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COMMON FIXED POINTS OF TWO MAPPINGS OF CONTRACTIVE TYPE

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The aim of this paper is to establish some new results on common fixed points of two mappings of contractive type and to show incorrectness in some theorems of other authors on common fixed points.

We shall use the following notations :

(X,d) is a metric space.

S, T are two (single-valued or multi-valued) mappings in X,

N is the set of all natural numbers,

$d(x,A) = \inf \{ d(x,y) : y \in A \}$, $(x \in X,\ A \subset X)$,

$D(A,B) = \max \{ \sup\limits_{x \in B} d(x,A),\ \sup\limits_{y \in A} d(y,B) \}$, $(A, B \subset X)$,

$\delta\,(A,B) = \sup \{ d(x,y) : x \in A,\ y \in B \}$, $(A; B \subset X)$,

$CB(X)$ is the family of all nonempty closed bounded subsets of X,

$$r(S, T ; x, y) = \max \{ d(x, y),\ d(x, Sx),\ d(y, Ty),\ \frac{1}{2}\,[d(x, Ty) + d(y, Sx)] \}.$$

$$r(T ; x, y) = r(T, T ; x, y).$$

Theorem 1. Let (X, d) be a complete metric space, S, T be two multi-valued mappings of X into $CB(X)$. Suppose there exists an upper semicontinuous from the right function $\alpha : [0,\infty) \to [0,1)$ such that :

$$D(Sx, Ty) \leqslant \alpha(d(x, y))\, r(S, T ; x, y) \tag{1}$$

for every $x, y \in X$, $x \neq y$.

Then either S or T has a fixed point.

If (1) holds for every $x, y \in X$ then S and T have a common fixed point.

Proof. Let $x_0 \in X$. If $x_0 \in Sx_0$ take r such that $d(x_0, Sx_0) < r$. Then there exists $x_1 \in Sx_0$ such that $d(x_0, x_1) < r$. Obviously $x_1 \neq x_0$. From (1) we have:

$$d(x_1, Tx_1) \leqslant D(Sx_0, Tx_1) \leqslant \alpha(d(x_0, x_1))\, r(S, T; x_0, x_1)$$
$$\leqslant \alpha(d(x_0, x_1))\, max\, \{d(x_0, x_1), d(x_0, Sx_0),$$
$$d(x_1, Tx_1), \frac{1}{2}[d(x_0, x_1) + d(x_1, Tx_1)]\}.$$

It is easy to see that this implies:

$$d(x_1, Tx_1) < min\, \{d(x_0, x_1), \alpha(d(x_0, x_1))\, r\}.$$

If $x_1 \notin Tx_1$ there exists $x_2 \in Tx_1$ such that

$$d(x_1, x_2) < min\, \{d(x_0, x_1), \alpha(d(x_0, x_1))\, r\}.$$

Similarly, select $x_3 \in Sx_2$ so that:

$$d(x_2, x_3) < min\, \{d(x_1, x_2), \alpha(d(x_1, x_2))\, \alpha(d(x_0, x_1))\, r\}.$$

Repeating the above argument with $c_n = d(x_n, x_{n+1})$ we obtain a sequence $\{x_n\}$ satisfying

(i) $x_{n+1} \in Sx_n$ *if n is even,*

$x_{n+1} \in Tx_n$ *if n is odd,*

$x_{n+1} \neq x_n \quad (n \in N)$

(ii) $c_n < \min\{c_{n-1}, \alpha(c_{n-1}) \ldots \alpha(c_0)\, r\}$.

From this $c_n \searrow b \geqslant 0$.

Denote $M = \overline{\lim}\, \alpha(c_n)$. Since $M \leqslant \alpha(b) < 1$ there exists $n_0 \in N$ such that $\alpha(c_n) \leqslant L = M + \varepsilon < 1$ for any $n \geqslant n_0$. From this it is easy to see that $\{x_n\}$ is a Cauchy sequence and hence converges to some $x^* \in X$.

