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# ON THE CONTRACTION PRINCIPLE IN UNIFORMIZABLE SPACES

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## I. INTRODUCTION

In the recent years, many authors have extended the contraction principle to probabilistic metric spaces and probabilistic locally convex spaces.

By the results of Cain and Kasriel [2], it is easy to show that these spaces are special cases of uniformizable and quasiuniformizable spaces (defined below).

This paper will present some new fixed point theorems in uniformizable and quasiuniformizable spaces. These theorems, on one hand, generalize the corresponding results in probabilistic metric spaces and probabilistic locally convex spaces, and, on the other hand, simplify their proofs.

## II. FIXED POINT THEOREMS FOR UNIFORMIZABLE SPACES

First of all we recall some definitions.

**Definition 1.** Let X be an arbitrary set. A mapping $d : X \times X \to R^+$ is called a *pseudo-metric* if for every x, y, z in X:

a) $d(x, y) \geqslant 0, d(x, x) = 0,$

b) $d(x, y) = d(y, x),$

c) $d(x, y) \leqslant d(x, z) + (z, y).$

**Definition 2.** A pair $(X, d_\alpha)$ where $d_\alpha$ is a pseudo-metric for each $\alpha$ in an arbitrary index set A, is called a *uniformizable space*.

In the sequel we suppose that the family of $d_\alpha$ has an additional condition:

$$d_\alpha(x, y) = 0. \ (\forall \alpha \in A) \Rightarrow x = y.$$

It is well known that a uniformizable space with this property is a Hausdorff topological space.

**Theorem 1.** *Let $(X, d_\alpha)$ be a complete uniformizable space. T be a mapping in X satisfying the condition: for each $\alpha \in A$ there is $k_\alpha < 1$ such that:*

$$d_\alpha(Tx, Ty) \leqslant k \max \{d_\alpha(x, y), d_\alpha(x, Tx)\ d_\alpha(y, Ty), d_\alpha(x, Ty), d_\alpha(y, Tx)\}$$
*for every* $x, y \in X$.

*Then T has a unique fixed point* $x^*$ *and* $T^n x \to x^*$ *as* $n \to \infty$ *for each* $x \in X$.

**Proof.** It suffices to repeat the proof of Theorem 3 in [3] for each $\alpha \in A$ and to use the separateness of X.

We denote

$$r_\alpha(x, y) = \max\{d_\alpha(x, y), d_\alpha(x, Tx), d_\alpha(y, Ty), \frac{1}{2}[d_\alpha(x, Ty) + d_\alpha(y, Tx)]\}.$$

**Theorem 2.** *Let* $(X, d_\alpha)$ *be a complete uniformizable space, T be a continuous mapping in X, satisfying the condition: for every* $\varepsilon > 0$ *and* $\alpha \in A$ *there is* $\delta = \delta(\varepsilon, \alpha) > 0$ *such that*

$$r_\alpha(x, y) < \varepsilon + \delta \Rightarrow d_\alpha(Tx, Ty) < \varepsilon. \tag{1}$$

*Then the conclusion of Theorem 1 still holds.*

**Proof.** First, we note that (1) implies the following condition

$$\left.\begin{array}{l} r_\alpha(x, y) > 0 \Rightarrow d_\alpha(Tx, Ty) < r_\alpha(x, y), \\ r_\alpha(x, y) = 0 \Rightarrow d_\alpha(Tx, Ty) = 0. \end{array}\right\} \tag{2}$$

Indeed, if $r_\alpha(x, y) > 0$ we take $\varepsilon = r_\alpha(x, y)$. Since $r_\alpha(x, y) < \varepsilon + \delta$, by (1) we have $d_\alpha(Tx, Ty) < \varepsilon = r_\alpha(x, y)$. If $r_\alpha(x, y) = 0$ then $r_\alpha(x, y) < \varepsilon + \delta$ for each $\varepsilon > 0$. Hence by (1) we have $d_\alpha(Tx, Ty) < \varepsilon$ for each $\varepsilon > 0$, i. e. $d_\alpha(Tx, Ty) = 0$.

We now take $x_0 \in X$ and put $x_{n+1} = Tx_n$, $n = 0, 1, 2, \ldots$, Fix $\alpha \in A$, it suffices to show that $\{x_n\}$ is a Cauchy sequence for $d_\alpha$. In the sequel N denotes the set of natural numbers.

Remark that for each $n \in N$ we have

$$r_\alpha(x_{n-1}, x_n) = \max\{d_\alpha(x_{n-1}, x_n), d_\alpha(x_n, x_{n+1})\}.$$

It follows that if $r_\alpha(x_{n-1}, x_n) = 0$ then $r_\alpha(x_n, x_{n+1}) = 0$. Indeed,

$r_\alpha(x_{n-1}, x_n) = 0 \Rightarrow d_\alpha(x_n, x_{n+1}) = 0 \Rightarrow r_\alpha(x_n, x_{n+1}) = d_\alpha(x_{n+1}, x_{n+2})$. If $r_\alpha(x_n, x_{n+1}) > 0$ by (2) we get a contradiction:

$$d_\alpha(x_{n+1}, x_{n+2}) < d_\alpha(x_{n+1}, x_{n+2}).$$

Thus, we may assume $r_\alpha(x_{n-1}, x_n) > 0$ for each n. Then it suffices to repeat the proof of Theorem 1.1 in [10].

