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LOCAL CONTROLLABILITY OF LINEAR SYSTEMS WITH RESTRAINED CONTROLS IN BANACH SPACE

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1. INTRODUCTION

Consider the abstract control system:

$$\dot{x} = Ax + Bu, \quad x \in X, \quad u \in \Omega \subset U \tag{1}$$

where X and U are the real Banach spaces and $B \in [U, X]$ the Banach space of all bounded linear operators from U into X. Unless otherwise stated, X always intended infinite-dimensional. Throughout this paper the operator A is assumed to be (closed linear with domain D(A) dense in X and range R(A) in X) the infinitesimal generator of a strongly continuous semigroup (of class C_0) of bounded linear operators S(t), $t \geqslant 0$ [1], Ω is a non-empty set of the space U.

The properties of controllability for systems of type (1) with X, U being finite-dimensional spaces are studied by a number of authors [2] – [4] under various assumptions on the restraint control set Ω. For the systems (1) in the infinite-dimensional Banach spaces, to our knowledge, the controllability problem was studied up to now only for the case, when $\Omega = U$, i, e., for the systems without restraints on control (see. e.g. [5] – [7])

The purpose of this paper is to present necessary and sufficient conditions for local controllability of the infinite-dimensional system (1) with the arbitrary restraint control set Ω, which satisfies, in general, only one requirement:

$$\exists u_0 \in \Omega \;\; ; \;\; Bu_0 = 0. \tag{2}$$

The main result of this paper, Theorem 6 and Theorem 7, can be considered as an extension to infinite-dimensional systems of the well-known controllability criteria, which have been obtaind by Brammer [3] for systems in R_n. It is also worth remarking, that the technique of [3] – [4], which based on the arguments of linear algebra and the theory of almost periodic functions, obviously, can not be applied to the case of infnite-dimensional systems. The method developed here for the studying controllability is quite different from the technique mentioned above: it is based on some fundamental results of functional analysis, such as Baire cathegory theorem, open mapping theorem, the properties of a measure with values in a Banach space, etc. Central to our analysis is the use of Krein-Rutman' s therem [8], which for the aim of our paper, can be stated as follows;

Theorem A (Krein-Rutman). Let C be a convex cone with non-empty interior in a Banach space X and let $\{S(t)\}$ be a family of commutative bounded linear operators, mapping the cone C into itself, i.e., $S(t)\, C \subset C$ for all t. Then there exists a positive functional $f \in C^* \subset X^*$, which is a common eigenvector of all dual operators :

$$S^*(t)\, f = \lambda(t)\, f, \quad \text{for all } t, \text{ where } \quad \lambda(t) > 0.$$

2. DEFINITIONS AND PRELIMINARY RESULTS

Let us consider the linear autonomous system (1) In this section we shall introduce some definitions aud auxiliary results, which will be used below.

Definition 1. A U-valued function $u(t)$ defined for $t \geqslant 0$ is said to be an admissible control if for every $T > 0$, $u(t)$ is strongly measurable on $[0, T]$, essentially bounded on this interval and takes its values in the restraint control set Ω, i.e., $u(t) \in \Omega$ for all $t \geqslant 0$.

For each admissible control $u(t)$ and each $x_0 \in X$ we will refer to the function $x(t)$, defined by the formula:

$$x(t) = S(t)\, x_0 + \int_0^t S(t-\tau)\, B\, u(\tau)\, d\tau \tag{3}$$

as mild solution of system (1). Here and in what follows the integral is understood in the sense of Bochner.

The term «mild solution» is motivated by the well-known fact, that if the function $u(t)$ $(t \geqslant 0)$ is strongly continuously differentiable and $x_0 \in D(A)$, then (3) is the unique solution of system (1).

For every $T > 0$ we denote by $\widetilde{\Omega}_T$ the set of all admissible controls $u(t)$, restricted on the interval $[0, T]$. It is clear, that $\widetilde{\Omega}_T$ is a subset of the Banach space $L_\infty([0, T], U)$, the space of all essentially bounded functions on $[0, T]$ with the norm $\| u(\cdot) \|_\infty = \operatorname{Vraimax}_{0 \leqslant t \leqslant T} \| u(t) \|_\Omega$.

