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⊗ – STRIC TAU CATEGORIES

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0. — INTRODUCTION

In 1965, in his article [4], MacLane has introduced the notion of category with an multiplication denoted by ⊗. Furthermore, this multiplication may satisfy the associative, commutative and unit constraints; these are the isomorphisms of functors:

$$a_{A;B;C}: A\otimes(B\otimes C) \xrightarrow{\sim} (A\otimes B)\ C, \otimes \ \textit{(associative constraint)}$$

$$c_{A,B}: A\otimes B \xrightarrow{\sim} B\otimes A. \ \textit{(commutative constraint)},$$

$$g_A: A \xrightarrow{\sim} 1\otimes A,\ d_A: A \xrightarrow{\sim} A\otimes 1,$$

the triplet $(1, g, d)$ forms an *unit constraint*; in which 1 is fixed object of the considered category, the associative constraint must verify the pentagon axiom, the commutative constraint must have the following property:

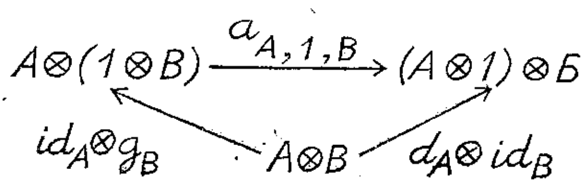
$$c_{B,A}.\,c_{A,B} = \mathrm{id}_{A\otimes B}:$$

and $g_1 = d_1$ for the unit constraint.

In this paper, by ⊗ — category we mean a category with a multiplication ⊗

An associative constraint a (commutative constraint c, unit constraint $(1, g, d)$ of a category A) is said to be *strict* if $a_{A,B,C} = \mathrm{id}_{A\otimes B\otimes C}$ for all $A, B, C \in$ ObA (resp, $c_{A,B} = \mathrm{id}_{A\otimes B}$, $g_A = \mathrm{id}_A = d_A$).

A ⊗-category A together with an associative constraint a and an unit constraint $(1, g, d)$ is a ⊗-AU category if the following triangle commutes



A $\otimes$-*AU* category is said to be *strict* if a and $(1, g, d)$ are strict, we also call A a $\otimes$-*strict AU category*.

A $\otimes$ - category together with an associative constraint a and commutative constraint c is a $\otimes$ - *AC* category if the hexagon axiom is fulfilled [4].

A $\otimes$ - category together with an associative a, commutative constraint c and an unit constraint $(1, g, d)$ is called a $\otimes$ - *ACU* category if A is a $\otimes$ - *AU* category and a $\otimes$ - *AC* category. A $\otimes$ - *ACU* category is said to be *strict* if a, c and $(1, g, d)$ are strict; e W also call A a $\otimes$ - *strict ACU category.*

A $\oplus$ - *functor* from a $\otimes$ - category A' to a $\otimes$ - category A is a pair $(F, \breve{F})$ of a functor $F : A \to A'$ and an isomorphism of bifunctors

$$\breve{F}_{X,Y} : F(X) \otimes F(Y) \to F(X \otimes Y).$$

A $\otimes$ - functor $(F, \breve{F})$ is said to be *strict* if $F(X \otimes Y) = F(X) \otimes F(Y)$ and $\breve{F}_{X,Y} = \mathrm{id}_{X \otimes Y}$ for all $X, Y \in \mathrm{Ob}A$.

A$\otimes$-fun ctor $(F, \breve{F})$ from a $\otimes$ - *AU* category A' to a $\otimes$ - *AU* category A' is said to be a $\otimes$-*AU* functor if there exists an isomorphism $\widehat{F} : 1_{A'} \xrightarrow{\sim} F(1_A)$ and the following diagrams commutes:

$$\begin{array}{ccccc}
FX \otimes (FY \otimes FZ) & \xrightarrow{\mathrm{id}_{FX} \otimes \breve{F}_{YZ}} & FX \otimes F(Y \otimes Z) & \xrightarrow{\breve{F}_{X,Y \otimes Z}} & F(X \otimes (Y \otimes Z)) \\
\uparrow a'_{FX,FY,FZ} & & & & \uparrow F(a_{X,Y,Z}) \\
(FX \otimes FY) \otimes FZ & \xrightarrow{\breve{F}_{X,Y} \otimes \mathrm{id}_{FZ}} & F(X \otimes Y) \otimes FZ & \xrightarrow{\breve{F}_{X \otimes Y, Z}} & F((X \otimes Y) \otimes Z)
\end{array}$$

(0.2)

$$\begin{array}{ccc}
FX & \xrightarrow{F(g_X)} & F(1 \otimes X) \\
\uparrow g'_{FX} & & \uparrow \breve{F}_{1,X} \\
1_{A'} \otimes FX & \xrightarrow{\widehat{F} \otimes \mathrm{id}_{FX}} & F1_A \otimes FX
\end{array}
\qquad
\begin{array}{ccc}
FX & \xrightarrow{F(d_X)} & F(X \otimes 1) \\
\uparrow d'_{FX} & & \uparrow \breve{F}_{X,1} \\
FX \otimes 1_{A'} & \xrightarrow{\mathrm{id}_{FX} \otimes \widehat{F}} & FX \otimes F1_A
\end{array}$$

(0.3) (0.4)

The purpose of this paper is to prove that every $\otimes$ - *AU* category is $\otimes$-*AU* equivalent to a $\otimes$ - strict *AU* category (in the sense of [2]. Hence we obtain the MacLane s coherence theorems [1].

