ACTA MATHEMATICA VIETNAMICA

Tom 5 N°2

GENERALIZED PERIODIC SYSTEMS OF DIFFERENTIAL EQUATIONS

NGUYỄN CHÂN

Institute of Computation sciences and cybernetics, Hanoi

Introduction

Let $k(.)$ be a function which maps from $[0, \infty) = R_+$ into $(0, \infty)$. We introduce the following notations:

$$\underbrace{k(k(\ldots k(t)\ldots))}_{\nu \text{ times}} = k^{[\nu]}(t),\ \nu = 1, 2, \ldots;\ k^{[0]}(t) \equiv t;$$

$$k^{[\nu]}(0) = k_\nu\ ;\quad [k_\nu, k_{\nu+1}) = K_\nu\ ;\quad k_{\nu+1} - k_\nu = l_\nu .$$

Suppose the function $k(.)$ has the following property which we call k-property.

k-Property. 1. $k(.) \in C(R_+)$, $k(0) > 0$.

2. If t increases from $k_{\nu-1}$ to k_ν, then $k(t)$ increases from k_ν to $k_{\nu+1}$, for every $\nu \geqslant 1$.

3. There exists the derivative $\dot{k}(t) > 0$ for $t \in R_+ \setminus \{k_\nu, \nu = \overline{0, \infty}\}$. It follows that there exists the inverse function $h(.)$ of $k(.)$ and its derivative $\dot{h}(.) > 0$ for $t \in [k_1, \infty) \setminus \{k_\nu, \nu = \overline{1, \infty}\}$.

Consider the following linear homogeneous system of differential equations:

$$\dot{x} = A(t)x\ ,\quad t \in R_+. \tag{0.1}$$

Here $A(t)$ is a $(n \times n)$ — matrix, the components of which are continuous (or piecewise-continuous) in R_+ and

$$A(k(t)) = \frac{A(t)}{\dot{k}(t)}\ ,\ t \in R_+, \tag{0.2}$$

$x(.)$ is a n-vetor-function.

As we have noted in [4] (0.2) is a natural condition for that the system (0.1) has a k-generalized periodic solution (k-g.p.s.) $x(.)$, i.e. a solution for which $x(k(t)) = x(t)\ \forall t \in R_+$.

In [4] we have considered the class $k, \alpha - GPF$ of all k, α — generalized periodic functions (k, α — g.p.f., α is a real number), i.e. the functions $x(.)$, for which $x(k(t)) = \alpha\, x(t)$ $\forall t \in R_+$. We have studied the structure and some important properties of the class k, α—GPF and proved that this class is independent of well-known classes of periodic or almost periodic functions.

In this paper we will consider k, $1-$g.p.systems (or, for simplicity, $k-$g.p.systems), i. e. the systems with the condition of the form (0.2). We first will give without proof the generalized Floquet theorem for the system (0.1). Then we will deal with some problems such as reducibility of the system, existence and unicity of a solution of the $k-$g.p.system. At last we will get one theorem of the Andronov-Vitt's type.

The functions in this paper may take real or complex values.

§ I. THE GENERALIZED FLOQUET THEORY

By standart arguments the following theorem, which is called the generalized Floquet theorem, can be proved (see also [1]).

Theorem 1.1. *The fundamental matrix $X(t)$ of solutions of the system (0.1), normalized at $t = 0$, can be written in one of the following forms:*

1) $X(t) = \Phi(t) \,.\, exp\,(\Lambda\mu(t))$,

$$\text{where } \Phi(k(t)) = \Phi(t),\ \mu(k(t)) = \mu(t) + \omega,\ t \in R_+,\ \Phi(.) \in C^1(R_+) \tag{1.1}$$

(or $\Phi(.)$ is piece-wise-smooth), $\mu(.)$ is a real function, $\omega = const$, Λ is a const-matrix.

2) $X(t) = \Phi(t) \,.\, exp\,(\Lambda t)$,

$$\text{where } \Phi(t + \tau(t)) = \Phi(t) \,.\, exp\,\{-\Lambda\,[\tau(t) - \tau(0)]\},\ \forall t \in R_+\ \ \tau(t) = k(t) - t. \tag{1.2}$$

Remarks. 1. The representations (1.1) and (1.2) hold also for arbitrary fundamental system of solutions (it is not necessary to normalize at $t = 0$).