We say that a point $x_0 \in X$ is reachable on $[0, T]$ from the origin (by system (1)) if there exists an admissible control $u(t) \in \widetilde{\Omega}_T$ such that;

$$x_0 = \int_0^T S(T-t)\, u(t)\, dt.$$

The set of all points of X reachable on $[0, T]$ from the origin will be denoted by R_T and called the attainable set of system (1) on $[0, T]$. Let $R = \bigcup_{T > 0} R_T$.

Definition 2. The control system (1) is said to be locally controllable (locally controllable on $[0, T]$) if the set R (respectively, the set R_T) contains the origin in its interior, i.e., $0 \in \text{int } R$ (respectively, $0 \in \text{int } R_T$), and is said to be locally ε-controllable (locally ε-controllable on $[0, T]$ if R (respectively, R_T) is dense in some neighbourhood of the origin i.e., $0 \in \text{int } \overline{R}$ (respectively, $0 \in \text{int } \overline{R}_T$).

Our method of studying the local controllability of system (1) is mainly based on geometrical properties of the attainable set R_T. We shall need two following lemmas:

Lemma 1. The closure of the attainable set R_T is convex.

The proof of this lemma is similar to the proof of the analogous result for the finite-dimensional case (see, e.g. [2]) and, by that reason will not be presented here. The only difference in the proof is that, instead of the well-known Lyapunov's lemma, we need use here its infinite-dimensional generalization obtained by Uhl in [9]. It is also necessary to remark, that in the analogous way we can even establish a more general property, than the previous lemma. Namely, for any measurable subset E of the interval $[0, T]$, the closure of the set R_E, defined by:

$$R_E = \left\{x_0 \in X : x_0 = \int_E S(T-t)\, B\, u(t)\, dt,\ u(t) \in \Omega \text{ for all } t \in E\right\} \tag{4}$$

is convex. We also omit the proof of this fact.

Lemma 2. The closure of the attainable set of system (1) remains unchanged when one replace the restraint control set Ω by its convex hull $\text{co}\Omega$, i.e. $\overline{\widetilde{R}}_T = \overline{R}_T$, where the symbol $\widetilde{R}_T$ denote the attainable set on $[0, T]$ of the system

$$\dot{x} = Ax + Bu, \quad u \in \text{co}\Omega. \tag{5}$$

Proof. Clearly, it suffices to prove the following inclusion:

$$\overline{\widetilde{R}}_T \subset \overline{R}_T. \tag{6}$$

Consider the attainable set $\widetilde{R}_T^{(s)}$ of system (5) by means of all simple controls $u^{(s)}(t)$, which are defined as follows:

$$u^{(s)}(t) \equiv u_k \quad \text{for all } t \in E_k \quad \text{and } k = 1, 2, \ldots, N,$$

where $u_k \in \text{co}\Omega$, E_k are measurable, $E_k \cap E_j = \phi$ if $j \neq k$ and $\bigcup_{k=1}^{N} E_k = [0, T]$.

Let $\widetilde{x}_0^{(s)}$ be an

arbitrary point of $\widetilde{R}_T^{(s)}$. Then by definition we have

$$\widetilde{x}_o^{(s)} = \sum_{k=1}^{N} \sum_{j=1}^{N_k} \lambda_{kj} \int_{E_k} S(T-t)\; Bu_{kj}\; dt \overset{d}{=} \sum_{k=1}^{N} \widetilde{x}_k, \tag{7}$$

where $\lambda_{kj} \geqslant 0$, $\lambda_{k1} + \lambda_{k2} + \dots + \lambda_{kN_k} = 1$, $u_{kj} \in \Omega$ for all $j = 1, 2,\dots, N_k$ and $k = 1, 2,\dots, N$. It is obvious that $\widetilde{x}_k \in \mathrm{co}R_{E_k}$, where the set R_{E_k} is defined as (4). By Lemma 1, as is has been noted above, the closure of R_{E_k} is convex. Hence $\mathrm{co}\, R_{E_k} \subset \overline{R}_{E_k}$. It implies that for any $\varepsilon > 0$ there exists an admissible control $u_k(t) \in \Omega$ (for all $t \in E_k$) such that

$$\| \widetilde{x}_k - \int_{E_k} S(T-t)\cdot B\; u_k(t)\; dt \| < \frac{\varepsilon}{N}. \tag{8}$$