1. A - CATEGORIES

Definition 1.1. Let **X** be a category, **A** a $\otimes$ - *AU* category. We say that a right action of **A** on **X** is given if we have;

a) a functor $\omega : \mathbf{X} \times \mathbf{A} \to \mathbf{X}$,
we write $\omega(X, A) = X . A$, $\omega(u, f) = u . f$ for all $X \in \mathrm{Ob}\mathbf{X}$, $A \in \mathrm{Ob}\mathbf{A}$ and $u : X \to Y$, $f : A \to B$;

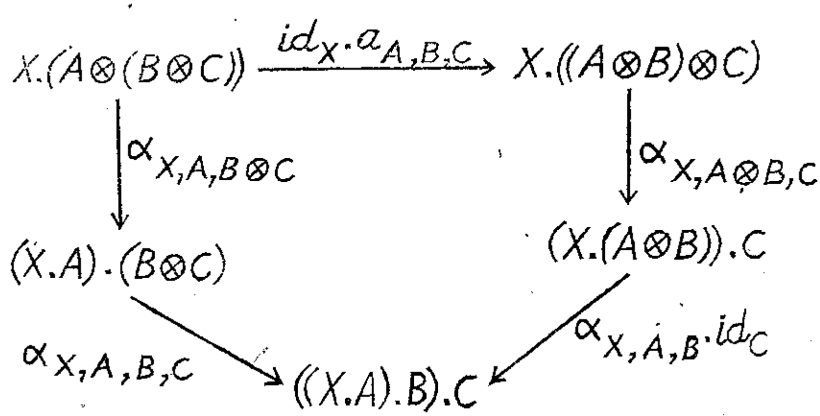
b) an isomorphism of trifunctors

$$\alpha_{X, A, B} : X . (A \otimes B) \xrightarrow{\sim} (X . A) . B ;$$

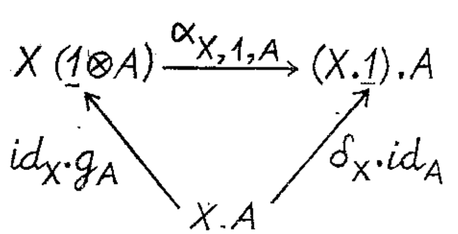
c) an isomorphism of functors

$$\delta_X : X \xrightarrow{\sim} X . 1$$

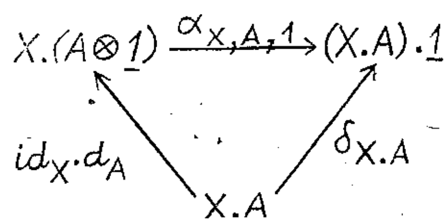
such that the following diagrams are commutative



(1.1.1)



(1.1.2)



(1.1.3)

Definition 1.2. Let **A** be a $\otimes$-AU category. A category **X** is said to be a *right* **A**-*category* if and only if a right action of **A** on **X** is given.

Example 1.2.1. Let R be a commutative ring with identity element and **Mod R** the category of all R-modules. It is easy to see that **Mod R** is a $\otimes$-AU category, in which the multiplication $\otimes$ is the tensor product of R-modules, the associative and unit constraints are the canonical isomorphisms:

$$a_{A, B, C} : A \otimes (B \otimes C) \xrightarrow{\sim} (A \otimes B) \otimes C,$$

$$g_A : A \xrightarrow{\sim} R \otimes A, \quad d_A : A \xrightarrow{\sim} A \otimes R.$$

Mod R will be a right **Mod R** - *category* if we define a right action of **Mod R** on itself as follows:

$$\omega(X, A) = X \otimes A,$$

$$\alpha_{X,A,B} = a_{X,A,B} : X \otimes (A \otimes B) \xrightarrow{\sim} (X \otimes A) \otimes B,$$
$$\delta_X = d_X : X \xrightarrow{\sim} X \otimes R.$$

In general, if $\mathbf{A}$ is a $\otimes$-AU category with an AU constraint $(a, (\mathbf{1}, g, d))$, $\mathbf{A}$ is a right $\mathbf{A}$-category if we define a right action of A on itself as follows:

$$\omega(X, A) = X \otimes A,$$
$$\alpha_{X,A,B} = a_{X,A,B},$$
$$\delta_X = d_X,$$

Then the right $\mathbf{A}$-category $\mathbf{A}$ is denote by $\mathbf{A}_d$.

Similarly, we can define a left $\mathbf{A}$-category.

Throughout the rest of this paper we will only consider the right $\mathbf{A}$-categories, which will simply be called $\mathbf{A}$-categories.

Definition 1.3. Let $\mathbf{X}$ be an $\mathbf{A}$-category, $\mathbf{X}'$ a $\mathbf{A}'$-category. A *functor* from $\mathbf{A}$-category $\mathbf{X}$ to $\mathbf{A}'$-category $\mathbf{X}'$ is a triplet $(F, \breve{F}, (T, \breve{T}))$, in which F is a functor from $\mathbf{X}$ to $\mathbf{X}'$; $(T, \breve{T}) : \mathbf{A} \to \mathbf{A}'$ is a $\otimes$-AU functor;

$$\breve{F}_{X,A} : F(X . A) \xrightarrow{\sim} FX . TA$$

is an isomorphism of bifunctors, such that the following diagrams are commutative:

$$\begin{array}{ccccc}
F(X.(A \otimes B)) & \xrightarrow{\breve{F}_{X,A \otimes B}} & FX.T(A \otimes B) & \xrightarrow{\mathrm{id}_{FX}.\breve{T}_{A,B}} & FX.(TA \otimes TB) \\
\downarrow{\scriptstyle F(\alpha_{X,A,B})} & & & & \downarrow{\scriptstyle \alpha_{FX,TA,TB}} \\
F((X.A).B) & \xrightarrow{\breve{F}_{X.A,B}} & F(X.A).TB & \xrightarrow{\breve{F}_{X,A}.\mathrm{id}_{TB}} & (FX.TA).TB
\end{array}$$

(1.3.1)

$$\begin{array}{ccc}
FX & \xrightarrow{F(\delta_X)} & F(X.1) \\
{\scriptstyle FX}\downarrow & & \downarrow{\scriptstyle \breve{F}_{X,1}} \\
FX.1_{A'} & \xrightarrow{\mathrm{id}_{FX}.\hat{T}} & FX.T1_A
\end{array}$$

(1.3.2)

If $\mathbf{A}' = \mathbf{A}$, $(T, \breve{T}) = (\mathrm{id}, \mathrm{id})$, then $(F, \breve{F}, (\mathrm{id}, \mathrm{id}))$ is called $\mathbf{A}$-functor and denoted by $(F, \breve{F})$.