The proper values λ_j of matrix Λ are called the charateristic indices (c.i.) of $k-$g.p.system (0,1) (For the correlation between λ_j and the Liapunov c.i. of nontrivial solutions of the system (0.1) see § II below). We also call the proper values ρ_j of matrix $X(k(0))$ multiplicators of the system (0.1) ($\rho_j \neq 0$ because det $X(k(0)) \neq 0$). It can be proved that

$\lambda_j = \dfrac{1}{\omega}\,\text{Ln}\,\rho_j$ ($\omega = k(0)$ in the representation (1.2)), and that every multiplicator ρ corresponds to a nontrivial normal solution $\xi(.)$ such that

$$\xi(k(t)) = \rho\,.\,\xi(t),\quad \forall t \in R_+, \tag{1.3}$$

and in opposite, if a nontrivial solution $\xi(.)$ satisfies (1.3), then ρ will be a multiplicator for the system (0.1). From this it follows that (0.1) has a $k-$g.p.s. iff at least one of its multiplicators is equal to 1. Moreover, the normal solution (1.3) may be written in the form

$$\xi(t) = \exp\,(\lambda\mu(t))\,.\,\varphi(t),\ \text{where } \varphi(k(t)) = \varphi(t),$$

or

$$\xi(t) = \exp\,(\lambda t)\,.\,\varphi(t),\ \varphi(k(t)) = \varphi(t)\,.\,\exp\,[-\lambda\,(\tau(t) - \tau(0))],$$

$$\forall t \in R_+,\ \lambda = \frac{1}{\omega}\,\text{Ln}\,\rho.$$

2. In Theorem 1.1 we have mentioned a function $\mu(.)$ for which

$$\mu(k(t)) = \mu(t) + \omega, \ \forall t \in R_+. \qquad (\mu)$$

Suppose now $\mu|_{K_0}(.) \in C^1(K_0)$. As in § V [4] we can prove that for $\mu(.) \in C^1(R)$ it is necessary and sufficient that

$$\mu'_+(0) = k'(0) \cdot \pi'_-(k_1). \qquad (1.4)$$

In what follows we will suppose in general that the function $\mu(.)$, besides the condition (μ), satisfies (1.4), too. Then clearly

$$\mu'(k(t)) = \frac{\mu'(t)}{\dot{k}(t)}, \ \forall t \in R_+. \qquad (1.4')$$

We note also that besides the conditions (μ) and (1.4) no other requirement on the function $\mu(.)$ is required. But for easiness we can choose this function satisfying, for example, $\mu'(t) > 0$, $\inf_t \mu'(t) > 0, \ldots$

§ II. REDUCIBILITY AND μ — REDUCIBILITY

First, using the representation (1.2), we examine the reducibility of the system (0.1). Put

$$\max_{t \in K_i} |\tau(t) - \tau(0)| = \bar{\tau}_i, \quad \max_{t \in K_i} \frac{|k(t) - 1|}{\dot{k}(t)} = m$$

$$\min_{t \in K_i} \dot{k}(t) = \underline{k}'_i .$$

Theorem 2.1. *Suppose the function $k(.)$ satisfies the following conditions :*

$$a) \ \sum_i \bar{\tau}_i = \tau_\Sigma < +\infty \ ; \qquad b) \ \inf_i \prod_{\nu=0}^{i} \underline{k}'_\nu > 0 \ ;$$

$$c) \ \sup_i \left\{ m_i + \frac{m_{i-1}}{\underline{k}'_i} + \frac{m_{i-2}}{\underline{k}'_i \underline{k}'_{i-1}} + \ldots + \frac{m_0}{\underline{k}'_i \ldots \underline{k}'_1} \right\} < +\infty.$$

Then the k-g.p.system (0.1) is reducible.

Proof. Let $\Phi(.)$, Λ be two matrices as in the representation (1.2). Using the exchange $x(t) = \Phi(t) y(t)$, the system (0.1) becomes into the form $\dot{y} = \Lambda y$.

Therefore in order to get the reducibility of the system (0.1) we have only to show that $\Phi(.)$ is a Liapunov matrix (see [2]) In the following we verify its properties.

1) $\sup_{t \in R_+} \|\Phi(t)\| < \infty$: For $t \in K_{i+1}$ we have

$$\|\Phi(t)\| \leqslant \|\Phi(h(t))\| \cdot \exp[\,\|\Lambda\| \cdot |\tau(h(t)) - \tau(t)|\,] \leqslant \|\Phi(h(t))\| \cdot \exp(\|\Lambda\| \cdot \bar{\tau}_i) \leqslant$$

$$\leqslant \|\Phi(h^{[2]}(t))\| \cdot \exp[\,\|\Lambda\| \cdot (\bar{\tau}_i + \bar{\tau}_{i-1})] \leqslant \ldots \leqslant$$

$$\|\Phi(h^{[i]}(t))\| \cdot \exp(\|\Lambda\| \cdot \sum_{\nu=1}^{i} \bar{\tau}_\nu) \leqslant \max_{t \in k_0} \|\Phi(t)\| \cdot \exp(\|\Lambda\| \cdot \tau_\Sigma) < \infty.$$

Since the last inequality is independent of t, the property 1) is established.