If $x^* = x_n$ for all n large enough then $x_n = x_{n+1}$, contradicting (i). Thus there exists a subsequence of $\{x_n\}$, denoted by $\{x_m\}$, with $x_m \neq x^*$ $(\forall m)$ and either all of m are even or they are odd.

When all of m are even we have:

$$d(x^*, Tx^*) \leqslant d(x^*, x_{m+1}) + d(x_{m+1}, Tx^*) \leqslant d(x^*, x_{m+1}) + D(Sx_m, Tx^*) \leqslant$$
$$\leqslant d(x^*, x_{m+1}) + \alpha(d(x_m, x^*))\, r(S, T; x_m, x^*) \leqslant$$
$$\leqslant d(x^*, x_{m+1}) + \alpha(d(x_m, x^*)),\, max\, \{d(x_m, x^*), d(x^*, Tx^*),$$
$$d(x_m, x_{m+1}), \frac{1}{2}[d(x_m, Tx^*) + d(x^*, x_{m+1})]\}.$$

Noting that $\lim_{m\to\infty} \alpha(d(x_m, x^*)) \leqslant \alpha(0) < 1$, we derive from this $d(x^*, Tx^*) = 0$, and since Tx^* is closed, $x^* \in Tx^*$.

Similarly, for the case when all of m are odd, we obtain $x^* \in Sx^*$. The first part of Theorem 1 is thus proved.

If (1) holds for any $x, y \in X$ then every fixed point of one mapping is also a fixed point of the other. Indeed, supposing $x \in Sx$ we have

$$d(x, Tx) \leqslant D(Sx, Tx) \leqslant \alpha(0)\, d(x, Tx).$$

hence $x \in Tx$. Similarly for the case $x \in Tx$. The proof is complete.

Remark 1. If S and T are single-valued the common fixed point mentioned in Theorem 1 is unique. Indeed, if x, y are two common fixed points of S and T then from (1) it follows:

$$d(x, y) \leqslant \alpha(d(x, y))\, d(x, y).$$

Hence $x = y$.

Lemma. Let $A \in CB(X)$ and $q \in [0,1)$. Then for any $x \in X$ there exists $a \in A$ such that:

$$q\delta(x, A) \leqslant d(x, a).$$

Proof. Assuming the contrary, there is $x_0 \in X$ such that:

$$q\delta(x_0, A) > d(x_0, a), \quad (\forall a \in A)$$

Then $q\delta(x_0, A) \geqslant \sup\limits_{a \in A} d(x_0, A)$, contradicting $q < 1$.

Theorem 2. Let (X, d) be a complete metric space, S, T two multivalued mappings of X into $CB(X)$. Suppose that there exists $q \in [0,1)$ such that

$$\delta(Sx, Ty) \leqslant q \max\left\{d(x, y), \delta(x, Sx), \delta(y, Ty), \frac{1}{2}[d(x, Ty) + d(y, Sx)]\right\}$$

for any $x, y \in X$.

Then S and T have a unique common fixed point x^* and $Sx^* = Tx^* = \{x^*\}$.

Proof. Select p satisfying $q < p < 1$. We construct two single-valued mappings s, t satisfying the following conditions

$$s(x) \in Sx,\ p\delta(x, Sx) \leqslant d(x, s(x))$$
$$t(x) \in Tx,\ p\delta(x, Tx) \leqslant d(x, t(x)),\ (\forall x \in X).$$

These mappings exist by the above lemma.

Then for any $x, y \in X$ we have

$$d(s(x), t(y)) \leqslant \delta(Sx, Ty) \leqslant q\max\Big\{d(x, y), \delta(x, Sx), \delta(y, Ty),$$
$$\frac{1}{2}[d(x, Ty) + d(y, Sx)]\Big\} \leqslant r \cdot r(s, t;\ x, y)$$

with $r = \dfrac{q}{p} < 1$

By Remark 1, s and t have a unique common fixed point. Obviously, it is also a common fixed point of S and T.