Now we define: $u_\varepsilon(t) \equiv u_k(t)$ for $t \in E_k$, and $k = 1, 2,\dots, N$. Then, clearly, $u_\varepsilon(t)$ is admissible on $[0, T]$ in the sense of. Definition 1, i. e., $u_\varepsilon(t) \in \widetilde{\Omega}_T$. Let $x_\varepsilon \in R_T$ be the corresponding reachable point. Then, by (7) and (8), we have:

$$\| \widetilde{x}_o^{(s)} - x_\varepsilon \| = \| \sum_{k=1}^{N} (\widetilde{x}_k - \int_{E_k} S(T-t)\, B\, u_k(t)\, dt) \|$$

$$\leqslant \sum_{k=1}^{N} \| \widetilde{x}_k - \int_{E_k} S(T-t)\; B\, u_k(t)\; dt \| < \varepsilon.$$

It means that R_T is dense in $\widetilde{R}_T^{(s)}$, or, in the other words

$$\widetilde{R}_T^{(s)} \subset \overline{R}_T. \tag{9}$$

On the other hand, for any point. $\widetilde{x}_o \in \widetilde{R}_T$, by definition

$$\widetilde{x}_o = \int_0^T S(T-t)\; B\, u(t)\; dt,$$

where the function $u(t)$ is strongly measurable, essentially bounded on $[0, T]$ and $u(t) \in \mathrm{co}\Omega$ for all $t \in [0, T]$. Since $u(t)$ is integrable in the sense of Bochner, there exists a sequence of simple functions $u_n^{(s)}(t)$ with values in $\mathrm{co}\Omega$ such that

$$\int_0^T \| u(t) - u_n^{(s)}(t) \| dt \to 0 \qquad \text{as } n \to \infty.$$

Next, from the strong continuity of the semigroup $S(t)$ and the boundedness of B, it follows that

$$\| \int_0^T S(T-t)\, Bu(t)\, dt - \int_0^T S(T-t)\, B\, u_n^{(s)}(t)\, dt \| \leqslant$$

$$\leqslant M\, e^{\alpha T'} \int_0^T \| u(t) - u_n^{(s)}(t) \| dt \rightarrow 0$$

as $n \rightarrow \infty$ (Above, $M = \| B \|$ and α is any constant greater than

$$w_0 = \lim_{t \to \infty} \ln \| S(t) \| / t,$$

see, e. g., [10]).

Therefore, $\widetilde{R}_T^{(S)}$ is dense is $\widetilde{R}_T$, i.e.

$$\widetilde{R}_T \subset \overline{\widetilde{R}_T^{(S)}}. \tag{10}$$

From (9), (10) it directly implies the desired inclusion (6) and thus the Lemma 2 is completely proved.

We now make some remarks concerning with the concepts of controllability introduced above by Definition 2. Firstly, we note that if the operator B is compact then system (1) can not be locally controllable. If, in addition, the control set Ω is bounded, then system (1) can not even be locally ε-controllable on any finite interval $[0, T]$. These properties can be easily shown by virtue of the fact, that, provided the compactness of B, the controllability operator K_T defined by:

$$K_T\, u(.) = \int_0^T S(T-t) B\, u(t)\, dt \tag{11}$$

from $L_\infty([0, T], U)$ into X, is compact [10].

It is obvious that, if the Banach spaces X and U are finite-dimensional, then the concept of local controllability (local contrallability on $[0, T]$) coincides with the concept of local ε-controlability (resp., local ε-controllability on $[0, T]$). Moreover, if the condition (2) is satisfied, then the local controllability implies the local controllability on some suitable finite interval $[0, T]$. This property, in general, is violated for infinite-dimensional systems. For example, it is not difficult to check that the system in $X = l_2$: $\dot{x} = u$, $u \in \Omega \subset l_2$, where Ω is the «Hilbert brick» (i. e., $\Omega = \left\{ u = (u_1, u_2, \ldots): \ |u_n| \leqslant \frac{1}{n} \right\}$) is even globally ε-controllable, i.e. $\overline{R} = X = l_2$. However, this system is not locally ε-controllable on any finite interval $[0, T]$, since for every $T > 0$ the attainable set R_T of this system equals to $T\Omega$ and hence $\text{int}\, \overline{R_T} = \phi$.