2) $\sup_t \| \dot{\Phi}(t) \| < \infty$: From (0,2) we have

$$\dot{\Phi}(t) = \frac{d}{dt} \{\Phi(h(t)) . \exp [\Lambda(\tau(0) - \tau(h(t)))]\} =$$

$$= \frac{d}{ds} \{\Phi(s) . \exp [\Lambda(\tau(0)-\tau(s))]\} \Big|_{s = h(t)} . \dot{h}(t) =$$

$$= \{\Phi'(h(t)) - \tau'(h(t))\Lambda\} . \exp [\Lambda(\tau(0) - \tau(h(t)))] . \dot{h}(t)$$

Therefore for $t \in K_i$ we get

$$\| \Phi'(k(t)) \| \leqslant \{ \| \Phi'(t) \| + | \tau'(t) | . \| \Lambda \| \} . \frac{\exp(\| \Lambda \| . \overline{\tau_i})}{\dot{k}(t)} =$$

$$= \frac{1}{\dot{k}(t)} . \| \Phi'(t) \| . \exp (\| \Lambda \| \overline{\tau_i}) + \frac{1}{\dot{k}(t)} . \| \Lambda \| . | \tau'(t) | . \exp (\| \Lambda \| . \overline{\tau_i}) \leqslant$$

$$\leqslant \| \Phi'(t) \| . \frac{\exp(\| \Lambda \| . \overline{\tau_i})}{\underline{k}_i'} + \| \Lambda \| . m_i . \exp(\| \Lambda \| . \overline{\tau_i}) \leqslant$$

$$\leqslant \{ \| \Phi'(h(t)) \| . \frac{\exp(\| \Lambda \| . \overline{\tau_{i-1}})}{\underline{k}_{i-1}'} + \| \Lambda \| . m_{i-1} . \exp(\| \Lambda \| . \overline{\tau_{i-1}})\} \frac{\exp(\| \Lambda \| . \overline{\tau_i})}{\underline{k}_i'} +$$

$$+ \| \Lambda \| . m_i . \exp (\| \Lambda \| . \overline{\tau_i}) = \| \Phi'(h(t)) \| . \frac{\exp [\| \Lambda \| . (\overline{\tau_i} + \overline{\tau_{i-1}})]}{\underline{k}_i' . \underline{k}_{i-1}'} +$$

$$+ \| \Lambda \| . \frac{m_{i-1}}{\underline{k}_i'} . \exp[\| \Lambda \| . (\overline{\tau_i}+\overline{\tau_{i-1}})] + \| \Lambda \| . m_i . \exp (\| \Lambda \| . \overline{\tau_i}) \leqslant \ldots$$

$$\ldots \leqslant \| \Lambda \| . m_i . \exp(\| \Lambda \| . \overline{\tau_i}) + \| \Lambda \| . \frac{m_{i-1}}{\underline{k}_i'} . \exp [\| \Lambda \| . (\overline{\tau_i} + \overline{\tau_{i-1}})] +$$

$$+ \| \Lambda \| . \frac{m_{i-2}}{\underline{k}_i' . \underline{k}_{i-1}'} . \exp[\| \Lambda \| . (\overline{\tau_i} + \overline{\tau_{i-1}} + \overline{\tau_{i-2}})] + \ldots +$$

$$+ \| \Lambda \| . \frac{m_1}{\underline{k}_i' \ldots \underline{k}_1'} . \exp [\| \Lambda \| . (\overline{\tau_i} + \ldots + \overline{\tau_0})] +$$

$$+ \| \Phi'(h^{[i]}(t)) \| . \frac{\exp(\| \Lambda \| . (\overline{\tau_i} + \ldots + \overline{\tau_0}))}{\underline{k}_i' \ldots \underline{k}_0'} .$$

Thus for any $t \in R_+$ we have

$$\| \Phi'(t) \| \leqslant \| \Lambda \| . \exp(\| \Lambda \| . \tau_\Sigma) . \sup_i \left\{ m_i + \frac{m_{i-1}}{\underline{k}_i'} + \ldots + \frac{m_0}{\underline{k}_i' \ldots \underline{k}_1'} \right\} +$$

$$+ \max_{t \in K_0} \| \Phi'(t) \| . \frac{\exp(\| \Lambda \| . \tau_\Sigma)}{\inf_i \prod_{\nu=0}^{i} \underline{k}_\nu'} < + \infty$$

because of a) – c). The property 2) is established.

3) $\inf_{t \in R_+} |\det \Phi(t)| > 0$: Indeed, we have ($Sp\,\Lambda$—spur of matrix Λ):

$$|\det \Phi(k(t))| = |\det \Phi(t)| \cdot \exp[(\tau(0) - \tau(t)) \cdot Re Sp \Lambda] \geqslant$$
$$\geqslant |\det \Phi(t)| \cdot \exp[-\overline{\tau_i} \cdot |Re Sp \Lambda|] \geqslant \ldots \geqslant \inf_{t \in k_0} |\det \Phi(t)| \cdot \exp(-\tau_{\Sigma} |Re Sp \Lambda|) > 0.$$

Thus, all three characteristic properties of a Liapunov matrix are satisfied. The proof is ended.

Remark. The conditions a) – c) will be satisfied, for example, in the case, when $k(t)$ is sufficiently near 1 for every $t \in R_+$, i.e. the system becomes «periodic» with almost constant period. In the special case when $k(t) = t + r$, we get the well-known Liapunov theorem.