Let x be some common fixed point of S and T. Then

$$\delta(x, Sx) \leqslant \delta(Sx, Tx) \leqslant q\max\{\delta(\bar{x}, Sx), \delta(x, Tx)\}.$$

Similarly,

$$\delta(x, Sx) \leqslant q\max\{\delta(x, Sx), \delta(x, Tx)\}.$$

Hence $\delta(x, Sx) = \delta(x, Tx) = 0$, i.e. $Sx = Tx = \{x\}$.

From this and the uniqueness of the common fixed point of s and t it follows that the common fixed point of S and T is also unique. The proof of Theorem 2 is complete.

Corollary 1. (Rus [8]). Let (X, d) be a complete metric space, S, T be two mappings of X into $CB(X)$. Suppose there exist $\alpha, \beta \geqslant 0$ such that $\alpha + 2\beta < 1$ and

$$\delta(Sx. Ty) \leqslant \alpha d(x, y) + \beta[\delta(x, Sx) + \delta(y, Ty)]$$

for any $x, y \in X$.

Then the conclusion of Theorem 2 still holds.

Corollary 2. Let (X, d) be a complete metric space, S, T two mappings of X into $CB(X)$. Suppose there exists $q \in [0, 1)$ such that ;

$$\delta(Sx, Ty) \leqslant q\max\left\{d(x, y), \delta(x, Sx), \delta(y, Ty), \frac{1}{4}[\delta(x, Ty) + \delta(y, Sx)]\right\} \tag{2}$$

for any $x, y \in X$

Then the conclusion of Theorem 2 still holds.

Proof. Since

$$\delta(x, Ty) \leqslant d(x, y) + \delta(y, Ty),$$
$$\delta(y, Sx) \leqslant d(y, x) + \delta(x, Sx),$$

it follows from (2) that

$$\delta(Sx, Ty) \leqslant q\max\{d(x, y), \delta(x, Sx), \delta(y, Ty)\}$$

for any $x, y \in X$.

It remains to apply Theorem 2.

Corollary 3. (Avram [2]). Let (X, d) be a complete metric space, S, T be two mappings of X into $CB(X)$. Suppose there exist $a, b, c \geqslant 0$ such that: $a + 2b + 4c < 1$ and

$$\delta(Sx, Ty) \leqslant ad(x, y) + b[\delta(x, Sx) + \delta(y, Ty)] + c[\delta(x, Ty) + \delta(y, Sx)]$$

for any $x, y \in X$.

Then the conclusion of Theorem 2 still holds.

Remark 2. In [6] Nguyễn Anh Minh established an example showing that in Theorems 1 and 2 we can not replace $\frac{1}{2}[d(x, Ty) + d(y, Sx)]$ by $\max\{d(x, Ty), d(y, Sx)\}$. Namely there exist two single-valued mapping S and T satisfying

$$d(Sx, Ty) \leqslant \frac{1}{2}\max\{d(x, y), d(x, Sx), d(y, Ty), d(x, Ty), d(y, Sx)\}$$

for every x, y and such that neither has a fixed point.

Theorem 3. Let (X, d) be a complete metric space, S, T be two multi-valued mappings of X into $CB(X)$. Suppose there exist $a_1, \ldots, a_5 \geqslant 0$ such that $a_1 + \ldots + a_5 < 1$, $a_1 = a_2$ or $a_3 = a_4$ and

$$\delta(Sx, Ty) \leqslant a_1 \delta(x, Sx) + a_2 \delta(y, Ty) + a_3 d(x, Ty) + a_4 d(y, Sx) + a_5 d(x, y)$$

for any $x, y \in X$.

Then the conclusion of Theorem 2 still holds.