**Theorem 3.** Let $(X, d_\alpha)$ be a complete uniformizable space, T be a mapping in X satisfying:

1) for each $\alpha \in A$ there exists a nondecreasing function $q_\alpha : R^+ \to [0,1]$ and $f(\alpha) \in A$ such that

$$d_\alpha(Tx, Ty) \leqslant q_\alpha(d_{f(\alpha)}(x, y))\, d_{f(\alpha)}(x, y)$$

for every $x, y$ in X.

2) for each $\alpha \in A$ and $t > 0$

$$\overline{\lim_{n\to\infty}}\ q_{f^n(\alpha)}(t) < 1,$$

3) there is $x_0 \in X$ such that for each $\alpha \in A$

$$K_\alpha = \sup \{ d_{f^n(\alpha)}(x_0, Tx_0) : n \in N \} < \infty.$$

Then there exists a unique $x^* \in X$ such that

4) $x^* = Tx^*$ and

5) for each $\alpha \in A$

$$\sup \{ d_{f^n(\alpha)}(x_0, x^*) : n \in N \} < \infty.$$

**Proof.** Put $x_{n+1} = Tx_n$, $n = 0, 1, 2, \ldots$. We shall show that $\{x_n\}$ is a Cauchy sequence. Fix $\alpha \in A$ then choose $n_\alpha$ by assumption 2) so that

$$q_{f^n(\alpha)}(K_\alpha) \leqslant Q_\alpha < 1$$

for each $n \geqslant n_\alpha$.

Take $n \geqslant n_\alpha$. From assumption 1) it follows

$$d_\alpha(x_n, x_{n+1}) = d_\alpha(Tx_{n-1}, Tx_n) \leqslant q_\alpha(d_{f(\alpha)}(x_{n-1}, x_n)) \times \ldots$$
$$\times\, q_{f^{n-1}(\alpha)}(d_{f^n(\alpha)}(x_0, x_1))\; d_{f^n(\alpha)}(x_0, x_1). \tag{3}$$

From assumptions 2) and 3) it follows that for each $m \in \{n_\alpha, \ldots, n\}$ we have

$$d_{f^m(\alpha)}(x_{n-m}, x_{n-m+1}) < d_{f^{m+1}(\alpha)}(x_{n-m-1}, x_{n-m}) < \ldots < d_{f^n(\alpha)}(x_0, x_1) \leqslant K_\alpha.$$

From (3) we have

$$d_\alpha(x_n, x_{n+1}) \leqslant Q_\alpha^{n-n_\alpha} K_\alpha.$$

Since $Q_\alpha < 1$, it is easily seen that $\{x_n\}$ is Cauchy and hence, $x_n \to x^* \in X$.

Further, for each $\alpha \in A$ we have

$$d_\alpha(x_{n+1}, Tx^*) = d_\alpha(Tx_n, Tx^*) \leqslant q_\alpha(d_{f(\alpha)}(x_n, x^*))\, d_{f(\alpha)}(x_n, x^*)$$
$$\leqslant M_\alpha\, d_{f(\alpha)}(x_n, x^*).$$

Hence $x_{n+1} \to Tx^*$, and consequently $x^* = Tx^*$.

To show that $x^*$ satisfies condition 5) we fix $\alpha \in A$, $m \geqslant n_\alpha$ (defined above) and take $n \geqslant m$. Then

$$d_{f^m(\alpha)}(x_n, x_0) \leqslant d_{f^m(\alpha)}(x_n, x_{n-1}) + \ldots + d_{f^m(\alpha)}(x_1\, x_0)$$
$$\leqslant q_{f^m(\alpha)}(d_{f^{m+1}(\alpha)}(x_{n-1}, x_{n-2})) \ldots q_{f^{m+n-2}(\alpha)}(d_{f^{m+n-1}(\alpha)}(x_1, x_0)) \times$$
$$\times\, d_{f^{m+n-1}(\alpha)}(x_1, x_0) + \ldots + d_{f^m(\alpha)}(x_1\, x_0).$$

From assumptions 2) and 3) it follows

$$d_{f^m(\alpha)}(x_n, x_0) \leqslant K_\alpha \sum_{j=0}^{n} Q_\alpha^j \leqslant \frac{K_\alpha}{1 - Q_\alpha}.$$

Hence $x^*$ satisfies condition 5).