We now introduce the following

Definition 2.1. Let $p(.): R_+ \to R$ be a given real function. A linear homogeneous system (0.1) (without the condition (0.2)) is called p—reducible if we can bring it into the form $\dot{y} = p(t)By$. (2.1)
(B is a constant matrix) by the substitution

$$y = L(t)x \qquad (2.2)$$

where L(.) is a Liapunov matrix.

Remarks. 1. If $p(t) > 0$ then by the substitution $s = \int_0^t p(\xi)d\xi$ the system (2.1) can be brought into a system having a constant matrix.

2. A p-reducible system is not always reducible in the usual sense. Indeed, consider the following equation ($n = 1$)

$$\dot{x} = p(t)x. \qquad (2.3)$$

Suppose that by the substitution $y = l(t)x$, (2.4)
where $l(.) \in C^1(R_+)$, $0 < \inf_{t \in R_+} |l(t)| \leqslant \sup_{t \in R_+} |l(t)| < \infty$ (2.5)

and $\sup_{t \in R_+} |\dot{l}(t)| < \infty$, we can bring the system (2.3) into the form $\dot{y} = by$. (2.6)

From (2.4), (2.6) and (2.3) we have $\dot{l}(t)x + l(t)\dot{x} = \dot{l}(t)x + l(t)p(t)x =$

$= bl(t)x$. Hence $\dot{l}(t) = l(t)(b - p(t))$ or $l(t) = \exp\{bt - \int_0^t p(s)ds\}$.

Let us consider the function $p(.)$ for which $p(k(t)) = \dfrac{p(t)}{\dot{k}(t)}$.

Then since $\int_{k_i}^{k_{i+1}} p(s)ds = \int_{k_0}^{k_1} p(k^{[i]}(s)) \cdot (k^{[i]})(s)ds = \int_{k_0}^{k_1} p(s)ds = J_0$, $\forall i$, in order to get (2.5) we must have, at least, $\sup_i |bk_i - iJ_0| < \infty$, or, putting $d = J_0/b$,

$\sup_i | k_i - id | < \infty$. It is clear that for some functions $k(.)$ there not exists such value d(it exists, for example, if $l_i = l_o + e_i$, $\sum_i | e_i | < \infty$ $(d = l_o)$).

3. It is possible to prove that the well-known Liapunov theorem on stability of the system $\dot{y} = By$ (of the quasi-linear system $\dot{y} = Ay + \varphi(t, y)$, $\frac{\varphi(t, x)}{\| x \|} \underset{t}{\rightrightarrows} 0$ when $x \to 0$, too) can be extended to the system (2.1) (and to the system $\dot{y} = p(t)By + \varphi(t, y)$).

By standart arguments we can prove the following theorem of the Erugin's type.

Theorem 2.2. *The system (0.1) will be p-reducible iff one of its fundamental matrices of solutions can be represented in the form*

$$X(t) = L(t) . \exp [q(t)B], \tag{2.7}$$

where $L(.)$ is a Liapunov matrix, B is a constant matrix and

$$q(t) = \int_0^t p(s)\, ds.$$

Corollary. Suppose a function $\mu(.)$ satisfies the conditions (μ) and (1.4). Then the k-g.p.system (0.1) – (0.2) is $\mu'(.)$-reducible.

This corollary follows from Theorem 2.2 and the representation (1.1). We also note that the derivative $\mu'(.)$ has property (1.4) (see Remark 2 in Definition 2.1).

§ III. THE EXISTENCE AND UNICITY THEOREMS

Almost all well-known theorems for usual periodic systems (see, for example, [2] and [3], Chapter XII) such as the Liapunov's theorems for linear homogeneous and nonhomogeneous periodic systems, the Massera's theorem, the Poincare's theorem. the Liapunov's theorem about asymptotic stability of a periodic solution,... can be extended to k-g.p.systems. For example, let us consider the following system

$$\dot{y} = A(t)\, y + f(t, y) \tag{3.1}$$

and the homogeneous system corresponding to it

$$\dot{y} = A(t)\, y, \tag{3.2}$$

where $A(t)$ is a $(n \times n)$-matrix, $A(.) \in C(R_+)$, $A(k(t)) = \frac{A(t)}{\dot{k}(t)}$, and such that (3.2) has no nontrivial k-g.p.s.

(Then it is possible to show that for every function $\varphi(.) \in C(R_+)$, $\varphi(k(t)) = \frac{\varphi(t)}{\dot{k}(t)}$, the linear nonhomogeneous system

$$\dot{y} = A(t)\,y + \varphi(t) \tag{3.3}$$

has a unique k-g.p.s. $y(.)$, moreover, there exists a number M independent of $\varphi(.)$ and for which

$$\| y(t) \| \leqslant M \cdot \int_{0}^{k(o)} \| \varphi(s) \| \, ds, \quad t \in K_o \tag{3.4}$$

Return to the system (3.1).