Proof. Put $\varepsilon = 1 - a_1 - a_2 - a_3 - a_4 - a_5$,

$$b_i = a_i + \frac{\varepsilon}{3}, \quad c_i = \frac{a_i}{b_i}, \quad (i = 1, 2).$$

Using the above lemma we construct two single-valued mappings s and t such that

$$s(x) \in Sx, \quad c_1 \delta(x, Sx) \leqslant d(x, s(x)),$$
$$t(x) \in Tx, \quad c_2 \delta(x, Tx) \leqslant d(x, t(x)).$$

For every $x, y \in X$ we have

$$d(s(x), t(y)) \leqslant b_1 d(x, s(x)) + b_2 d(y, t(y)) +$$
$$+ a_3 d(x, t(y)) + a_4 d(y, s(x)) + a_5 d(x, y).$$

In view of a theorem due to Wong [11], s and t have a unique common fixed point.

The rest of the proof is similar to the proof of Theorem 2 and can be omitted.

Remark 3. When S and T are single-valued Theorem 3 coincides with the above mentioned theorem of Wong. Another generalization of Wong's theorem can be found in [10].

Theorem 1 contains a rather strong assumption, namely $\alpha(0) < 1$. In the following theorem, this assumption will be relaxed.

Theorem 4. Let (X, d) be a complete metric space, S, T be two single-valued mappings of X into itself, at least one of which is continuous. Suppose there exists a upper-semicontinuous function $\alpha : (0, \infty) \to (0, 1)$ satisfying

$$d(Sx, Ty) \leqslant \alpha(d(x, y) \;\; r(S, T; x, y))$$

for any $x, y \in X$, $x \neq y$.

Then S or T has a fixed point.

Proof. Let $x_0 \in X$ and put

$$x_{2n+1} = Sx_{2n}, \; x_{2n+2} = Tx_{2n+1}, \quad (n \geqslant 0).$$

We may assume that $x_m \neq x_{m+1}$ for each m. We shall prove that $\{x_m\}$ is a Cauchy sequence. Putting $b_m = d(x_m, x_{m+1})$ it is easy to verify that $b_{m+1} \leqslant \alpha(b_m) b_m$ for each $m \in N$ and hence $b_m \searrow b \geqslant 0$.

If $b > 0$ then by the upper semicontinuity of α we obtain a contradiction $b \leqslant \alpha(b)\, b < b$. Thus $b = 0$.

Now, using the method of Wong [12], we assume the contrary that $\{x_m\}$ is not a Cauchy sequence. Then there exists $\varepsilon > 0$ such that for each $m \in N$ there are $p_m > q_m \geqslant m$ satisfying

$$d(x_{p_m}, x_{q_m}) \geqslant \varepsilon,\ d(x_{p_m-1}, x_{q_m}) < \varepsilon.$$

Denoting

$$c_m = d(x_{p_m}, x_{q_m}),\ a_m = d(x_{p_m-1}, x_{q_m})$$

we have

$$\varepsilon \leqslant c_m = d(x_{p_m}, x_{q_m}) \leqslant d(x_{p_m}, x_{p_m-1}) + d(x_{p_m-1}, x_{q_m}) < b_{p_m-1} + \varepsilon.$$

Hence $c_m \searrow \varepsilon$. On the other hand,

$$\varepsilon > a_m = d(x_{p_m-1}, x_{p_m}) \geqslant d(x_{p_m}, x_{q_m}) - d(x_{p_m}, x_{p_m-1}) \geqslant \varepsilon - b_{p_m-1}.$$

Hence $a_m \nearrow \varepsilon$. Thus $a_m > 0$ for all m large enough.