Example 1.3.3. Assume that $\mathbf{A}$ is a $\otimes$-AU category and A is an object of $\mathbf{A}$ Consider the functor $\Phi_A : \mathbf{A}_d \to \mathbf{A}_d$ defined by

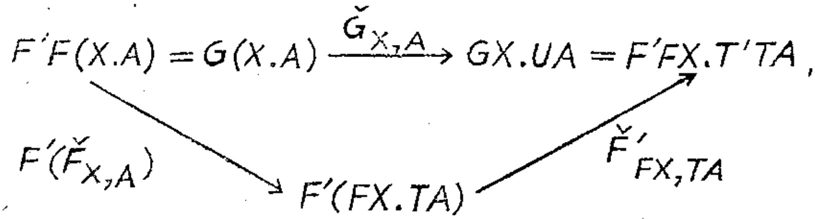
$$\Phi_A(X) = A \otimes X, \ X \in Ob\mathbf{A}_d$$
$$\Phi_A(u) = \mathrm{id}_A \otimes u, \ u \in Fl\mathbf{A}_d.$$

Set $(\breve{\Phi}_A)_{X,B} = a_{A,X,B} : \Phi_A(X.B) \xrightarrow{\sim} \Phi_A(X).B$. It is easy to verify that $(\Phi_A, \breve{\Phi}_A)$ is an $\mathbf{A}$-functor.

Proposition 1.4. Let $(F, \widehat{F}\ (T, \breve{T}))$ be a functor from an A-category X to an A'-category X' and $(F', \breve{F}', (T', \breve{T}'))$ a functor from the A'-category X' to an A"-category X". Then $(G, \breve{G}, (U, \breve{U}))$, in which

$$G = F'F,\ (U, \breve{U}) = (T'T, \widetilde{T'T}),$$

and $\breve{G}_{X,A}$ is defined by the following commutative diagram

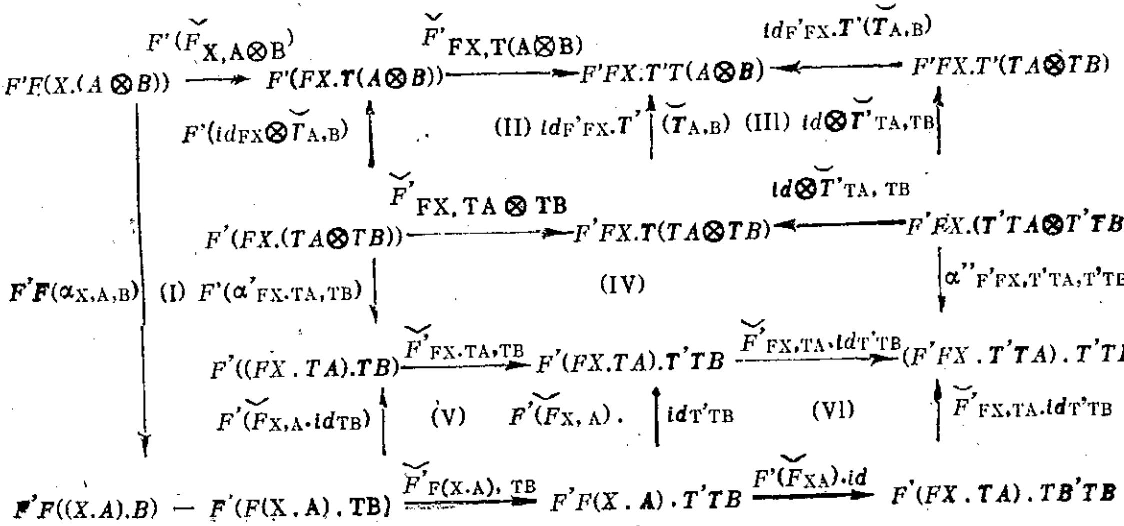


(1.4.1)

is a functor from the A-category X to the A"-category X".

Proof.—It is sufficient to check the commutativity of (1.3.1) and (1.3.2) for $(G, \breve{G}, (U, \breve{U}))$. First, we consider the following diagram (1.4.2), in which (I) is commutative by the definition of $(F, \breve{F}, (T, \breve{T}))$; (II) and (V) are commutative since F' is an isomorphism of bifunctors; (III) and (VI) commute obviously; (IV) commutes by the definition of $(F', \breve{F}', (T', \breve{T}'))$. Therefore the outer border is commutative and this is just the diagram (1.3.1) for $(G, \breve{G}, (U, \breve{U}))$.

Now consider the diagram (1.4.3), in which (I) commutes obviously; (II) is commutative by the definition of $(F, \breve{F}, (T, \breve{T}))$; (III) by the definition of $(F', \breve{F}'$ $(T'\ \breve{T}'))$; (IV) is commutative since F' is an isomorphism of bifunctors. Therefore the outer border is commutative and it is just the diagram (1.3.2) for $(G, \breve{G}, (U, \breve{U}))$.