Let $f(t, y) \in C(R_+, \{ \| y \| \leqslant r \})$, $f(k(t), y) = \frac{f(t, y)}{\dot{k}(t)}$, $\forall t \in R_+$, $\| y \| \leqslant r$ and furthermore

$$Mk(0) \cdot \| f(t, y) \| \leqslant r, \ \forall t \in K_o, \ \| y \| \leqslant r. \tag{3.5}$$

We suppose also $\inf_{\nu} (k^{[\nu]})'(t) = k' > 0$, $\nu = 1, 2, \ldots$; $t \in K_o$.

Theorem 3.1. *Suppose the conditions that are mentioned above hold. Then the system (3.1) has at least one k-g.p.s.*

Proof. Consider the space k-CGP of all continuous k-g.p. f.s $g(.)$ with the norm $|g| = \max_{R_+} \| g(t) \|$. This space is a Banach space with uniform convergence on K_o. Assume $g(.) \in k$-CGP and $|g| \leqslant r$. As we have just noted above the system

$$\dot{y} = A(t)\,y + f(t, g(t)) \tag{3.6}$$

has a unique k-g.p.s. $y_g(.)$ which satisfies (3.4) with $\varphi(s) = f(s, g(s))$ (take into account that $f(k(t), g(k(t))) = f(k(t), g(t)) = \frac{f(t, g(t))}{\dot{k}(t)}$).

From (3.4) and (3.5) it follows that $|y_g| \leqslant r$. We define the mapping T: k-CGP $\rightarrow$ k-CGP, putting $T[g] = y_g(.)$.

We must show T has a fixed point.

As we have just seen, T maps the ball $S_r = \{ |g| \leqslant r \}$ into itself. Moreover, from (3.4) we get

$$| T[g] - T[h] | \leqslant M \cdot \int_{K_o} \| f(t, g(t)) - f(t, h(t)) \| \, dt.$$

From this the continuity of T follows. Furthermore, for every $g \in S_r$ we have (because $\| f(k^{[\nu]}(t), g(k^{[\nu]}(t))) \| = \frac{1}{(k^{[\nu]})'(t)} \cdot f(t, g(k^{[\nu]}(t))) \leqslant \frac{1}{k'} \cdot \max_{(K_o, S_r)} \| f(t,y) \|$).

Consequently, if $y = T[g]$, then from (3.6) (since $\| y(t) \| \leqslant r$) it follows that $\| \dot{y}(t) \| \leqslant N$ (N is independent of $g \in S_r$). Thus $T[S_r]$ is totally bounded and equally continuous, hence its closure is compact. From the corollary of the Tikhonov's theorem (see [3]) we conclude that T has a fixed point.

The proof is ended.

§IV. THE THEOREM OF ANDRONOV-VITT'S TYPE

We first note that the autonomous system

$$\dot{y} = f(y) \tag{4.1}$$

has no k-g.p.s. if $\dot{k}(.) \not\equiv 1$, Indeed, assume $\eta(.)$ is a solution of (4.1) and $\eta(k(t)) = \eta(t)$, $\forall t \in R_+$. Then, since $\eta(k(t))$ is also a solution, we have $\overline{\eta}(k(t)) \equiv \equiv \eta^{\cdot}(k(t)) . \dot{k}(t) = f(\eta(k(t))) \equiv f(\eta(t))$. But ($\eta(t)$ is a solution) $\eta^{\cdot}(k(t)) = f(\eta(k(t))) \equiv \equiv f(\eta(t))$. From this it follows that $\dot{k}(t) = 1$, $\forall t \in R_+$. This is a contradiction with the hypothesis $\dot{k}(.) \not\equiv 1$.

Together with (4.1) we will also consider the system in variations corresponding to the solution $\eta(.)$

$$\dot{x} = f'_y(\eta(t))x. \tag{4.2}$$

Suppose the matrix $f'_y(y)$ is nonsingular for the values of y that are being considered. Then $\dot{\eta}(t)$ cannot be a $k-g$. p.s. of the system (4.2) if $\dot{k}(.) \not\equiv 1$, Indeed, assume $\dot{\eta}(t)$ is a $k-$g.p.s. From $\dot{\eta}(t) = f(\eta(t))$ it follows that $\eta(t) = g(\dot{\eta}(t))$. Thus $\eta(t)$ is a $k-$g.p.s., too. But it is impossible as we have noted above.

The three following lemmas may be proved analogously as in [2], using the representation (1.1);

Lemma 4.1. Assume the $k-$g.p. system $\dot{x} = P(t)x$, (4.3)

where $P(.) \in C(R_+)$, $P(k(t)) = \dfrac{P(t)}{\dot{k}(t)}$, has a multiplicator $\rho_1 = 1$, and the orthers satisfy $|\rho_j| < 1$, $j = 2, ..., n$.

Then (4.3) has a fundamental matrix of solutions of the form

$$X(t) = \Phi(t) . \mathrm{diag}\,(E_1, \exp(C_1 . \mu(t))), \tag{4.4}$$

where $\Phi(t)$ is a real nonsingular matrix, $\Phi(t) \in C^1(R_+)$, $\Phi(k(t)) = \Phi(t)$, $\forall t \in R_+$; $E_1 = 1$; C_1 is a real matrix of order $(n-1)$ by $(n-1)$ having the proper values λ_j with Re $\lambda_j < 0, j = 1, 2, ..., n-1$; $\mu(.)$ is a function satisfying the conditions (μ) and (1.4) (see §1).