Consider the sequence $\{c_m\}$. We can distinguish two cases:

(i) There is a subsequence of $\{c_m\}$, denoted again by $\{c_m\}$, with one of p_m, q_m being odd and the other even. First, consider the case when p_m is even, q_m is odd. Then we have

$$d(x_{p_m+1}, x_{q_m+1}) = d(Sx_{p_m}, Tx_{q_m}) \leqslant$$

$$\leqslant \alpha(c_m) \max \left\{ c_m, b_{p_m}, b_{q_m}, \frac{1}{2} [d(x_{p_m}, x_{q_m+1}) + d(x_{q_m}, x_{p_m+1})] \right\}$$

$$\leqslant \alpha(c_m)\, [c_m + b_{p_m} + b_{q_m}]. \tag{3}$$

Noting that

$$c_m - b_{p_m} - b_{q_m} \leqslant d(x_{p_m+1}, x_{q_m+1}) \leqslant c_m + b_{p_m} + b_{q_m},$$

and letting m tend to the infinity in (3) we obtain a contradiction: $\varepsilon \leqslant \alpha(\varepsilon) < \varepsilon$. Similarly for the case when p_m is odd, q_m is even.

(ii) There exists a subsequence $\{c_m\}$, with p_m q_m being both even or both odd. Consider the case when both p_m, q_m are even. Then we have

$$d(x_{p_m+1}, x_{q_m+1}) \leqslant d(x_{p_m+1}, x_{p_m}) + d(x_{p_m}, x_{q_m+1}) \leqslant$$

$$\leqslant b_{p_m} + d(Tx_{p_m-1}, Sx_{q_m}).$$

For m so large that $a_m > 0$ we can write

$$d(x_{p_m+1}, x_{q_m+1}) \leqslant b_{p_m} + \alpha(a_m) \max \left\{ a_m, b_{p_m-1}, b_{q_m}, \frac{1}{2} [d(x_{p_m-1}, x_{q_m+1}) + c_m] \right\}$$

$$\leqslant b_{p_m} + \alpha(a_m) \max \{a_m, c_m + b_{p_m-1} + b_{q_m}\} = b_{p_m} + \alpha(a_m)[c_m + b_{p_m-1} + b_{q_m}].$$

Letting m tend to the infinity we obtain a contradiction $\varepsilon \leqslant \alpha(\varepsilon)\varepsilon < \varepsilon$.

Similarly for the case when both p_m, q_m are odd. Thus $\{x_m\}$ is a Cauchy sequence and hence $x_m \to x^* \in X$.

Now assume that S is continuous. Then from the fact $x_{2n} \to x^*$ it follows that $Sx_{2n} \to Sx^*$. On the other hand, $Sx_{2n} = x_{2n+1} \to x^*$. Since X in separated, we obtain $x^* = Sx^*$.

Similarly for the case when T is continuous. The proof is complete.

Remark 4. The case when $S = T$ and α is upper semicontinuous from the right has been considered in [9].

Remark 5. The following example shows that in Theorem 4 $\frac{1}{2}[d(x, Ty) + d(y, Sx)]$ can not be replaced by $\max\{d(x, Ty), d(y, Sx)\}$.

Let X be the set of all integers,

$$Sx = Tx = x + 1, \ (\forall x \in X)$$

Then it is easy to see that

$$d(Tx, Ty) \leqslant \alpha(d(x, y)) \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}$$

for any $x, y \in X$, $x \neq y$, where

$$\alpha(t) = 1 - \frac{1}{t+1}$$

but T has no fixed point.

Remark 6. The following example shows that the assumption about the continuity of S or T can not be omitted.

$$\text{Let } X = \left\{1, \frac{1}{2}, \dots, \frac{1}{2^n}, \dots, 0\right\}, \ S = T \text{ with}$$

$$T\left(\frac{1}{2^n}\right) = \frac{1}{2^{n+1}}; \ (n \geqslant 0); \ T(0) = 1.$$

It is easy to verify that

$$d(Tx, Ty) \leqslant \alpha(d(x, y))\, r(T; x, y)$$

for any $x \neq y$, where

$$\alpha(t) = \max\left\{1 - \frac{t}{2}, \frac{1}{2}\right\}$$

but T has no fixed point.