(1 . 4 . 2)

$$\begin{array}{ccccc}
F'FX & = & F'FX & \xrightarrow{F'F(\delta_X)} & F'F(X.1) \\
\| & (I) & \downarrow{\scriptstyle F'(\delta'_{FX})} \ (II) & & \downarrow{\scriptstyle F'(\breve{F}_{X,1})} \\
F'FX & \xrightarrow{F'(\delta'_{FX})} & F'(F.1_{A'}) & \xrightarrow{F'(id.\widehat{T})} & F'(FX.T1_A) \\
\downarrow{\scriptstyle \delta''_{F'FX}} & (III) & \downarrow{\scriptstyle \breve{F}'_{FX,1_{A'}}} & (IV) & \downarrow{\scriptstyle \breve{F}'_{FX,T1}} \\
F'FX.1_{A''} & \xrightarrow[id.\widehat{T'}]{} & F'FX.T'1_{A'} & \xrightarrow[F'(id).T'(\widehat{T})]{} & F'FX.T'T1
\end{array}$$

(1 . 4 . 3)

Definition 1.5. Let $(F, \breve{F}, (T, \breve{T}))$ and $(G, \breve{G}, (U, \breve{U}))$ be two functors from an **A**-category **X** to an **A'**-category **X'**. A pair (φ, τ), in which $\varphi : F \to G$, $\tau : (T, \breve{T}) \to (U, \breve{U})$, is called a *morphism* from $(F, \breve{F}, (T, \breve{T}))$ to $(G, \breve{G}, (U, \breve{U}))$ if the following diagram

$$\begin{array}{ccc}
F(X.A) & \xrightarrow{\breve{F}_{X,A}} & FX.TA \\
\downarrow{\scriptstyle \varphi_{X.A}} & & \downarrow{\scriptstyle \varphi_X.\tau_A} \\
G(X.A) & \xrightarrow{\breve{G}_{X,A}} & GX.UA
\end{array}$$

(1.5.1)

is commutative.

When $\mathbf{A} = \mathbf{A'}$, $(T, \breve{T}) = (U, \breve{U}) = (\mathrm{id}, \mathrm{id})$, (φ, id) is denoted by φ and called an **A**-*morphism*.

Example 1.5.2. Assume that $(\Phi_A, \breve{\Phi}_A)$ and $(\Phi_B, \breve{\Phi}_B)$ are **A**-functors from $\mathbf{A}_d$ to itself, $f: A \to B$ is a morphism in **A**, (example 1.3.3). Set

$$\Phi_f(X) = f \otimes \mathrm{id}_X : \Phi_A(X) \to \Phi_B(X).$$

Then Φ_f is an **A**-morphism. In fact, from the fact that the associative constraint a is an isomorphism of trifunctors it follows that the following diagram

$$\begin{array}{ccc}
\Phi_A(X.D) = A \otimes (X \otimes D) & \xrightarrow{(\breve{\Phi}_A)_{X,D}} & (A \otimes X) \otimes D = \Phi_A(X).D \\
\downarrow{\scriptstyle \Phi_f(X.D)} & & \downarrow{\scriptstyle \Phi_f(X).\mathrm{id}_D} \\
\Phi_B(X.D) = B \otimes (X \otimes D) & \xrightarrow[(\breve{\Phi}_B)_{X,D}]{} & (B \otimes X) \otimes D = \Phi_B(X).D
\end{array}$$

is commutative; i.e we obtain the diagram (1.5.1).

Definition 1.6. A functor $(F, \breve{F}, (T, \breve{T}))$ is called an equivalence if and only if F and T are equivalences.

2.1. In this section we only use A-categories A_d and A-functors. Then the diagrams (1.3.1) and (1.3.2) have the simple forms:

$$\begin{array}{ccccc}
F(X.(A\otimes B)) & & \xrightarrow{\quad \breve{F}_{X,\otimes A B} \quad} & & FX.(A\otimes B) \\
\Big\downarrow F(\alpha_{X,A,B}) & & & & \Big\downarrow \alpha_{FX,A,B} \\
F((X.A).B) & \xrightarrow[\breve{F}_{X.A,B}]{} & F(X.A).B & \xrightarrow[\breve{F}_{X,A}.\mathrm{id}_B]{} & (FX.A).B
\end{array}$$

(2.1.1)

$$\begin{array}{ccc}
FX & \xrightarrow{\quad F(\delta_X) \quad} & F(X.\underline{1}) \\
 & {}_{\delta_{FX}}\searrow \qquad \swarrow_{\breve{F}_{F,\underline{1}}} & \\
 & FX.\underline{1} &
\end{array}$$

(2.1.2)

Definition 2.2. A $\otimes$-AU category **A** with an AU constraint $(a, (\mathit{1}, g, d))$ is called *strict* if and only if

$$a_{X,B,C} = \mathrm{id}_{A,B,C} \quad \text{for all } A, B, C \in \mathrm{Ob}\mathbf{A},$$
$$g_A = \mathrm{id}_A = d_A \quad \text{for all } A \in \mathrm{Ob}\mathbf{A}.$$

Proposition 2.3. The category **End** $(\mathbf{A}_d)$ of all **A**-functors from $\mathbf{A}_d$ to itself with the multiplication $\otimes$ defined by the following relations:

$$(F', \breve{F}') \otimes (F, \breve{F}) = (F'F, \breve{F'F}) \text{ for all } (F, \breve{F}), (F', \breve{F}') \text{ in } \mathbf{End}\ (\mathbf{A}_d) \qquad (2.3.1)$$

$$(\varphi' \otimes \varphi)_X = \varphi'_{GX} F'(\varphi_X) = G'(\varphi_X)\varphi'_{FX} \qquad (2.3.2)$$

for $\varphi: (F, \breve{F}) \to (G, \breve{G})$, $\varphi': (F', \breve{F}') \to (G', \breve{G}')$,
is a $\otimes$ - strict AU category.