Lemma 4.2. Let a fundamental matrix of solutions of the system (4.3) $X(t)$ have the form (4.4) and

$$G(t, s) = \begin{cases} X(t) . \mathrm{diag}(0, E_{n-1}) . X^{-1}(s) & \text{for } t > s \\ -X(t) . \mathrm{diag}(E_1, 0) . X^{-1}(s) & \text{for } t < s. \end{cases} \tag{4.5}$$

Then 1) $G(t, t-0) - G(t, t+0) = E_n$;

2) $\dot{G}_t((t, s) = P(t).G(t, s)$ for $t \neq s$;

3) $\| G(t, s) \| \leqslant \begin{cases} m . \exp[\alpha(\mu(s) - \mu(t))] & \text{for } t > s, \\ m \quad \text{for} \quad t < s, \quad \alpha > 0, \quad m > 0; \end{cases}$

4) the vector-function $y(t) = X(t)d + \int\limits_0^\infty G(t,s)f(s)\,ds$. (4.6)

where d is an arbitrary vector of order n such that its first component is equal to $0((d, e_1) = 0,\ e = \text{colon}\ (1, 0, \ldots, 0))$, $f(.) \in C(R_+)$, $\int\limits_0^\infty \| f(t) \|\, dt < \infty$, is a solution of the nonhomogeneous system

$$\dot{y} = P(t)y + f(t) \quad (4.7)$$

for which $\lim\limits_{t \to +\infty} y(t) = 0$

Lemma 4.3. Let $P(t)$, $X(t)$, d, $G(t, s)$ be defined as in two lemmas above. Consider the system

$$\dot{z} = P(t)z + \varphi(t, z) \quad (4.8)$$

and together with it the system

$$z(t, d) = X(t)d + \int\limits_0^\infty G(t, s) \cdot \varphi(s, z(s, d))\, ds \quad (4.9)$$

where the function $\varphi(t, z)$ satisfies the Lipschitz condition

$$\| \varphi(t, \bar{z}) - \varphi(t, z) \| \leqslant L \| \bar{z} - z \|,\ t \in R_+,\ \| \bar{z} \| \leqslant \Delta,$$
$$\| z \| \leqslant \Delta \quad , \quad \varphi(t, 0) \equiv 0$$

If $L \leqslant \alpha\mu'/8m$, then for $\| d \| < d_0 = \dfrac{\Delta}{2m_1}$ ($\| X(t) \| \leqslant m_1 \cdot \exp\,[-\alpha\mu(t)]$, see (4.1)), the integral system (4.9) has a solution $z(t, d)$, which is a $(n-1)$ – parametric family of solutions of the system (4.8) and has the property

$$\lim_{t \to \infty} z(t, d) = 0. \quad (4.10)$$

It is necessary to note that the function $\mu(.)$ and its properties play a very important part in the proof of the Lemmas 2 and 3.

Let us now consider the system

$$\dot{y} = a(t) f(y) \quad (4.11)$$

and together with it the system in variations

$$\dot{x} = a(t) f'_y(\eta(t))\, x, \quad (4.12)$$

corresponding to the solution $\eta(.)$ of the system (4.11). Here $a(.) : R_+ \to R$, $a(k(t)) = \dfrac{a(t)}{\dot{k}(t)}$, $\inf\limits_{t \in K_0, v} (k^{[v]})'\,(t) > 0$.

Definition 4.1. (see also [2]). Let S be a given continuous surface (with its equation $\Psi(y) = 0$, $\Psi(y) \in C$). We say a solution $y_0(.)$ of the system (4.12) is S – conditional stable iff for any its solution $y(.)$ starting from the point $y(0) \in S$ we have

$$\lim_{t \to \infty} [y(t) - y_0(t)] = 0.$$

Theorem 4.1. (of Andronov-Vitt's type).

Assume the system (4.11) has a k-g.p.s. $\eta(.)$ to which the corresponding system in variations (4.12) has one zero characteristic index (of the first order) and the orthers have negative real parts.