Thus, it is an interesting problem to study the relationship between the concepts of local controllability for infinite-dimensional systems. However, it is not the purpose of this paprer and so we will restrict ourselves here by formulating some results obtained in this direction. The reader is refered to [11] for the details.

Theorem 1. Suppose the restaint control set Ω is bounded and contains O in its interior: $O \in \text{int}\,\Omega$. Then, system (1) is locally controllable on [0, T] if and only if it is locally ε-controlable on [0, T].

Theorem 2. Assume that for some finite $T_o > 0$ the closure of the attainable set R_{T_o} has a nonempty interior: int $\overline{R}_{T_o} \neq \phi$. Let, additionally, the condition (2) be satisfied. If system (1) is locally ε-controllable, then for a suitable finite $T > 0$ the system is also locally ε-controllable on the interval [0, T].

Corollary 3. If system (1) with additional condition (2) is locally controllable, then for a suitable finite $T > 0$ the system is locally ε-controllable on the interval [0, T].

Theorem 4. Assume that the restraint control set Ω is convex and for some $T > 0$ the attainable set R_T has a non-empty interior: int $R_T \neq \phi$. Let, additionally, (2) be satisfied. If system (1) is locally controllable then there exists a finite time $T_o > 0$ such that the system is also locally controllable on the interval $[0, T_o]$.

Corodlary 5. Suppose Ω is convex, has a non-empty interior and satisfies (2). If system (1) is locally controllable then there exists a finite time $T_o > 0$ such that the system is also locally controllable on the interval $[0, T_o]$.

3. THE MAIN RESULTS

We begin this section by making the following remarks. It is well-known from the analytical theory of semigroups that if a Banach space X is reflexive and A is an infinitesimal generator of a strongly continuous semigroup S(t) on X, then the dual operator A^* is also (linear, closed, with domain $D(A^*)$ strongly dense is X^*) and the infinetesimal generator of the strongly continuous semigroup $S^*(t)$ on X^*. If the space X is not reflexive then $D(A^*)$ need not be strongly dense in X^*. However, in this cas, a dual semigroup theory with the desired continuity properties can be carried out on the closure of $D(A^*)$. So, for the sake of simplicity, we assume in this section that the Banach state space X is reflexive.

We are now in a position to state our main controllability result.

Theorem 6. Suppose Ω is convex set with non-empty interior and satisfies the condition (2). Let, in addition, the semigroup S(t) be differentiable. Then system (1) is locally controllable if and only if:

a) the system with unconstrained control

$$\dot{x} = Ax + Bu, \quad x \in X, \quad u \in U \tag{12}$$

is globally (exactly) controllable;

b) there is not any eigenvector $f \in X^*$ of the dual operator A^*, corresponding to a real eigenvalue λ and supporting to the set $B\Omega$: $f \in X^*$: $A^*f = \lambda f$ and $\langle f, Bu\rangle \geq 0$ for all $u \in \Omega$.

Proof. Necessity: The necessity of a), obviously, follows from the definitions. We shall prove the necessity of b) by contradiction.

Let $f_o \in X^*$ be an eigenvector of A^* with a real eigenvalue λ and f_o is supporting to $B\Omega$: $\langle f_o, Bu\rangle \geq 0$ for all $u \in \Omega$. Then for every $T > 0$ and any $x_o \in R_T$ we have:

$$\langle f_o, x_o\rangle = \int_0^T \langle f_o, S(T-t)\, Bu(t)\rangle\, dt = \int_0^T \langle S^*(T-t) f_o, Bu(t)\rangle dt$$