Proof. It is easy to see that **End** $(\mathbf{A}_d)$ is a category. Now we prove that the multiplication is a bifunctor. First, we verify that $\varphi' \otimes \varphi$ is an **A** - morphism. In fact, we have the following diagram

$$\begin{array}{ccccc}
F'F(X.A) & \xrightarrow{\quad F'(\breve{F}_{X,A}) \quad} & F'(FX.A) & \xrightarrow{\quad \breve{F}'_{FX,A} \quad} & F'FX.A \\
F'(\varphi_{X.A})\Big\downarrow & \text{(I)} & \Big\downarrow F'(\varphi_X.\mathrm{id}_A) & \text{(II)} & F'\varphi_X.\Big\downarrow \mathrm{id}_A \\
F'G(X.A) & \xrightarrow[F(\breve{G}_{X,A})]{} & F'(GX.A) & \xrightarrow{\quad \breve{F}_{GX,A} \quad} & F'GX.A \\
\varphi'_{G(X.A)}\Big\downarrow & \text{(III)} & \varphi'_{GX.A}\Big\downarrow & \text{(IV)} & \Big\downarrow \varphi'_{GX}.\mathrm{id}_A \\
G'G(X.A) & \xrightarrow[G'(\breve{G}_{X,A})]{} & G'(GX.A) & \xrightarrow[\breve{G}'_{GX,A}]{} & G'GX.A
\end{array}$$

in which (I) and (IV) are commutative since φ and φ' are the **A**-morphisms; (II) commutes since $\breve{F'}$ is an isomorphism of bifunctors; (III) commutes since φ' is an isomorphism of functors. Therefore the outer border is commutative and this proves that $\varphi' \otimes \varphi$ is an **A**-morphism.

From (2.3.2) it follows that:

$$(id_{(F', \breve{F'})} \otimes id_{(F, \breve{F})})_X = id_{F'FX} \cdot F'(id_{FX}) = id_{F'FX} \cdot id_{F'FX} =$$

$$= id_{F'FX} = (id_{(F', \breve{F'}) \otimes (F, \breve{F})})_X ;$$

$$id_{(F', \breve{F'})} \otimes id_{(F, \breve{F})} = id_{(F', \breve{F'}) \otimes (F, \breve{F})} .$$

Let $\varphi : (F, \breve{F}) \to (G, \breve{G}), \varphi'' : (F', \breve{F'}) \to (G', \breve{G'})$,

$\psi : (G, \breve{G}) \to (H, \breve{H}), \psi' : (G', \breve{G'}) \to (H', \breve{H'})$.

We have:

$$[(\psi' \otimes \psi)(\varphi' \otimes \varphi)]_X = (\psi' \otimes \psi)_X (\varphi' \otimes \varphi)_X = \psi'_{HX} G'(\psi_X) \varphi'_{GX} F'(\varphi_X) =$$

$$= \psi'_{HX} \varphi'_{HX} F'(\psi_X) F'(\varphi_X) = (\psi' \varphi')_{HX} F'((\psi\varphi)_X) = (\psi' \varphi'' \otimes \psi\varphi)_X, \text{ i. e}$$

$$(\psi' \otimes \psi)(\varphi' \otimes \varphi) = \psi' \varphi' \otimes \psi\varphi.$$

Thus **End** ($\mathbf{A}_d$) is a category. Furthermore, from the diagram (1.4.1), it follows that:

$$(F'' \breve{F'}) F = F'' (\breve{F'} F).$$

Hence we have:

$$((F'', \breve{F''})(F', \breve{F'}))(F, \breve{F}) = ((F'' F') F, (\breve{F'' F'}) F)$$

$$(F'' (F' F), F'' \widetilde{(F' F)}) (F'', \breve{F''}) ((F', \breve{F'}) (F, \breve{F})).$$

We also have:

$$(id, id) \otimes (F, \breve{F}) = (F, \breve{F}) = (F, \breve{F}) \otimes (id, id).$$

Moreover, for $\varphi : F \to G, \varphi'' : F' \to G', \varphi'' : F'' \to G''$, we have:

$$((\varphi'' \otimes \varphi) \otimes \varphi)_X = (\varphi'' \otimes \varphi')_{GX} F'' F(\varphi_X) = \varphi''_{G'GX} F''(\varphi'_{GX}) F''(F' \varphi_X) =$$

$$= \varphi''_{G'GX} F''(\varphi'_{GX} F'(\varphi_X)) = \varphi''_{G'GX} F''((\varphi' \otimes \varphi)_X) = (\varphi'' \otimes (\varphi' \otimes \varphi))_X$$

and for $\varphi : F \to G$, id : id $\to$ id, we have obviously:

$$(id \otimes \varphi)_X = \varphi_X = (\varphi \otimes id)_X.$$

Thus, **End** ($\mathbf{A}_d$) is a $\otimes$-strict AU category.

Theorem 2.4. Let **A** be a $\otimes$-AU category with $(a, (\mathbf{1}, g, d))$ as AU constraint. Then **A** is $\otimes$-AU equivalent to the $\otimes$-strict AU category **End** ($\mathbf{A}_d$).

Proof. We define a $\otimes$-AU functor $(\Phi, \breve{\Phi})$ from the $\otimes$-AU category **A** to the $\otimes$-AU category **End** ($\mathbf{A}_d$) as follows:

$$A \mapsto (\Phi_A, \breve{\Phi}_A)$$

$$f \mapsto \Phi_f$$

with $(\Phi_A, \breve{\Phi}_A)$ and Φ_f given by the examples 1.3.3 and 1.5.2 respectively ; and

$$\breve{\Phi}_{A,B} : \Phi_A \otimes \Phi_B \to \Phi_{A \otimes B}$$

is an isomorphism of functors define by

$$(\breve{\Phi}_{A,B})_X = a_{A,B,X} : (\Phi_A \otimes \Phi_B)(X) \to \Phi_{A \otimes B}(X).$$

One can easily verify that $(\Phi, \breve{\Phi})$ so defined is a $\otimes$ - AU functor.

