ACTA MATHEMATICA VIETNAMICA
TOM 5, N° 1 (1980)

DUALITY IN THE INFINITE SYMMETRIC PRODUCTS *

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Introduction

It appears that there exists a dual parallelism between the homological analysis of the Eilenberg—MacLane spaces $K(\pi,q)$ and that of the iterated loop spaces $\Omega^q\Sigma^q X$. In particular, there is an analogy between the Steenrod cohomology operations and the Dyer—Lashof homology operations on these spaces (Serre [29], Araki—Kudo [2], Dyer—Lashof [11])

This is the first in a series of papers to afford some examinations to a more sufficient understanding of this phenomenon and, further to make a review of the (co) homology operations with a particular conviction that some new tools to handle them efficiently may arise from this. In the present paper, we shall show that the above mentioned dual parallelism has the origin in the classical Poïncaré—Lefschetz duality.

Let CG' denote the category of compactly generated Hausdorff spaces with nondenerate base points. Recall that, by a theorem of Dold—Thom, the infinite symmetric product $SP^\infty X$ is weakly homotopy equivalent to a direct product of the Eilenberg—MacLane spaces if X is an arcwise connected space in CG'. On the other hand, in this case, there exists a weak homotopy equivalence α_q: $C(X,q)\to\Omega^q\Sigma^q X$, where $C(X,q)$ is a space built up from the configuration space $F(\mathbf{R}^q,n)$ and the smash product $X^{[n]}$ (see May [17]). These fundamental results, lead us to make a review of the infinite symmetric product $SP^\infty\Sigma^q X$ of the q—iterated suspension $\Sigma^q X$ of X, and consider the relation between $C(X,q)$ and $SP^\infty\Sigma^q X$. It will be easily observed that $SP\Sigma^q X$ is homeomorphic to the space built up from the nq—dimentional sphere $(\times_n \mathbf{R}^q)$ and $X^{[n]}$ by a similar way in the construction of $C(X,q)$. From this remark, with a little care, $C(X,q)$ can be imbedded naturally in $SP(X,q)$ as a subspace. Consequently, it is natural to construct the space $B(X,q)$ built up from $F(R^q,n)^\cdot = (\times_n R^q)^\cdot /T(R^q,n)$ with $T(\mathbf{R}^q,n) = (\times_n \mathbf{R}^q)^\cdot - F(\mathbf{R}^q,n)$, and $X^{[n]}$, and then to consider the relation between $B(X,q)$ and $C(X,q)$.

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*) This work was announced at the second Vietnamese Mathematical Congress, August 15—19, 1977.

From the Poincaré — Lefschetz duality in the sphere $(\underset{*}{\times}\mathbf{R}^q)^{\cdot}$ with respect to $((\underset{*}{\times}\mathbf{R}^q)^{\cdot}, T(\dot{\mathbf{R}}^q.n))$, we shall obtain correspondingly a duality in the infinite symmetric. product $SP(X,q)$ with respect to $(SP(X,q), D(X,q))$ i.e. a duality between $B(X,q)$ and $C(X,q)$. Here $D(X,q)$ is a a subspace of $SP(X,q)$ builtup from $T(\mathbf{R}^q,n)$ and $X[^n]$; we note that $B(X,q) = SP(X,q)/D(X,q)$.

$B(X,q)$ and $C(X,q)$ are topological monoids in CG' (see § 4). So their singular chain complexes $S_*(B(X,q))$ and $S_*(C(X,q))$ are DG algebras. Now, the duality between $B(X,q)$ and $C(X,q)$ appears in the following

Main Theorem. Let $X \in CG'$. Then there exist the morphism of DG algebras

$$\text{(i) } B^q\mathcal{S}^0C_*(X) \to S_*(B(X,q))$$

$$\text{(ii) } F^q\mathcal{S}^qC_*(X) \to S_*(C(X,q))$$

which are chain equivalences for any DGA-module $C_*(X)$ such that $C_*(X) \approx S_*(X)$. Here, B^q and F^q denote the q-iterated bar construction and cobar construction respectvetly and—$\mathcal{S}^q$ the q-iterated suspension.