Then 1) If $\int_0^\infty |\dot{a}(t)|\,dt < \infty,$ (4.13)

then for every $\varepsilon > 0$ *there exists* t_ε *and a number* $\beta = \beta(\varepsilon)$ *such that*

$$\|\eta(t) - \beta\xi(t)\| < \varepsilon, \quad \forall t > t_\varepsilon. \tag{4.14}$$

2) There exists a continuous surface S_0 *in a neighbourhood of the point* $\eta(0) \in S_0$ *such that* $\eta(.)$ *is* S_0*—conditional stable.*

Proof. First we transfer the origin of coordinates into the point $\eta(0)$. Then the systems (4.11) and (4.12) become into the forms

$$\dot{v} = a(t)\, g(v), \tag{4.14'}$$

$$\dot{u} = a(t)\, g'_v(v_0(t))\, u, \tag{4.12'}$$

moreover, $v_0(.)$ is a k—g.p.s. of (4.11'), $v_0(0) = 0$ while the c.i. of (4.12) remain unchanged and the unique k — g.p.s. $\xi(.)$ of (4.12) goes over into the unique k — g.p.s. $u_0(.)$ of (4.12'). Now if we carry out the turning transformation of the coordinate system around its origin so that the first axis coincides with the direction of the vector $u_0(0)$, then the form of the systems, together with the properties mentioned above, are unchanged (of course, with orther function $g(.)$). Therefore we can suppose $u_0(0) = e_1 \| u_0(0) \|$.

The system (4 12') satisfies all the conditions of Lemma 4.1. Hence it has a real fundamental matrix of solutions of the form (4.4)

$$U(t) = \Phi(t) \,.\, \mathrm{diag}(E_1, \exp(C_1\mu(t))),$$
$$\Phi(k(t)) = \Phi(t), \ \forall t \in R_+, \ \mathrm{Re}\ \lambda_i(C_1) < 0.$$

From this it follows that the first column of matrix is a k-g.p.s. of (4.12'). However, this system has only a unique k-g.p.s. $u_0(.)$. Thus we can write (exactly to a constant multiplier)

$$U(t) = \left[\frac{u_0(t)}{\| u_0(t) \|},\ U_1(t)\right], \tag{4.15}$$

where each component of $U_1(t)$ is a sum of products of the components in Φ, except its first column, by the components of $\exp(C_1\mu(t))$.

1) Differentiating both sides of (4.11)' with $v(.) = v_0(.)$ (it is possible because the right-hand-side of (4.11') belongs to C^1), we get

$$\ddot{v}_0 = a(t) g'_v(v_0(t))\dot{v}_0 + \dot{a}(t) g(v_0(t)).$$

We consider this system as a linear nonhomogeneous system for $\dot{v}_0(.)$ corresponding to the homogeneous system (4.12') with the free summand $\dot{a}(t) g(v_0(t))$. Therefore, by the method of variation of constants we get

$$\dot{v}_0(t) = U(t)U^{-1}(t_0)\dot{v}(t_0) + \int_{t_0}^{t} U(t)U^{-1}(s)\dot{a}(s)g(v_0(s))ds, \text{ or, using (4.4)}$$

$$\dot{v}_0(t) = \Phi(t)\text{diag}(E_1,\ \exp C_1[\mu(t) - \mu(t_0)])\ .\ \Phi^{-1}(t_0)\dot{v}(t_0) +$$

$$+ \int_{t_0}^{t} \dot{a}(s)\ .\ \Phi(t)\ .\ \text{diag}(E_1,\ \exp C_1[\mu(t) - \mu(s)])\ .\ \Phi^{-1}(s)\ .\ g(v_0(s)ds.$$

Taking into account that the first column of matrix $\Phi(t)$ is $\frac{u_0(t)}{\| u_0(t) \|}$, we can write the last correlation into the form

$$\dot{v}_0(t) = \beta(t_0)u_0(t) + \psi_1(t, t_0) + \int_{t_0}^{t} \dot{a}(s)\, \psi_2(t, s)ds, \tag{4.16}$$

where $\beta(t_0) = (\Phi^{-1}(t_0)\ .\ \dot{v}(t_0))_1\ .\ \| u_0(0) \|^{-1}$ (we denote the first component of a vector (.) by $(.)_1$);

$\| \psi_1(t, t_0) \| \leqslant r_1\ .\ \exp\ [-\alpha(\mu(t) - \mu(t_0))],\ 0 > -\alpha > \max \text{Re}\ \lambda_j(C_1)$;

$\| \psi_2(t, s) \| \leqslant r_2\ .\ \exp\ [-\alpha(\mu(t) - \mu(s))] \leqslant r_3$ for $t \geqslant s$.

Therefore from (4.16) we get

$$\| \dot{v}_0(t) - \beta(t_0)u_0(t) \| \leqslant r_1\ .\ \exp[-\alpha(\mu(t) - \mu(t_0))] +$$

$$+ r_3 \int_{t_0}^{t} \| \dot{a}(s) \| ds \leqslant r_1\ .\ \exp\ [-\alpha\underline{\mu}'(t - t_0)] + r_3 \int_{t_0}^{\infty} | \dot{a}(s) | ds. \tag{4.17}$$

For $\varepsilon > 0$ take t_0 such that $\int_{t_0}^{\infty} | \dot{a}(s) | ds < \varepsilon/2$. Then choose t_ε so large that $t_\varepsilon > t_0$ and

$$r_1\ .\ \exp[-\alpha\underline{\mu}'\ .\ (t - t_0)] < \varepsilon/2,\ \forall t > t_\varepsilon.$$

Putting $\beta = \beta(t_0)$, from (4.17) we get

$$\| \dot{v}_0(t) - \beta u_0(t) \| < \varepsilon.$$

Translating all this into the language of the initial system of coordinates, we get (4.14).