Now we define a quasi-inverse

$$\psi : \mathbf{End}\ (\mathbf{A}_d) \to A$$

of Φ by the following relations :

$$\psi\ (F, \breve{F}) = F1, \text{ for all } (F, \breve{F}) \in \text{Ob } \mathbf{End}\ (\mathbf{A}_d), \tag{2.4.1}$$

$$\Psi\ (\varphi) = \varphi_1, \text{ for all } \varphi \in \text{Fl } \mathbf{End}\ (\mathbf{A}_d), \tag{2.4.2}$$

ψ is a functor because :

$$\Psi\ (\text{id}_{(F,\breve{F})}) = \text{id}_{F1} = \text{id}_{\Psi\ (F,\breve{F})}, \text{ and}$$

$$\Psi\ (\psi\varphi) = (\psi\varphi)_1 = \psi_1\,\varphi_1 = \psi\ (\psi)\ \Psi\ (\varphi).$$

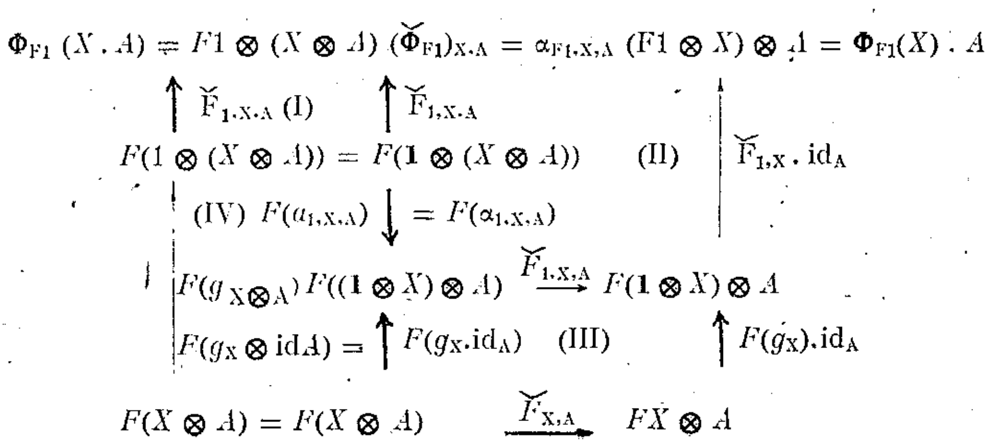
We define the isomorphisms $\tau : \psi\,\Phi \xrightarrow{\sim} \text{id}_A$; $\tau' : \Phi\Psi \xrightarrow{\sim} \text{id}_{\mathbf{End}\ (\mathbf{A}_d)}$ as follows :

$$\tau_A = d_A^{-1}, \tag{2.4.3}$$

$$(\tau'_{(F,\breve{F})})_X = F\ (g_X^{-1})\ \breve{F}^{-1}_{1,X}. \tag{2.4.4}$$

We see that $\tau'_{(F,\breve{F})}$ is an $\mathbf{A}$ - morphism. In fact, we have the following diagram,

in which (I) commutes obviously ; (II) is just the commutative diagram (2. 1. 1) ; (III) is commutative since F is an isomorphism of bifunctors ; (IV) commutes since $\mathbf{A}$ is a $\otimes$ $-AU$ category. Therefore the outer border is commutative and this proves that $\tau'_{(F,\breve{F})}$ is an $\mathbf{A}$-morphism.



Thus Ψ is a quasi-inverse of Φ and $(\Phi, \breve{\Phi})$ is a $\otimes - AU$ equivalence, so the theorem is proved.

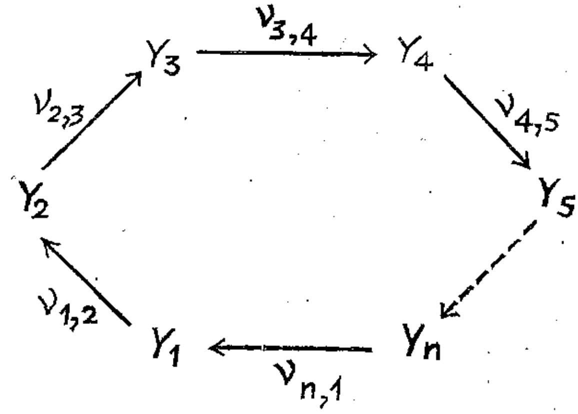
Lemma 2.5. Let A and A' to be the $\otimes$ - AU categories, $(\Phi, \breve{\Phi})$: $\mathbf{A} \to \mathbf{A'}$ a $\otimes - AU$ functor, X a product of a finite family $(X_i)_{i \in I}$ of objects in A and X' a product in $\mathbf{A'}$, in which when instend of X_i we have $\Phi(X_i)$. Then every morphism $\mu : \Phi X \to X'$ constructed from $\breve{\Phi}^{-1}$, $\widehat{\Phi}^{-1}$, id and $\otimes$ in $\mathbf{A'}$ is equal.

Proof. We proceed the proof of lemma by induction on the numbers of elements of I. For $I = \{ \alpha \}$ our lemma holds obviously. Assume that I has $n > 1$ elements and the lemma is true for every $k < n$. We always can write $X = Y \otimes Z$. Since μ constructed from $\breve{\Phi}^{-1}$, $\widehat{\Phi}^{-1}$, id and $\otimes$ in $\mathbf{A'}$, we see that μ must be the composition of the following morphisms:

$\Phi(X) = \Phi(Y \otimes Z) \xrightarrow{\breve{\Phi}^{-1}_{Y,Z}} \Phi(Y) \otimes \Phi(Z) \xrightarrow{\nu \otimes \lambda} Y' \otimes Z'$, here Y and Z are the products of $(X_i)_{i \in I_1}$ and $(X_i)_{i \in I_2}$ respectively, $I_1 \perp\!\!\perp I_2 = I$. Y', Z' defined as the same of X' and ν, λ are constructed from $\breve{\Phi}^{-1}$, $\breve{\Phi}^{-1}$, id and $\otimes$ in A', $X' = Y' \otimes Z'$. By assumption ν, λ are unique. Therefore the lemma is proved.