Finally we shall prove the uniqueness of $x^*$. Let y satisfy the condition 4) and 5). Fix $\alpha \in A$ and denote

$$P_\alpha(y) = \sup \{ d_{f^n(\alpha)}(x_0, y) : n \in N \}$$

From assumption 1) we have

$$d_\alpha(x^*, y) = d_\alpha(Tx^*, Ty) \leqslant q_\alpha(d_{f(\alpha)}(x^*, y))\, d_{f(\alpha)}(x^*, y) \leqslant \dots$$

$$q_\alpha(d_{f(\alpha)}(x^*, y)) \dots q_{f^{n-1}(\alpha)}(d_{f^n(\alpha)}(x^*, y))\, d_{f^n(\alpha)}(x^*, y). \tag{4}$$

Note that by condition 5) for each $j \in N$ we have

$$d_{f^j(\alpha)}(x^*, y) \leqslant d_{f^j(\alpha)}(x^*, x_0) + d_{f^j(\alpha)}(x_0, y) \leqslant P_\alpha(x^*) + P_\alpha(y).$$

Choose $n_\alpha$ so that for $n \geqslant n_\alpha$ $q_{f^n(\alpha)}(P_\alpha(x^*) + P_\alpha(y)) \leqslant \widetilde{Q}_\alpha < 1$, from (4) we have

$$d_\alpha(x^*, y) \leqslant \widetilde{Q}_\alpha^{n-n_\alpha} [P_\alpha(x^*) + P_\alpha(y)].$$

Letting $n \to \infty$ we get $d_\alpha(x^*, y) = 0$ for each $\alpha = A$, i.e. $x^* = y$ and the proof is completed.

**Remark 1.** If we require the inequality in Condition 2) of Theorem 3 to be uniform in t then in Condition 1) we can suppose that $q_\alpha$ are arbitrary bounded functions of $R^+$ into iself. So in this form our theorem generalizes a result of Hadzic and Stankovic [6] where $X$ is assumed to be locally convex and $q_\alpha$ to be constant.

**Theorem 4.** Let $(X, d_\alpha)$ be a complete uniformizable space, T be a continuous mapping in $X$. Suppose that

1) for each $\alpha \in A$ there are a nondecreasing function $q_\alpha : R^+ \to [0,1]$ and $f(\alpha) \in A$ satisfying the condition: for each $x \in X$ there exists $m(x) \in N$ such that:

$$d_\alpha(T^{m(x)} x, T^{m(x)} y) \leqslant q_\alpha(d_{f(\alpha)}(x, y))\, d_{f(\alpha)}(x, y)$$

for every $y \in X$,

2) for each $\alpha \in A$ and $t > 0$

$$\overline{\lim_{n\to\infty}}\; q_{f^n(\alpha)}(t) < 1,$$

3) there is $x_0 \in X$ such that for each $\alpha \in A$ there is $n_\alpha \in N$ with

$$K_\alpha = \sup\{d_{f^n(\alpha)}(x_0, T^s x_0) : n \geqslant n_\alpha,\ s \in N\} < \infty.$$

Then there exists a unique $x^* \in X$ such that

4) $x^* = Tx^*$,

5) for each $\alpha \in A$

$$\sup\{d_{f^n(\alpha)}(x_0, x^*) : n \in N\} < \infty.$$

**Proof.** Put $m_i = m(x_i)$, $x_{i+1} = T^{m_i} x_i$ $(i = 0, 1, 2, ..)$. Take $k \in N$, $\alpha \in A$, by assumption 1) it follows that for each $n \in N$ we have

$$d(T^k x_n, x_n) = d_\alpha(T^k T^{m_{n-1}} x_{n-1}, T^{m_{n-1}} x_{n-1}) \leqslant$$

$$\leqslant q_\alpha(d_{f(\alpha)}(T^k x_{n-1}, x_{n-1})) \dots q_{f^{n-1}(\alpha)}(d_{f^n(\alpha)}(T^k x_0, x_0))\, d_{f^n(\alpha)}(T^k x_0, x_0). \tag{5}$$

Choose $n_\alpha$ such that the condition 3) holds and simultaneously

$$q_{f^n(\alpha)}(K_\alpha) \leqslant Q_\alpha < 1$$

for $n \geqslant n_\alpha$.

Put $n \geqslant n_\alpha$ and $p \in N$. Then $x_{n+p} = T^{m_{n+p-1}+\dots+m_n}x_n$.
With $k = m_{n+p-1} + \dots + m_n$ in (5), by an argument analogous to that used in the proof of Theorem 3 we have

$$d_\alpha(x_{n+p}, x_n) \leqslant Q_\alpha^{n-n_\alpha} K_\alpha = c_\alpha(n)$$

for each $p \in N$. From this $\{x_n\}$ is Cauchy and hence $x_n \to x^* \in X$.