2) In the system (4.11') we use the substitution $v = v_0(t) + z$. We have

$$\dot{z} = a(t)g'_v(v_0(t))\,z + a(t)\,\varphi(t, z), \tag{4.18}$$

where $\varphi(t, z) = [g(v_0(t) + z) - g(v_0(t))] - g'_v(v_0(t))z$.

Thus $\varphi'_z(t, z) = g'_v(v_0(t) + z) - g'_v(v_0(t))$. But $a(.)$ and $v_0(.)$ are bounded functions. Hence

$$a(t)\ .\ \varphi(t, z) \underset{t}{\rightrightarrows} 0,\ a(t)\ .\ \varphi'_z(t, z) \underset{t}{\rightrightarrows} 0 \text{ when } z \to 0.$$

Moreover $\varphi(t,0) \equiv 0$ and $\varphi'_z(t, 0) \equiv 0$, consequently in the sufficiently small ball $\| z \| \leqslant \Delta$ the function $a(t) \cdot \varphi(t, s)$ satisfies the Lipschitz condition with a sufficiently small coefficient L. Thus the system (4.18) satisfies the conditions of the Lemma 4.3 (pay our attention on that $a(k(t))g'_v(v_0(k)(t))) = = \frac{a(t)}{\dot{k}(t)} g'_v(v_0(t))$ because $v_0(k(t)) = v_0(t)\ \forall t \in R_+$).

Therefore the system (4.18) has a $(n-1)$ – parametric family of solutions $z(t, d)$ ($d=(0, d_2,\dots, d_n)$, $\| d \|$ is sufficiently small) with the property $\lim\limits_{t\to\infty} z(t, d)=0$.

Consider the corresponding solutions of the system (4.11')

$$v(t) = v_0(t) + z(t, d). \tag{4.19}$$

Since $v_0(0) = 0$, we have $v(0) = z(0, d) = z^{(0)}(d)$. By [2], Chapter IV, §20, we conclude that in a n-dimensional space $(v_1, v_2, \dots, v_n)\left((z_1^{(o)}, z_2^{(o)}, \dots, z_n^{(o)}) \right)$ there exists a continuous surface $S_0^{(v)}$ with its equation $v_1^{(o)} = \psi(v_2^{(o)}, \dots, v_n^{(o)})$, defined in a neighbourhood of the origin $O_n^{(v)} \in S_0^{(v)}$ and such that every point $(v_1^{(o)}, \dots, v_n^{(o)}) \in S_0^{(v)}$ corresponds to one vector $d = (0, d_2, \dots, d_n)$, $\| d \|$ is sufficiently small, and inversely, the solution $v(t)$. $v(0) = (v_1^{(o)}, \dots, v_n^{(o)})$ corresponds to the solution $z(t, d)$ (see (4.19)). Thus, if the trajectory $v(t)$ initiates from the point $v(0) \in S_0^{(v)}$ then $v(t) - v_0(t) = z(t, d) \to 0$ when $t \to \infty$.

Translating all this into the language of the initial system of coordinates, we conclude that the solution $\eta(.)$ is S_0-conditional stable, where, S_0 is $S_0^{(v)}$ in the initial coordinates, i. e. it has the properties mentioned in the point 2).

The proof is ended.

Remarks. 1. As we knew, in the case $a(.) \equiv 1$, $\dot{k}(.) \equiv 1$, $\dot{\eta}(.) \neq 0$, $\eta(t)$ coincides with $\xi(t)$. However, it $\dot{k}(.) \not\equiv 1$, then, saying in general, $\dot{\eta}(t)$ cannot be a k-g.p.s. of the corresponding system in variations (see the Remark in §IV).

The evalution (4.14) shows that, though $\dot{\eta}(t)$ does not coincide with $\xi(t)$, these vectors almost lie on the same line for sufficiently large t.

2. In [4] § V we had

$$a'(k(t)) = \frac{\dot{a}(t)}{[\dot{k}(t)]^2} = \frac{a(t)\ddot{k}(t)}{[\dot{k}(t)]^3}.$$

The condition (4.13) is satisfied, for example, if the order of $k(.)$ is higher than the second one (then the boundedness of $a(.)$, i. e. $\inf_{t \in k_0, v} (k^{[v]})'(t) > 0$, is ensured, too).

Acknowledgement. I wish to thank the teachers and the participants of the seminar «Differential equations» in the Hanoi University for their helpful comments.

Received October 10th, 1977.

REFERENCES

1. Burton T. A., Muldowney J. S. *A generalized Floquet Theory*. Arch. Math., 1968, 19, N° 2, 188 – 194.

2. Demidovitch B. P. *Lessons on mathematical theory of stability*, M, 1967 (in Russian).

3. Hartman F. *Ordinary differential equations*, M, 1973 (in Russian).

4. Nguyễn Chân. *The class of generalized periodic functions*. Acta Math. Viet. T 5, N° 1 (1980). 3–20.