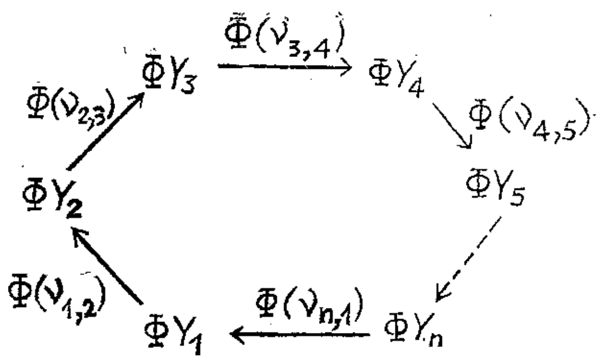
Corollary 2.6. (Maclane's coherence theorem). Let $\mathbf{A}$ be a $\otimes$-AU category with an AU constraint $(a, (\mathbf{1}, g, d))$. Then a, g, d are coherent (in the sense of [4]).

Proof. Assume that we have the diagram



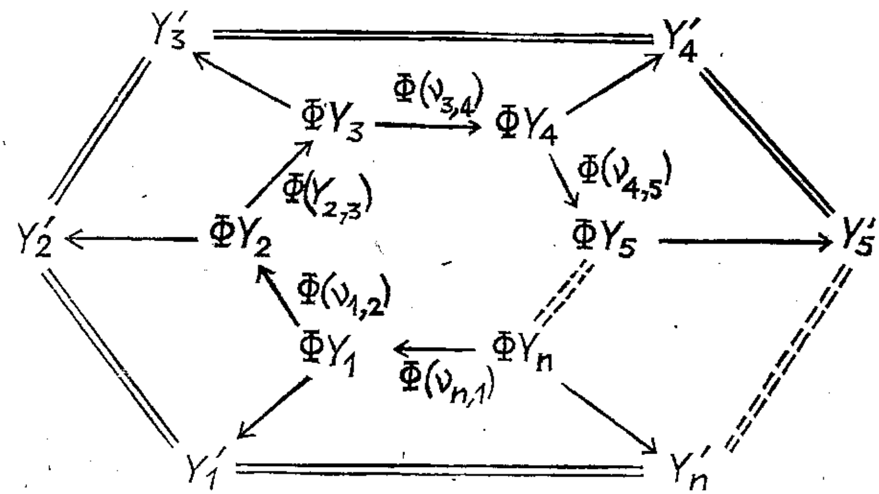
in which Y_i is the product of a finite family $(X_j)_{j \in I_i}$, $I_i \neq \phi$ and the set of j's, $j \in I$ such that $X_j \neq \mathbf{1}$ is the same for $i = 1, \ldots, n$; $\nu_{i,i+1}$ and $\nu_{n,1}$ are constructed from $a, a^{-1}, g, g^{-1}, d, d^{-1}$, id and $\otimes$ in $\mathbf{A}$, $i = 1, \ldots, n-1$. We must provet hat this diagram is commutative. We can always assume that every ν consists only one of $a, a^{-1}, g, g^{-1}, d, d^{-1}$.

To prove the commutative of the given diagram, we will prove that the following diagram commutes in **End** $(\mathbf{A}_d)$



(2.6.1)

here $(\Phi, \breve{\Phi})$ is the ⊗-AU equivalence in the proof of theorem 2.2. By the compatibility of $(\Phi, \breve{\Phi})$ with AU constraints we can construct on each edge of the diagram (2.6.1) a rectangular which opposition edge of $\Phi(\nu)$ is identity. Thus we can extend the diagram (2.6.1) to the diagram



in which Y_i' is the product in **A'**, in which when instend of X_j, $j \in I_i$ we have $\Phi(X_j)$, $i = 1, \ldots, n$ and the morphism $\Phi Y_i \to Y_i'$ is constructed from $\breve{\Phi}^{-1}, \widehat{\Phi}^{-1}$, id and ⊗ in **End** $(\mathbf{A}_d)$. By the lemma 2.5 the rectangulars are commutative. The outer border is commutative obviously. Therefore the diagram (2.6.1) is commutative. Since Φ is an equivalence, so the given diagram is.

Theorem 2.7. Every ⊗-ACU category is ⊗-ACU equivalent to a ⊗-ACU category which is a ⊗-strict AU category.

Proof. Assume that **A** is a ⊗-ACU category. Then **End** $(\mathbf{A}_d)$ is also a ⊗-ACU category, furthermore it is a ⊗-strict AU category. The ⊗-AU equivalence $(\Phi, \breve{\Phi})$ is compatible with the commutative constraints (Ch. I, 1, prop. 5, [2]); i. e ⊗-ACU category **A** is ⊗-ACU equivalent to the ⊗-ACU category **End** ($\mathbf{A}_d$).

Theorem 2.8. Every ⊗-associative category is ⊗-associative equivalent to a ⊗-strict associative category.

