTS

Theorem 1. Let X be a topological linear Hausdorff space, or a metric space, K be a subset of X. Let for each $v \in I$, $F_v : K \to 2^X$ be a multivalued mapping having a fixed point x_v in K. Suppose that $F_v \overset{(*)}{\to} F$, where $F : K \to 2^X$ is a closed multivalued mapping.

Then every limit point in K of $\{x_v\}$ is a fixed point of F in K.

Proof: Let $x_v \to x_o$, we shall verify that $x_o \in F(x_o)$.

Since $F_v \overset{(*)}{\to} F$. $x_v \to x_o$ and $x_v \in F_v(x_v)$ there exists a net $\{z_v\} \subset X$, $z_v \in F(x_v)$ such that $z_v - x_v \to 0$ in the case, where X is a topological linear Hausdorff space and $d(z_v, x_v) \to 0$ in the case where X is a metric space. Hence, in both cases, it implies: $z_v \to x_o$.

From the closedness of F we have $x_o \in F(x_o)$.

This completes the proof of Theorem.

Applying this result we prove the following theorem, from which it can be easily obtained the results of K. Fan [5], C. J. Himmelberg [7] (Theorem 1 and Theorem 2).

Theorem 2. Let K be a nonempty subset of a topological linear locally convex Hausdorff X, $F : K \to 2^K$ be a closed multivalued mapping such that $F(K)$ is contained in a compact subset C of K and $F(x)$ is nonempty convex for all x in some dense almost convex subset A of K. Then F has a fixed point in C.

Proof. Let $\mathcal{U} = \{U_v\}$ be a local base of neighborhoods of O consisting of closed convex symmetric sets, and $U_{v_1} \supseteq U_{v_2}$ if $v_2 \geqslant v_1$, $\{U_v\} \to 0$.
Set

$$F_v(x) = F(x) + U_v, \ x \in K.$$

Let $\{x_v\} \subset K$, $x_v \to x$ and $\{y_v\} \subset X$, $y_v \in F_v(x_v)$ we have

$$y_v = z_v + u_v, \text{ where } z_v \in F(x_v) \text{ and } u_v \in U_v$$

Therefore

$$y_v - z_v = u_v \in U_v.$$

Since $\{U_v\} \to 0$, hence $y_v - z_v \to 0$, and so $F_v \overset{(*)}{\to} F$ on K.

We now show that for any v, the multivalued mapping F_v has a fixed point in K.

Indeed, since C is a compact subset and A is dense almost convex subset of K, hence for any U_v there exists a finite subset $\{v_1, \ldots, v_n\} \subseteq A$ such that:

$$C \subseteq \bigcup_{i=1}^{n} v_i + \frac{1}{2} U_v.$$

By the almost convexity of A for $\frac{1}{2} U_v$ and $\{v_1, \ldots, v_n\} \subseteq A$ there exists a finite subset $\{z_1, \ldots, z_n\} \subseteq A$ such that

$$\text{co}\{z_1, \ldots, z_n\} \subseteq A \text{ and } v_i - z_i \in \frac{1}{2} U_v, \ i = 1, \ldots, n.$$

Put

$$C_v = co\{z_1, \ldots, z_n\}$$

and

$$H_v(x) = F_v(x) \cap C_v = (F(x) + U_v) \cap C_v \; , \; x \in C_v.$$

It is easy to verify that H_v is a closed multivalued mapping from C_v into itself and $H_v(x)$ is convex compact for all $x \in C_v$.

From the fact that

$$F(x) \subseteq C \subseteq \bigcup_{i=1}^{n} \{v_i + \frac{1}{2} U_v\} \subseteq \bigcup_{i=1}^{n} v_i - z_i + z_i + \frac{1}{2} U_v\}$$

$$\subseteq \bigcup_{i=1}^{n} \{z_i + U_v\} \subseteq C_v + U_v \text{ for all } x \in K.$$

we can deduce

$$(F(x) + U_v) \cap C_v \neq \phi, \text{ for all } x \in K,$$

and so $H_v(x) \neq \phi$ for all $x \in C_v$. Applying Kakutani's fixed point theorem [8] implies that H_v has a fixed point $x_v \in C_v$.

We have

$$x_v \in H_v(x) \subseteq F_v(x_v) \subseteq C + U_v$$

t means that x_v is a fixed point of F_v in C_v. Since $F_v \overset{(*)}{\rightarrow} F$ on K, and further rom the compactness of C it implies that $\{x_v\}$ has at least one limit point $x \in C$. and Theorem 1 shows that x is a fixed point of F in C.

This completes the proof of Theorem 2.

Remark. This theorem can be proved by constructive method if we apply Scarf's algorithm [4], [9] to find a fixed point of multivalued mappings H_v on C_v.