Since, obviously, $f_o \in \bigcap_{n=1}^{\infty} D(A^{*n})$, the function $S^*(T-t)\, f_o$ is infinitely differentiable in t and, therefore, it can be represented by the Taylor formula as follows [1]:

$$S^*(T-t) f_o = \sum_{k=0}^{n-1} \frac{(T-t)^k}{k!} A^{*k} f_o + \frac{1}{(n-1)!} \int_0^{T-t} (T-t-\tau)^{n-1}\, S^*(\tau)\, A^{*n} f_o\, d\tau$$

$$(n = 1, 2, 3, \ldots)$$

Hence, for every natural n we obtain:

$$\langle S^*(T-t) f_o, Bu\rangle = \sum_{k=0}^{n-1} \frac{\lambda^k (T-t)^k}{k!} \langle f_o, Bu\rangle + \frac{\lambda^n}{(n-1)!} \int_0^{T-t} (T-t-\tau)^{n-1} \times$$

$\times \langle S^*(\tau) f_o, Bu\rangle\, d\tau$, for all $t \in [0, T]$ and $u \in \Omega$.

If $\langle f_o, Bu\rangle = 0$, then it follows from the above formula that:

$\langle S^*(T-t) f_o, Bu\rangle = 0$ for all $t \in [0, T]$. If $\langle f_o, Bu\rangle > 0$,

then by the boundedness of $S(t)$ on $[0, T]$, the above expression can be written in the following form:

$$\langle S^*(T-t) f_o, Bu\rangle = \langle f_o, Bu\rangle \left(\sum_{k=0}^{n=1} \frac{\lambda^k (T-t)^k}{k!} + 0\left(\frac{|\lambda|^{n-1}}{(n-1)!}\right)\right) \text{ as } n \to \infty. \tag{13}$$

Since the sum $\sum_{k=0}^{n-1} \frac{\lambda^k (T-t)^k}{k!}$ uniformly converges to $e^{\lambda(T-t)}$ as $n \to \infty$, we obtain, by virtue of (13), that $\langle S^*(T-t) f_o, Bu\rangle > 0$ for all $t \in [0, T]$.

So, in any case, $\langle S^*(T-t)f_o, Bu\rangle \geq 0$ for all $t \in [0, T]$ and $u \in \Omega$. Therefore, $\langle S^*(T-t)f_o, Bu(t)\rangle \geqslant 0$ for all $t \in [0, T]$ and all $u(.) \in \widetilde{\Omega}_T$. Consequently, $\langle f_o, x_o\rangle \geqslant 0$ for all $x_o \in R_T$ and hence, by the abitrariness of T, for all $x_o \in R$. This contradiction to the local controllability of system (1) proves the necessity of b)).

Sufficiency: Consider the controllability operator K_T defined by (11). Then it implies from the condition a) that $X = \bigcup_{k=1}^{\infty} K_{T_k}(L_\infty([0, T_k], U))$ where $0 < T_1 < T_2 < T_3 \dots < \infty$ and $T_k \to \infty$ as $k \to \infty$. By the Baire theorem, there exists T_k such that $K_{T_k}(L_\infty([0, T_k], U))$ is of the second cathegory in X and thus, K_{T_k} is an open mapping from $L_\infty([0, T_k], U)$ onto X, By assumption, Ω is convex and int $\Omega \neq \phi$, hence the set $\widetilde{\Omega}_{T_k}$ of all admissible controls on $[0, T_k]$ is also convex and has a non-empty interior: int $\widetilde{\Omega}_{T_k} \neq \phi$. Therefore, the attainable set R_{T_k}, being the image of Ω_{T_k} by the open linear mapping K_{T_k}, is also convex and has a non-empty interior. Thus, by (2), the set B is convex and int $R \neq \phi$.

We now prove that $0 \in$ int R. Let $0 \notin$ int R. Then by the Hahn-Banach theorem, there exists a nontrivial functional $f_1 \in X^*$ such that $\langle f_1, x_o\rangle \geqslant 0$ for all $x_o \in R$. Consider the cone C generated by R, i.e., $C = \bigcup_{\lambda > 0} \lambda R$ Then the dual cone C^* of positive functionals is non-trivial since C^* contains f_1. It is quite easy to verify that for every $f \in C^*$ $\langle f, S(t)Bu\rangle \geqslant 0$ for all $t \in [0, T]$ and $u \in \Omega$. We will show that the semigroup $S(t)$, $t \geqslant 0$ maps the cone C into itself. Indeed since a bounded operator commutes with the Bochner integral, for every $x_o \in R$ and $t \geqslant 0$ we have:

$$S(t)\, x_o = \int_0^T S(t)\, S(T-\tau)\, Bu(\tau)\, d\tau =$$

$$= \int_0^T S(t+T-\tau)\, Bu(\tau)\, d\tau = \int_0^{t+T} S(t+T-\tau)\, B\widehat{u}(\tau)\, d\tau,$$

where the control $\widehat{u}(\tau)$ is defined as follows:

$$\widehat{u}(\tau) = \begin{cases} u(\tau), & 0 \leqslant \tau < T, \\ u_o, & T \leqslant \tau \leqslant T+t. \end{cases}$$

Obviously, $\widehat{u}(.) \in \widetilde{\Omega}_{T+t}$ Hence, $S(t)x_o \in R_{T+t} \subset R$ and thus $S(t).R \subset R$ for a ll $t \geqslant 0$. Furthermore, by the property of semigroups, the operators $S(t_1)$ and $S(t_2)$ are commutative one with another for any $t_1, t_2 > 0$. Applying the Krein-Rutman theorem (Theorem A) we find in the dual cone C^* a common eigenvector f_o of all dual operators $S^*(t)$, $t \geqslant 0$, with, correspondingly, nonnegative eigenvalues

$$\lambda(t),\ t \geqslant 0, \text{ i.e. } S(t)\, f_o = \lambda(t)\, f_o \text{ (for all } t \geqslant 0)\ \lambda(t) \geqslant 0, \tag{14}$$

Since the Banach state space X is reflexive we have, by the differentiability of $S(t)$, that $S^*(t)\ X^* \subset D(A^*)$ for all $t > 0$ [12]. Thus, $f_o \in D(A^*)$ and so, $S^*(t)\ f_o$ is defferentiable for $t > 0$. By differentiating the equality (14) we obtain:

$$A^*S^*(t)\ f_o = S^*(t)\ A^*f_o = \lambda'(t)\ f_o \text{ for all } t > 0.$$

It implies from the strong continuity of $S^*(t)$ that

$$\exists \lim_{t \to +0} S^*(t)\ A^*\ f_o = A^*f_o = \lambda_o f_o,$$

where $\lambda_o = \lim\ \lambda'(t)$ as $t \to +0$. Thus f_o is the eigenvector of A^* with real eigenvalue λ_o. On the other hand, according to our construction, $f_o \in C^*$ and so $\langle f_o,\ S(t)\ Bu\rangle \geqslant 0$ for all $t \geqslant 0$ and $u \in \Omega$. It implies, particularly, that f_o is supporting to $B\Omega$. This contradicts b) and completes the proof of the theorem.

In the case, when the convexity of the control set Ω is not assumed we have the following theorem.

Theorem 7. Assume that convex hull of Ω has a non-empty interior and Ω satisfies the condition (2). Let, in addition, the semigroup $S(t)$ be differentiable. Then the condition a) and b) of Theorem 6 are sufficient for local ε-controllability of system (1).

Proof. Consider the set $\widetilde{co}\Omega_T$ of all admissible controls on $[o, T]$ with values in the convex hull of Ω: $u(t) \in co\Omega$ for all $t \in [0, T]$. Let the symbol $\widetilde{R}_T$ stand for the attainable set of the system $\dot{x} = Ax + Bu$ by means of all admissible controls $u(.) \in \widetilde{co}\ \Omega_T$. Then, according to Lemma 1 and Lemma 2: $\overline{\widetilde{R}_T} = \overline{R_T}$ and $\overline{R_T}$ is convex. In the analogous way as the proof of Theorem 6, we find that, for some $T > 0$, $\text{int}\ \widetilde{R}_T \neq \phi$ and so, $\text{int}\ \overline{R_T} \neq \phi$. On the other, hand, it is easy to show that $\overline{R} = \overline{\bigcup_{T>o} R_T} = \overline{\bigcup_{T>o} \overline{R_T}}$ and by (2), $\overline{R}_{T_1} \subset \overline{R}_{T_2}$ if $T_1 < T_2$.