Proof. Let **A** be a ⊗-category with an associative constraint a. We construct a ⊗-AU category **C** as follows:

$$\mathrm{Ob}\mathbf{C} = \mathrm{Ob}\mathbf{A} \bigcup \{1\}.$$

where **1** is a symbol.

$$\mathrm{Hom}_C(A, B) = \mathrm{Hom}_A(A, B), \text{ if } A, B \in \mathrm{Ob}\mathbf{A}, \tag{2.8.1}$$

$$\mathrm{Hom}_C(\mathbf{1}, A) = \mathrm{Hom}_C(A, \mathbf{1}) = \phi, \text{ for all } A \in \mathrm{Ob}\mathbf{A} \tag{2.8.2}$$

$$\mathrm{Hom}_C(\mathbf{1}, \mathbf{1}) = \mathrm{id}_1, \tag{2.8.3}$$

$$A \otimes_C B = A \otimes_A B, \text{ if } A, B \in \mathrm{Ob}\mathbf{A}, \tag{2.8.4}$$

$$A \otimes \mathbf{1} = \mathbf{1} \otimes A = A, \text{ for all } A \in \mathrm{Ob}\mathbf{C} \tag{2.8.5}$$

$$f \otimes_C h = \otimes f_A h, \text{ if } f, h \in Fl\mathbf{A} \tag{2.8.6}$$

$$f \otimes \mathrm{id}_1 = \mathrm{id}_1 \otimes f = f, \text{ for all } f \in Fl\mathbf{C} \tag{2.8.7}$$

From (2.8.6) and (2.8.7), it follows that:

$$\mathrm{id}_A \otimes_C \mathrm{id}_B = \mathrm{id}_A \otimes_A \mathrm{id}_B = \mathrm{id}_{A \otimes B}, \text{ if } A, B \in \mathrm{Ob}\mathbf{A},$$

$$\mathrm{id}_A \otimes_C \mathrm{id}_1 = \mathrm{id}_A = \mathrm{id}_1 \otimes_A = \mathrm{id}_1 \otimes_C \mathrm{id}_A,$$

$$(f \otimes_C g)(h\, {}_C\!\otimes k) = fh \otimes gk, \text{ if } f, g, h, k \in Fl\mathbf{A}.$$

$$(f \otimes_C \mathrm{id}_1)(h \otimes_C id_1) = fh = fh \otimes \mathrm{id}_1$$

$$(\mathrm{id} \otimes_C f)(\mathrm{id} \otimes_C h) = fh = \mathrm{id} \otimes fh.$$

In **C** we define the AU constraint $(a, (1, g, d))$ as follows:

$$a'_{A,B,C} = a_{A,B,C}, \text{ if } A, B, C \in \mathrm{Ob}\mathbf{C},$$

$$a'_{1,A,B} = a'_{A,1,B} = a'_{A,B,1} = \mathrm{id}_{A \otimes B}, \text{ for all } A, B \in \mathrm{Ob}\mathbf{C}$$

$$g_A' = \mathrm{id}_A = d_A'.$$

It is easy to see that a', g' d' are isomorphisms of functors and satisfy the [pent]agon axiom and the diagram (0.1)

Furthermore, they are compatible.

Now we can establish a ⊗-AU equivalence

$$(\Phi, \breve{\Phi}) : \mathbf{C} \underset{\rightarrow}{\approx} \mathbf{End}\ (\mathbf{C}_d).$$

[Th]e full subcategory (**A**) ⊗-stable generated by **Φ** (**A**) [2], in which each [...] the form:

$(F, \breve{F}) = (\Phi_{A_1}, \breve{\Phi}_{A_1}) \otimes \dots \otimes (\Phi_{A_n}, \breve{\Phi}_{A_n})$, $A_1, \dots, A_n \in ObA$. We denote it by $\underline{<\Phi(\mathbf{A})>}$. It is easy to see that $\underline{<\Phi(A)>}$ is a -$\otimes$strict associative category and the restrict on A of $(\Phi, \breve{\Phi})$ is $\otimes$-associative functor from $\mathbf{A}$ to $\underline{<\Phi(A)>}$. The restrict on $\underline{<\Phi(\mathbf{A})>}$ of $(\Psi, \breve{\Psi})$ is also a $\otimes$-associative functor from $\underline{<\Phi(\mathbf{A})>}$ to $\mathbf{A}$ since

$$\psi((\Phi_{A_1}, \breve{\Phi}_{A_1}) \otimes \dots \otimes (\Phi_{A_n}, \breve{\Phi}_{A_n})) = \Phi_{A_1} \dots \Phi_{A_n}(1) =$$
$$= A_1 \otimes (A_2 \otimes \dots \otimes (A_n \otimes 1) \dots) = A_1 \otimes (A_2 \otimes (\dots \otimes (A_{n-1} \otimes A_n) \dots) \in Ob\mathbf{A}.$$

Thus $\mathbf{A} \approx \underline{<\Phi(\mathbf{A})>}$.

Corollary 2.9. Let $\mathbf{A}$ be a $\otimes$-associative category with the associative constraint a. Then a is coherent (in the sense of [4]).

Proof. It follows immediately from the theorem 2.8 and the corollary 2.6.

Theorem 2.10. Every $\otimes$-AC category is $\otimes$-AC equivalent to a $\otimes$-AC category which is a $\otimes$ – strict associative category.

Proof. Assume that $\mathbf{A}$ is a $\otimes$-AC category with associative constraint a and commutative constraint $\mathbf{c}$. We construct a $\otimes$-category $\mathbf{C}$ as in the proof of the theorem 2.8, on which we define the commutative constraint $\mathbf{c}'$ as follows:

$$c'_{A,B} = c_{A,B}, \quad \text{if } A, B \in Ob\mathbf{A},$$
$$c'_{A,1} = c'_{1,A} = id_A \quad \text{for all } A \in Ob\mathbf{C}.$$

It is easy to see that c' is an isomorphism of bifunctors and $\mathbf{C}$ is a $\otimes$-ACU category.

As the proof of the theorem 2.8, we obtain:

$$\mathbf{A} \sim \underline{<\Phi(\mathbf{A})>}$$

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