In both examples S can be choosen different from T. For instance, in the first example take $Sx = x + 2$ and $\alpha(t) = 1 - \frac{1}{t+2}$, while in the second example take

$$S(1) = \frac{1}{4}, \ Sx = Tx \quad (\forall x \neq 1).$$

In [1] Achari proved the following theorem

Theorem of Achari. Let F be a nonempty family of continuous mappings of a compact metric space (X, d) into itself. Suppose for each pair T_1, T_2 in F there exist $m = m(T_1, T_2)$ $n = n(T_1, T_2) \in N$ such that

$$d(T_1^m x, T_2^n y) < r(T_1^m, T_2^n; x, y)$$

for any $x, y \in X$, $x, \neq y$.

Then each mapping $T_i \in F$ has a unique fixed point which is also the unique common fixed point of the whole family F.

The following simple example due to Wong [11] shows that this Theorem is false.

Let $X = \{0,1\}$, $F = \{T_1, T_2\}$, where $T_1(0) = T_2(1) = 0$, $T_1(1) = T_2(0) = 1$, $m = n = 1$. Then F satisfies all conditions in the above Theorem but it has no common fixed point.

Using an idea of Jaggi [5] we can prove the following theorem.

Theorem 5. Let (X, d) be a complete metric space, S, T be two continuous mappings of X into itself. Assume that S and T satisfy the following conditions:

(i) $$d(Sx, Ty) < r(S, T; x, y)$$

for any $x,y \in X$, $x \neq y$.

(ii) There exists $x_0 \in X$ such that the sequence $\{x_n\}$ with

$$x_n = \begin{cases} Sx_{n-1} & \text{if } n \text{ is odd,} \\ Tx_{n-1} & \text{if } n \text{ is even,} \end{cases}$$

contains a convergent subsequence.

(iii) $ST = TS$

(iv) S and T have each at most one fixe point.

Then S and T have each a unique fixed point which is also their unique common fixed point.

Proof. First, note that we may assume $x_n \neq x_{n+1}$ for each $n \in N$. In fact, suppose there is $m \in N$ such that $x_m = x_{m+1}$. Without loss of generality we may assume that m is odd. Then $x_m = x_{m+1} = Tx_m$ and by (iii), $Sx_m = STx_m = TSx_m$. From (iv) it follows that $x_m = Sx_m$, i.e. x_m is a common fixed point common of S and T. The uniqueness follows immediately from (i).

Now denoting $b_n = d(x_n, x_{n+1})$, if n is even, we have

$$b_n = d(Tx_{n-1}, Sx_n) < \max\left\{b_{n-1}, b_n, \frac{1}{2} d(x_{n-1}, x_{n+1})\right\} = b_{n-1}.$$

Similarly for the case when n is odd. Hence, the sequence $\{b_n\}$ converges to $b \geq 0$, By assumption, $\{x_n\}$ contains a subsequence $\{x_k\}$ converging to $x^* \in X$. nsider two possible cases.

1) There exists a subsequence $\{x_m\}$ of $\{x_k\}$, with all m's even. Then, by continuity of S and T,

$$Sx_m = x_{m+1} \to Sx^*,$$
$$TSx_m = x_{m+2} \to TSx^*.$$

$x^* \neq Sx^*$ then from (i) it follows

$$d(Sx^*, TSx^*) < r(S, T; x^*, Sx^*) = d(x^*, Sx^*).$$

n the other hand,

$$d(Sx^*, TSx^*) = \lim_m d(x_{m+1}, x_{m+2}) = \lim_m b_{m+1} = \lim_n b_n = b =$$
$$= \lim_m b_m = \lim_m d(x_m, x_{m+1}) = d(x^*, Sx^*).$$

his contradiction shows that $x^* = Sx^*$. Hence, as above, we get by (iii) and (iv), $^* = Tx^*$.

2) The case when all m are odd is treated in a similar way. This completes he proof.

Remark 7. The above theorem immediately generalizes a theorem of aggi [5]. The above mentioned Wong's example shows that the condition (iv) can not be omitted.