Hence, $\overline{R}$ is convex and has a non-empty interior: $\text{int}\ \overline{R} \neq \phi$. We now only have to show that $0 \in \text{int}\ \overline{R}$. Let $0 \notin \text{int}\ \overline{R}$, then by using the analysis developed previously in the proof of Theorem 6, we may establish that the convex cone C_1 generated by $\overline{R}$ is invariant under the action of the semigroup $S(t)$. $t \geqslant 0$ i.e. $S(t)C_1 \subset C_1$ for all $t \geqslant 0$. Further, applying, similarly, the Krein – Rutman theorem we obtain a contradiction to the condition b) which completes the proof. We omit the details.

In conclusion, we make some remarks, concerning with Theorems 6 and 7 Firstly, it is important to note that the requirement $\text{int}\ \Omega \neq \phi$ and $\text{int}\ co\Omega \neq \phi$ is essential for these theorems. Indeed, for example, consider the following system in $X = l_2$: $\dot{x} = Ax + u$, $u \in \Omega$, where A is the left shift operator, i.e., $A(x_1, x_2\ x_3, ...) = (x_2, x_3\ x_4, ..)$ and Ω is the set of all positive vectors in l_2, Since the dual operator A^* has no eigenvector, the condition b) is satisfied for this system. Further, it can be directly verified that the corresponding system with unconstrained control $\dot{x} = Ax + u$ is globally (exactly) controllable in l_2, and so a) also holds. However, the system under consideration is not locally controllable, since, clearly $x_0 \geqslant 0$ for every $x_0 \in R$. Notice that in this case the interior of the control set is empty.

In spite of the above remark Theorem 6 and 7 can be actually strengthened by replacing conditions int $\Omega \neq \phi$ and int co$\Omega \neq \phi$ by the weaker assumptions ri $\Omega \neq \phi$ and, respectively, ri co$\Omega \neq \phi$, where the symbol riM denotes the relative interior of a set M $(O \in M)$ in the induced topology of the closed subspace $\overline{sp}$ M, which is generated by M. And what is more, under these weaker conditions one must only, instead of a), require that the system:

$$\dot{x} = Ax + Bv,\ x \in X,\ v \in \overline{sp}\ (\Omega - u_0)$$

be globally (exactly) controllable in X. This property is not difficult to be shown from the proof of Theorem 6. Finally, we also notice that the necessity part of Theorem 6 is proved without using the differentiability of the semigroup S(t).

REFERENCES

1. Hille and Fillips R. *Functional analysis and semigroups*. American Mathematical Society, Providence, R.I., 1958.

2. Lee E., Marcus P., *Foundations of the optimal control theory*. Nauka 1972.

3. Brammer R. *Controllability in linear autonomous systems with positive controllers.* SIAM J. Control, 10 (1972), 339 – 353.

4. Korobov V.I., Marinich A.P., Podol'skii E. N. *Controllability of linear autonomous systems with restrained controls.* Differen. Uravnenia. 11, (1975), pp. 1970 – 1982.

5. Fattorini H.O. *Some remarks on compete controllability.* SIAM J. Control, 4 (1966), pp, 686–694.

6. Kurjanskii A.B., *On controllability in Banach spaces.* Differen. Uravnenia., 5 (1969). pp. 1269–1271.

7. Triggiani R. *Controllability and observability in Banach space with bounded operators.* SIAm J. Control 13 (1975), pp. 462–491.

8. Krein M.G., Rutman M.A. *Linear operators leaving invariant a cone in a Banach space.* Uspehi Matemat. Nauk (N.S). 1 (23) (1948) pp. 3–95.

9. Ulh J., *The range of vector-valued measure.* Proc. Amer. Soc. 23 (1969) pp. 158-163.

10. Triggiani R. *On the lack of exact controllability for mild solutions in Banach spaces.* J. Math. Anal. Appl. 50 (1975), pp. 438–446.

11. Korobov V.I., Nguyen Khoa Son. *Controllability of linear systems in Banach space with restrained control.* Differen. Uravnenia. (to appear)

12. Butzer P.L. and Berens H., *Semigroups of operators and approximations.* Springer-Verlag., Berlin 1967.