If in (5) we take $k = 1$ then we get

$$d_\alpha(Tx_n, x_n) \leqslant c_\alpha(n)$$

for each $\alpha \in A$. Consequently, $Tx_n \to x^*$.

Since T is continuous, $Tx_n \to Tx^*$. Thus $x^* = Tx^*$. Putting

$$P_\alpha(x^*) = \max \{K_\alpha, d_\alpha(x_0, x^*), \dots, d_{f^{n-1}(\alpha)}(x_0, x^*)\}$$

we see that $x^*$ satisfies condition 5).

The proof of the uniqueness of $x^*$ is similar to that used in the proof of Theorem 3, here we note that if $x$ and y satisfy condition 4) then $T^{m(x^*)}x^* = x^*$, $T^{m(x^*)}y = y$. The rest of the proof is obvious and it can be omitted.

Note that Remark 1 still holds for Theorem 4.

**Remark 2.** *Under the hypotheses of Theorem 4 we may claim that* $\lim_{n\to\infty} T^n x_0 = x^*$.

Indeed, every $n \in N$ is of the form $n = r \cdot m(x^*) + p$, $0 \leqslant p < m(x^*)$. Fix $\alpha$ and choose $n_\alpha$ so that $d_{f^n(\alpha)}(x_0, T^s x_0) \leqslant K_\alpha$ $(s = 1, 2, \dots)$ and $q_{f^n(\alpha)}(K_\alpha + P_\alpha(x^*)) \leqslant Q_\alpha < 1$. $(\forall n \geqslant n_\alpha)$. Taking $n \geqslant n_\alpha$ we have

$$d_\alpha(T^n x_0, x^*) = d_\alpha(T^{rm(x^*)+p}x_0, T^{m(x^*)}x^*) \leqslant \dots \leqslant$$

$$\leqslant q_\alpha(d_{f(\alpha)}(T^{(r-1)m(x^*)+p}x_0, x^*)) \dots q_{f^{r-1}(\alpha)}(d_{f^r(\alpha)}(T^p x_0, x^*))\, d_{f^r(\alpha)}(T^p x_0, x_0). \quad (6)$$

Note that for each $j \in \{n_\alpha, \dots, r\}$, we have

$$d_{f^j(\alpha)}(T^{(r-j)m(x^*)+p}x_0, x^*) \leqslant d_{f^j(\alpha)}(T^{(r-j)m(x^*)+p}x_0, x_0) + d_{f^j(\alpha)}(x_0, x^*)$$
$$\leqslant K_\alpha + P_\alpha(x^*).$$

From (6) we get

$$d_\alpha(T^n x_0, x^*) \leqslant Q_\alpha^{r-n_\alpha}[K_\alpha + P_\alpha(x^*)].$$

Since $r \to \infty$ as $n \to \infty$ we have $d_\alpha(T^n x_0, x^*) \to 0$ for each $\alpha \in A$ and this completes the proof.

## III — APPLICATIONS TO PROBABILISTIC METRIC SPACES WITH $\Delta(a, a) \geqslant a$

Recall some definitions

A function $F : R \to [0, 1]$ is called a *distribution function* if it is nondecreasing, left-continuous, $\inf F = 0$, $\sup F = 1$. By $\mathcal{L}$ we denote the family of distribution functions.

Let $X$ be an arbitrary set, $\mathcal{F}$ be a mapping of $X \times X$ into $\mathcal{L}$. In what follows we shall denote by $F_{xy}(t)$ the value of $\mathcal{F}(x, y)$ at $t$.

**Definition 3.** A pair $(X, \mathcal{F})$ is called a *probabilistic metric space* (or briefly, PM-space) if for every $x, y, z \in X$

1) $F_{xy}(t) = 1 \quad (\forall t > 0) \Leftrightarrow x = y,$
2) $F_{xy}(0) = 0,$
3) $F_{xy} = F_{yx},$
4) $F_{xy}(t) = 1, \; F_{yz}(s) = 1 \Rightarrow F_{xz}(t + s) = 1.$

**Definition 4.** A mapping $\Delta: [0, 1]^2 \to [0, 1]$ is called a $\Delta$-*norm* if for every $a, b, c \in [0, 1]$

1) $\Delta(0, 0) = 0, \quad \Delta(a, 1) = a,$
2) $\Delta(a, b) = \Delta(b, a),$
3) $\Delta(a, b) \geqslant \Delta(c, d)$ if $a \geqslant c, \; b \geqslant d,$
4) $\Delta(\Delta(a, b), c) = \Delta(a, \Delta(b, c)).$

**Definition 5.** A triple $(X, \mathcal{F}, \Delta)$ is called a *Menger space* if $(X, \mathcal{F})$ is a PM-space, $\Delta$ is a $\Delta$-norm and moreover

$$F_{xz}(t + s) \geqslant \Delta(F_{xy}(t), F_{yz}(s))$$

for every $x, y, z \in X$, $t, s \in R$.