Lemma. Let (X,d) be a metric space, K be a nonempty subset of X. Let $F_n : K \rightarrow \mathcal{B}(X)$, n = 1, 2,..., and $F : K \rightarrow \mathcal{B}(X)$ be multivalued mappings so that F_n converges uniformly to F in the topology determined by the Hausdorff distance H on K.

Then $F_n \overset{(*)}{\rightarrow} F$

Proof. Since $\{F_n\}$ converges uniformly to F in the topology determined by the Hausdorff distance H on K, we have

$$\lim_{n \to +\infty} \sup_{x \in K} H(F_n(x), F(x)) = 0.$$

Let $\{x_n\}$ be a convergent sequence in K, we conclude

$$\overline{\lim_{n \to +\infty}} H(F_n(x_n), F(x_n)) \leqslant \lim_{n \to +\infty} \sup_{x \in K} H(F_n(x), F(x)) = 0.$$

From the definition of H we imply that for any sequence $\{y_n\} \subset X$, $y_n \in F_n(x_n)$ for any $\varepsilon_n > 0$, there exists a sequence $\{z_n\} \subset X$, $z_n \in F(x_n)$ such that

$$d(y_n, z_n) - \varepsilon_n \leqslant H(F_n(x_n), F(x_n)).$$

Take $\varepsilon_n \searrow 0$, we obtain

$$\overline{\lim_{n\to\infty}}\, d(y_n, z_n) \leqslant \overline{\lim_{n\to\infty}}\, H(F_n(x_n), F(x_n)) = 0,$$

and so $F_n \xrightarrow{(*)} F$.

Theorem 3. Let (X, d) be a metric space, K be a nonempty complete subset of X having ψ-contraction structure. Suppose that $F\colon K \to \mathcal{B}(K)$ is a nonexpansive multivalued mapping with $\overline{F(K)}$ compact. Then F has a fixed point in K.

Proof. For any natural number n we define

$$F_n(x) = F(\psi_n(x)),\ x \in K,$$

where ψ_n is from the property of ψ-contraction structure of K.

We have

$$\overline{\lim_{n\to\infty}} \sup_{x\in K} H(F_n(x), F(x)) = \overline{\lim_{n\to\infty}} \sup_{x\in K} H(F(\psi_n(x)), F(x))$$
$$\leqslant \overline{\lim_{n\to\infty}} \sup_{x\in K} d(\psi_n(x), x) = 0.$$

It shows that $\{F_n\}$ converges uniformly to F in the topology determined by the Hausdorff distance H on K.

From the Lemma, this implies that $F_n \xrightarrow{(*)} F$. Now, we verify that for any n, F_n has a fixed point in K.

Indeed, from the fact that:

$H(F_n(x), F_n(y)) = H(F(\psi_n(x)), F(\psi_n(y))) \leqslant d(\psi_n(x), \psi_n(y)) \leqslant a_n d(x, y)$, for all $x, y \in X$ and $a_n \in (0,1)$ is from the definition of the property of ψ-contraction structure of K.

Consequently, for any n, F_n is a contraction multivalued mapping from K into itself. Hence, by Nadler's theorem [10], there exists a point $x_n \in K$ such that $x_n \in F_n(x_n)$.

From the compactness of $\overline{F(K)}$ it implies that $\{x_n\}$ has at least one limit point x_0 in K. Theorem 1 shows that x_0 is a fixed point of F in K and then the proof of theorem is completed.

Corollary. Let X be a normed space and K be a nonempty bounded complete and starshaped at a point x_0 subset of X. Let $F\colon K \to \mathcal{B}(K)$ be a nonexpansive multivalued mapping with $\overline{F(K)}$ compact. Then F has a fixed point in K.

Proof. Since K is starshaped at x_0, set

$$\psi_n(x) = \frac{1}{n} x_0 + \left(1 - \frac{1}{n}\right) x,\ x \in K.$$

We obtain

$$\| \psi_n(x) - \psi_n(y) \| = \left(1 - \frac{1}{n}\right) \| x - y \| = a_n \| x - y \|,$$
$$\text{for all } x, y \in X,$$

where $a_n = \left(1 - \frac{1}{n}\right) \in (0,1)$. It means that for any n, ψ_n is a contraction mapping from K into itself. Because

$$\| \psi_n(x) - x \| = \frac{1}{n} \| x_0 - x \|, \text{ for all } x \in K$$

we have

$$\overline{\lim_{n \to \infty}} \sup_{x \in K} \| \psi_n(x) - x \| \leqslant \overline{\lim_{n \to \infty}} \sup_{x \in K} \frac{2}{n} \| x \| = 0,$$

and so $\{\psi_n\}$ converges uniformly to id on K.

It implies that K has the ψ-contraction structure with $\psi = \{\psi_n\}$.

Therefore the corollary follows immediatly from Theorem 3.

Remark. In the case, where F is a single mapping, our theorem 3 and its corollary include the results of W.G. Dotson [2], [3], L.F. Guseman and B,C. Peters [6], L.A. Talman [11].

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