Moreover, from the uniqueness of fixed point it follows that Theorem 5 still holds for a family of mappings satisfying the conditions stated in Achari's heorem, together with conditions (iii), (iv).

Remark 8. Wong's example also shows that the following theorem is false.

Theorem of Istratescu. ([3], Theorem 1.4, p.97)

Let (X, d) be a complete metric space, S, T be two mappings of X into itself. Suppose there are $\alpha, \beta \geqslant 0$ such that $\alpha + \beta < 1$ and

$$d(Sx, Ty) \leqslant \alpha\, d(x, Sx) + \beta\, d(y, Ty)$$

for any $x, y \in X$, $x \neq y$.

Then S and T have a unique common fixed point.

By the first part of Theorem 1, the conditions of the above theorem ensure only that either S or T ha a fixed point. In order to ensure that S and T have a unique common fixed-point we need one of the two following assumptions:

1) Inequality (4) holds for every $x, y \in X$ or

2) S and T satisfy (iii) and (iv) in Theorem 5.

Indeed, in case 1) this follows from the second part of Theorem 1, and in case 2), by (iii) and (iv) we need only consider the sequence $\{x_n\}$ with $x_n \neq x_{n+1}$ and show as in [3] that it is Cauchy.

Remark 9. Wong's example also shows that the following theorem is false.

Theorem of V. Istratescu and A. Istratescu. ([4], Theorem 2.1). Let (X, d) be a complete metric space, S, T two mappings of X into itself. Suppose there exists $k \in [0,1)$ such that

$$d(Sx, Ty) \leqslant k\, d(x, y)$$

for every $x, y \in X, x \neq y$. Then S and T have a unique common fixed point.

By an argument similar to the previous one we see that here, as well as in Theorem 2.2, [4], we need the assumptions (iii), (iv). For Theorem 3.1, 3.2 [4] similar modifications are also necessary.

Remark 10. In [7] Ray and Rhoades proved the following theorem.

Theorem of Ray and Rhoades. Let (X, d) be a complete metric space, S, T be two mappings of X into itself. Suppose there exists $k \in [0, 1)$ with the following property: for every $x, y \in X$ there exist $n = n(x)$, $m = m(y) \in N$ such that

$$d(S^n x, T^m y) \leqslant k\, r(S^n, T^m; x, y)$$

Then S and T have a unique common fixed point.

The following example shows that this theorem is false.

$$\begin{aligned} \text{Let } X = \{0,1,2\},\ & S(0) = 1, \quad S(1) = 2, \quad S(2) = 0 \\ & T(0) = 2, \quad T(1) = 0, \quad T(2) = 1 \\ & n(0) = 2, \quad n(1) = 1, \quad n(2) = 3 \\ & m(0) = 1, \quad m(1) = 2, \quad m(2) = 3. \end{aligned}$$

Then for every $x, y \in X$ the left hand side of (5) is always equal to 0, hence (5) always holds, but neither of S, T has a fixed point.

The above theorem obviously holds if n and m are constants (by combining Remarks 1 and 7 above). This theorem still holds if n is constant on each set $\{x, Sx, \ldots, S^n x, \ldots\}$ and m is constant on each set $\{x, Tx, \ldots, T^n x, \ldots\}$ for any $x \in X$.

Indeed, in [7] the authors called a point $\bar{x}$ periodic if $\bar{x} = S^{n(\bar{x})}\bar{x} = T^{n(\bar{x})}\bar{x}$ and proved that such a point exists.

Moreover from (5) it is obvious that if $x = S^{n(x)}x$ and $y = T^{m(y)}y$ then $x = y$. Now by the above additional condition,

$$S\bar{x} = S^{n(\bar{x})} S\bar{x} = S^{n(S\bar{x})} S\bar{x}.$$

$$T\bar{x} = T^{m(\bar{x})} T\bar{x} = T^{m(T\bar{x})} T\bar{x}.$$

Thus, $\bar{x} = S\bar{x} = T\bar{x}$.

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