Throughout this section we assume that for each $a \in [0, 1]$:

$$\Delta(a, a) \geqslant a. \; (^1)$$

Menger spaces have been detailly considered by Cain and Kasriel in [2]. Here we recall only some important facts that used in the sequel.

A sequence $\{x_n\}$ is said to be convergent to $x$ in X if for every $\varepsilon > 0$ and $\lambda \in (0,1)$ there is $n_0 \in N$ such that $F_{x_n x}(\varepsilon) > 1 - \lambda$ for all $n \geqslant n_0$. Similarly, for the definition of a Cauchy sequence. The completeness is defined naturally.

Denote $d_\lambda(x,y) = \sup\{t: F_{xy}(t) \leqslant 1 - \lambda\}$ for every $x, y \in X$; $\lambda \in (0,1)$. Then $d_\lambda$ is a pseudometric on X and

$$d_\lambda(x, y) = 0 \; (\forall \lambda \in (0, 1)) \Leftrightarrow x = y.$$

Moreover, we have

$$F_{xy}(d_\lambda(x, y)) \leqslant 1 - \lambda. \tag{7}$$

Metric spaces are special cases of Menger spaces with $\Delta(a, a) \geqslant a$.

**Theorem 5.** Let $(X, \mathcal{F}, \Delta)$ be a complete Menger space with $\Delta(a, a) \geqslant a$, T be a mapping in X satisfying the following condition: there is $k < 1$ such that

$$F_{TxTy}(kt) \geqslant \min\{F_{xy}(t), F_{xTx}(t), F_{yTy}(t), F_{xTy}(t), F_{yTx}(t)\} \tag{8}$$

for every $x, y \in X$, $t > 0$.

Then there exists a unique fixed point $x^*$ of T. Moreover $T^n x \to x^*$ as $n \to \infty$ for each $x \in X$.

(1) In fact, this condition in combination with 1) in Def. 4 gives $\Delta = \min$. However in the sequel condition 1) in Def. 4 is used nowhere.

**Proof.** It suffices to show that T satisfies conditions in Theorem 1 for above defined $d_\lambda$.

In the contrary case there would exist $\lambda$, $x$, y such that

$$d_\lambda(Tx, Ty) > k\max\{d_\lambda(x, y), d_\lambda(x, Tx), d_\lambda(y, Ty), d_\lambda(x, Ty), (d_\lambda(y, Tx)\}.$$

Put $t = d_\lambda(Tx, Ty)/k$. Since $t > \max\{d_\lambda(x, y), \dots, d_\lambda(y, Tx)\}$, $F_{xy}(t) > 1 - \lambda, \dots, F_{yTx}(t) > 1-\lambda$. From this and (8) it follows

$$F_{TxTy}(kt) = F_{TxTy}(d_\lambda(Tx, Ty)) > 1 - \lambda.$$

contradicting (7). The proof is completed.

Remark that this theorem implies Theorem 3 in [9].

**Theorem 6.** Let $(X, \mathcal{F}, \Delta)$ be a complete Menger space with $\Delta(a, a) \geqslant a$, T be a continuous mapping in X, satisfying the following condition: for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$F_{TxTy}(\varepsilon) \geqslant \min\{F_{xy}(\varepsilon + \delta), F_{xTx}(\varepsilon + \delta), F_{yTy}(\varepsilon + \delta), \max[F_{xTy}(\varepsilon + \delta), F_{yTx}(\varepsilon + \delta)]\} \tag{9}$$

for every $x, y \in X$.

Then the conclusion of Theorem 5 still holds.

**Proof.** It suffices to show that T satisfies the conditions in Theorem 2. Let $\varepsilon > 0$ we choose $\delta > 0$ such that (9) holds. Let $r_\lambda(x, y) < \varepsilon + \delta$ then by (7), $F_{xy}(\varepsilon + \delta) > 1 - \lambda$, $F_{xTx}(\varepsilon + \delta) > 1 - \lambda$, $F_{yTy}(\varepsilon + \delta) > 1 - \lambda$. To show that $\max\{F_{xTy}(\varepsilon + \delta), F_{yTx}(\varepsilon + \delta)\} > 1 - \lambda$ we note that in the contrary case we would have

$$\frac{1}{2}[d_\lambda(x, Ty) + d_\lambda(y, Tx)] \geqslant \varepsilon + \delta,$$

contradicting the fact that $r_\lambda(x, y) < \varepsilon + \delta$. Thus by (9), $F_{TxTy}(\varepsilon) > 1 - \lambda$ hence $d_\lambda(Tx, Ty) < \varepsilon$.

Applying Theorem 2, the theorem follows.

**Corollary.** Let $(X, \mathcal{F}, \Delta)$ be a complete Menger space with $\Delta(a,a) \geqslant a$, T be a mapping in X satisfying the condition: there exists an upper semicontinuous from the right function $k : (0, \infty) \to (0, 1)$ such that

$$F_{TxTy}(k(t)t) \geqslant F_{xy}(t) \tag{10}$$

for every $x, y \in X$, $t > 0$.

Then the conclusion of Theorem 5 still holds.

**Proof.** We shall show that (10) implies the following condition: for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$F_{TxTy}(\varepsilon) \geqslant F_{xy}(\varepsilon + \delta) \tag{11}$$

for every $x, y \in X$.

Take $\varepsilon > 0$ we have $k(\varepsilon)\varepsilon < \varepsilon$. Since the function $t \mapsto k(t)t$ is upper semicontinuous from the right, there is $\delta > 0$ such that

$$k(t)t < \varepsilon \text{ if } \varepsilon \leqslant t < \varepsilon + \delta.$$

Hence for each $t < \varepsilon + \delta$ we have

$$F_{TxTy}(\varepsilon) \geqslant F_{TxTy}(k(t)t) \geqslant F_{xy}(t).$$

By the left-continuity of $F_{xy}$, from this we have (11). Since (11) implies (9), the corollary follows. Remark that, this corollary generalises Theorem 2.1 in [8].

**Theorem 7.** Let $(X, \mathcal{F}, \Delta)$ be a complete Menger space with $\Delta(a, a) \geqslant a$, T be a mapping in X satisfying the condition

$$F_{TxTy}(t) > F_{xy}(t) \tag{12}$$

for every $x \neq y$, and $t > 0$.

Suppose in addition that there exists $x_0 \in X$ such that the sequence $\{T^n x_0\}_{n=0}^{\infty}$ contains a subsequence converging to $x^* \in X$.

Then $x^* = Tx^*$ and $T^n x_0 \to x^*$ as $n \to \infty$.

**Proof.** Put $d_\lambda$ as above, from (12) it follows

$$d_\lambda(Tx, Ty) \leqslant d_\lambda(x, y)$$

for every $x, y, \lambda$. Furthermore, if $x \neq y$, there exists $t > 0$ such that $F_{xy}(t) = 1 - \delta$ for some $\delta > 0$ Then $t \leqslant d_\delta(x, y)$ and hence $F_{TxTy}(t) > 1 - \delta$. From this $d_\delta(Tx, Ty) < t \leqslant d_\delta(x, y)$.

Applying a theorem of Ang and Daykin [1], the theorem follows.

Now let X be an arbitrary set, I be any index set, $\mathcal{F}^i$ $(i \in I)$ be a mapping of X into $\mathcal{L}$. As above the value of $\mathcal{F}^i(x)$ at t is denoted by $F^i_x(t)$.

**Definition 6.** A triple $(X, \mathcal{F}^i, \Delta)$ is called a probabilistic locally convex space (briefly, PLC-space), if $\Delta$ is a $\Delta$-norm and for every $x, y, z \in X$, $i \in I$

1) $F^i_x(t) = 1$ $(\forall t > 0,\ \forall i \in I) \Leftrightarrow x = 0$,

2) $F^i_x(0) = 0$,

3) $F^i_{cx}(t) = F^i_x\left(\frac{t}{|c|}\right)$ $(\forall t > 0,\ \forall c \neq 0)$

4) $F^i_{x+y}(t+s) \geqslant \Delta(F^i_x(t), F^i_y(s))$.

Throughout this section we assume that $\Delta(a, a) \geqslant a$ for each $a \in [0, 1]$. It is known that locally convex spaces are special cases of PLC-spaces.

In PLC-spaces the convergence, Cauchy net and completeness are defined similarly to that in PM-spaces, but here a net stands for a sequence.

The notion of PLC-spaces has been introduced in [7](1) and recalled in [4]. Following the scheme in [2] we easily obtain the following facts:

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1) The author is very sorry that he has no opportunity to see this book.

i) the topology in a PLC-space is a locally convex Hausdorff topology which is generated by the family of pseudometrics

$$d_{i\lambda}(x, y) = \sup\{t : F^{i}_{x-y}(t) \leqslant 1 - \lambda\} \qquad (i \in I,\ \lambda \in (0, 1)),$$

(ii) The family of pseudo-metrics $\{d_{i\lambda} : i \in I,\ \lambda \in (0, 1)\}$ has the following properties:

1) $d_{i\lambda}(x, y) = 0\ (\forall i, \lambda) \Leftrightarrow x = y$,

2) for fixed $x$, y, i the function $d_{i\lambda}(x, y)$ is nonincreasing and left-continuous in $\lambda$.

3) $F^{i}_{x-y}(d_{i\lambda}(x, y)) \leqslant 1 - \lambda$,

4) $d_{i\lambda}(\lambda x, \lambda y) = |\lambda|\, d_{i\lambda}(x, y)$,

5) $d_{i\lambda}(x + z, y + z) = d_{i\lambda}(x, y),\ (\forall z \in X)$

(iii) The previous properties characterises a PLC-space.

**Theorem 8.** Let $(X, \mathcal{F}^{i}, \Delta)$ be a complete PLC-space with $\Delta(a, a) \geqslant a$, T be a mapping in X with properties:

1) for each $i \in I$ there exists a nondecreasing bounded right-continuous function $q_i : R^{+} \to [0, 1]$ and $f(i) \in I$ such that

$$F^{i}_{Tx-Ty}(q_i(t)t) \geqslant F^{f(i)}_{x-y}(t)$$

for every $x$, $y \in X$, $t > 0$.

2) for each $i \in I$

$$\overline{\lim_{n\to\infty}}\, q_{f^{n}(i)}(t) < 1,$$

3) there is $x_0 \in X$ such that for each $i \in I$

$$\lim_{t\to\infty} F^{f^{n}(i)}_{Tx_0-x_0}(t) = 1$$

uniformly in $n \in N$.

Then there exists a unique $x^* \in X$ such that

4) $x^* = Tx^*$.

5) for each $i \in I$

$$\lim_{t\to\infty} F^{f^{n}(i)}_{x^*-x_0}(t) = 1$$

uniformly in $n \in N$.

**Proof.** Put $\lambda \in (0, 1)$, $\alpha = (i, \lambda)$, $q_\alpha = q_i$, $f(\alpha) = (f(i), \lambda)$. We shall show that for every $\alpha$, $x$, y

$$d_\alpha(Tx, Ty) \leqslant q_\alpha(d_{f(\alpha)}(x, y))\, d_{f(\alpha)}(x, y).$$

In the contrary case there would exist $\alpha$, $x$, y such that

$$d_\alpha(Tx, Ty) > q_\alpha(d_{f(\alpha)}(x, y))\, d_{f(\alpha)}(x, y).$$

By the right-continuity of $q_\alpha$ there is $t < d_{f(\alpha)}(x, y)$ such that

$$d_\alpha(Tx, Ty) \geqslant q_\alpha(t)t = q_i(t)t.$$

Then we have

$$F^i_{Tx-Ty}(d_\alpha(Tx, Ty)) \geqslant F^i_{Tx-Ty}(q_i(t)t) \geqslant F^i_{x-y}(t) > 1 - \lambda,$$

a contradiction.

To apply Theorem 3 it suffices to show that condition 5) is equivalent to the following one:

$$\sup\{d_{f^n(\alpha)}(x_o, x^*) : n \in N\} < \infty.$$

Indeed,

$$\lim_{t\to\infty} F^{f^n(i)}_{x_0-x^*}(t) = 1 \text{ (uniformly in } n \in N) \Leftrightarrow \forall(i, \lambda)\ \exists P_{i\lambda}(x^*) \text{ such that}$$

$$F^{f^n(i)}_{x_0-x^*}(t) > 1 - \lambda\ (\forall t \geqslant P_{i\lambda}(x^*)) \Leftrightarrow d_{f^n(i),\lambda}(x_o, x^*) < t \quad (\forall t \geqslant P_{i\lambda}(x^*))$$
$$\Leftrightarrow d_{f^n(\alpha)}(x_o, x^*) < P_\alpha(x^*) \text{ with } \alpha = (i, \lambda).$$

The proof is completed.

Note that Remark 1 applied to Theorem 8 extends a result of Hadzic [4], where $q_i$ are constant.

**Theorem 9.** Let $(X, \mathcal{F}^i, \Delta)$ be a complete PLC-space with $\Delta(a, a) \geqslant a$, T be a continuous mapping in X. Assume

1) for each $i \in I$ there are a nondecreasing bounded right-continuous function $q_i: R^+ \to [0,1]$ and $f(i) \in I$ satisfying the condition: for each $x \in X$ there is $m(x) \in N$ such that

$$F^i_{T^{m(x)}x - T^{m(x)}y}(q_i(t)\, t) \geqslant F^{f(i)}_{x-y}(t)$$

for every $y \in X$, $t > 0$.

2) for each $i \in I$

$$\overline{\lim_{n\to\infty}}\ q_{f^n(i)}(t) < 1,$$

3) there is $x_o \in X$ such that for each $i \in I$ there exists $n_i \in N$ with

$$\lim_{t\to\infty} F^{f^n(i)}_{T^s x_0 - x_0}(t) = 1$$

uniformly in $n \geqslant n_i$ and $s \in N$.

Then there exists a unique $x^* \in X$ with

4) $x^* = Tx^*$,

5) for each $i \in I$

$$\lim_{t\to\infty} F^{f^n(i)}_{x_0-x^*}(t) = 1$$

uniformly in $n \in N$.

The proof of this theorem is analogous to that of Theorem 8 using Theorem 4 instead of Theorem 3, and it can be ommitted.

Let X be an arbitrary set, $\{d_\alpha : \alpha \in A\}$ be a family of mappings of $X \times X$ into $R^+$, $\varphi$ be a mapping of A into itself.

**Definition 7.** A tripple $(x, d_\alpha, \varphi)$ is said to be a quasi-uniformizable space if for every $x, y, z \in X$ and $\alpha \in A$ we have

(i) $d_\alpha(x, y) \geqslant 0$, $d_\alpha(x, x) = 0$,

(ii) $d_\alpha(x, y) = d_\alpha(y, x)$.

(iii) $d_\alpha(x, y) \leqslant d_{\varphi(\alpha)}(x, z) + d_{\varphi(\alpha)}(z, y)$.

In the sequel we assume in addition that

$$d_\alpha(x, y) = 0, \quad (\forall \alpha \in A) \Leftrightarrow x = y.$$

Then it is easily seen that $(X, d_\alpha, \varphi)$ becomes a Hausdorff topological space with a basis of neighbourhoods consisting of the balls

$$B(x, \varepsilon, \alpha) = \{y \in X : d_\alpha(x, y) < \varepsilon\}$$

$(x \in X, \varepsilon > 0, \alpha \in A)$ and their finite intersections.

A standard example of quasi-uniformizable spaces is PLC-spaces with $\Delta$ continuous (1) (without assumption $\Delta(a, a) \geqslant a$).

Indeed, let $(X, \mathcal{F}^i, \Delta)$ be a PLC-space with $\Delta$ continuous. Take $\lambda \in (0, 1)$ and put $\alpha = (i, \lambda) \in I \times (0,1) = A$, $d_\alpha(x, y) = \sup\{t : F^i_{x-y}(t) \leqslant 1 - \lambda\}$.

We shall construct the mapping $\varphi$ as follows. Since $\Delta$ is continuous and $\Delta(1, 1) = 1$, for each $\lambda \in (0,1)$ there is $\delta_\lambda \in (0, 1)$ such that

$$\Delta(1-\delta, 1-\delta) \geqslant 1 - \frac{\lambda}{2} \quad (\forall \delta \leqslant \delta_\lambda) \tag{13}$$

Now we put

$$\varphi(\lambda) = \sup\{\delta_\lambda : (13) \text{ holds}\},$$

$$\varphi(\alpha) = \varphi(i, \lambda) = (i, \varphi(\lambda)).$$

We shall verify that this $\varphi$ satisfies the condition (iii) of Definition 7 Indeed, in the contrary case, there would exist $i, \lambda, x, y, z$ such that

$$d_{i\lambda}(x, z) > d_{i\varphi(\lambda)}(x, y) + d_{i\varphi(\lambda)}(y, z).$$

Then there exists t, s such that

$$d_{i\varphi(\lambda)}(x, y) < t, \; d_{i\varphi(\lambda)}(y, z) < s, \; d_{i\lambda}(x, z) > t + s.$$

Consequently, by (7) we have

$$F^i_{x-y}(t) > 1 - \varphi(\lambda), \quad F^i_{y-z}(s) > 1 - \varphi(\lambda), \quad F_{x-z}(t+s) \leqslant 1 - \lambda.$$

This contradicts the fact that

$$F^i_{x-z}(t+s) \geqslant \Delta(F_{x-y}(t), F_{y-z}(s)) \geqslant \Delta(1-\varphi(\lambda), 1-\varphi(\lambda)) > 1 - \lambda$$

by (13). Thus, (iii) holds.

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(1) In the sequel we need only the weaker condition: $\sup\limits_{a<1} \Delta(a, a) = 1$.

Now we state one fixed point theorem for quasi-uniformizable spaces. Let $(X, d_\alpha, \varphi)$ be a complete quasiuniformizable space, T be a continuous mapping in X satisfying condition 1), 2), 3) in Therorem 4. Repeating the argument in the proof of Theorems 3), 4) we obtain the conclusion of Theorem 4. Thus we have

**Theorem 10.** Theorem 4 still holds for quasiuniformizable spaces if $f\varphi = \varphi f$.

**Corollary 1.** Theorem 9 still holds for PLC-spaces with a continuous $\Delta$-norm. Remark 1 applied to this corollary extends a recent result of Hadzic [5].

**Corollary 2.** Let $(X, \mathcal{F}, \Delta)$ be a complete Menger space with $\Delta$ continuous, T be a continuous mapping in X. Suppose there exists a nondecreasing right-continuous function $q : R^+ \to [0,1)$ satisfying the condition : for each $x \in X$ there is $m(x) \in N$ such that: $F_{T^{m(x)}x T^{m(x)}y}(q(t)t) \geqslant F_{xy}(t)$ for every $y \in X$, $t > 0$.

Furthermore, suppose there exists $x_0 \in X$ such that

$$\lim_{l \to \infty} F_{T^s x_0, x_0}(t) = 1$$

uniformly in $s \in N$.

Then T has a unique fixed point in X.

The proofs of these corollaries are obvious and they can be omitted.

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