A direct consequence of this theorem is that the homology algebras $H_*(B(X,q)\,;\,\mathbf{Z}_p)$ and $H_*(C(X,q)\,;\,\mathbf{Z}_p)$ are completely determined by the method of Cartan construction (see Cartan [6], Milgram [21]), whenever $H_*(X : \mathbf{Z}_p)$ is computed. We shall present the computation in details in a subsequent paper. The computation of $H_*(C(X,q)\,;\,\mathbf{Z}_p)$ has been done in May [19] and Cohen [7] for $q=\infty$ and $q<\infty$ respectively by a different lines. Mainly, they applied the Dyer—Lashof's computation of $H_*(\Omega^q\Sigma^q X\,;\,\mathbf{Z}_p)$ and used the approximation between $C(X,q)$ and $\Omega^q\Sigma^q X$. So, the assertion for $C(X,q)$ here is of independent interest. Note that, instead of for $C(X,q)$ in (ii), Milgram [21] has proved a similar relation for $J_q(X)$, a space of the same homotopy type with $\Omega^q\Sigma^q X$ when X is connected.

The paper contains 11 section. In §1, we recall some basic facts on the point set topology for the spaces in CG' in a convenient form to construct in § 2 a class of spaces $L(X,q)$ including $SP(X,q,)$ $B(X,q)$ and $C(X,q)$ as special cases. In § 3, we prove a Steenrod's decomposition theorem for the homology of $L,(X,q)$. The notion of *mDGA* — algebras (*DGA* — algebras with multiplicity) formulated by Nakamura [26] will be recalled in §4, and the notion of *mDGA* — coalgebras will be introduced in § 5. These notions are of particular importance, since they are compatible with respect to the Steenrod's decomposition theorem.

The main tool of our study will be found in § 6. That is the decomposition of the space $\underset{*}{\times}\mathbf{R}^q$ given by Nakamura in [26] from which we obtain automatically a CW — decomposition for the space $F(\mathbf{R}^q,n)^{\cdot}$. The section 7 is a homological study of the space $B(S^0,q)$, and § 8 is a general study of the space $B(X,q)$. Here the part (i) of the main theorem is proved. By a dual analogy, we proceed to study homologically the space $C(X,q)$ in the two next sections § 9 and § 10, and we shall prove there the second part of the main theorem. Note that we shall prove more than what we stated in the main theorem : the morphisms of DGA algebras in (i) and (ii) are compatible with the Steenrod's decompositions for the homology groups of $B(X,q)$ and $C(X,q)$.

The paper is concluded by section § 11 with a proof of the well known group completion theorem for $\alpha_q : C(X,q) \to \Omega^q \Sigma^q X$ for every X in CG'.

It is a great pleasure to acknowledge the gratitude of the author to Prof. Tokushi Nakamura from whom he has obtained among the green leaves of the gingkos many valuable explanations about the notion of $mDGA$-algebras and the decomposition of the space $\underset{n}{\times} \mathbf{R}^q$.

§ 1. PRELIMINARIES ON THE CATEGORY CG

The paper of Steenrod [34] shows why it is convenient to work in the category of compactly generated Hausdorff spaces. In this spirit, we shall work in this category. All products, mapping spaces et cetera are always assumed to be given the compactly generated topology. Correspondingly the notion of topological monoids, group actions, fiber spaces... are modified in coherence with this notion of products. The point set topology required here can be found in [34].

Let CG denote the category of compactly generated Hausdorff spaces and continuous maps. Let CG' denote the category of based spaces in CG and base point preserving maps. We shall denote by $*$ the base point unless otherwise specified, and base points are always assumed to be non-degenerate, in the sense that $\{*\}$ is a neighborhood deformation retract in X (briefly an NDR in X) for each X in CG'.

1.1 Let G be a finite group, and X a G-space in CG. Recall that a subspace A of X is a *G-equivariant NDR* in X if A is invariant under the action of G and if there exists a representation (u, h) of (X, A) as an NDR-pair (i.e. a pair of maps $u : X \to I = [0,1]$ such that $A = u^{-1}(0)$, $h : I \times X \to X$ such that $h(0, x) = x, x \in X$ and $h(1, x) \in A$ whenever $u(x) < 1$) is a pair of G-maps. For instance, as it is well known, a pair of G-equivariant CW-complexes is a G-equivariant NDR-pair. If (X, A) is a G-equivariant NDR-pair, then $(X/G, A/G)$ is obviously an NDR-pair with the representation induced from that of (X,A).

A G-equivariant NDR-pair (X,A) will be said to be *relatively G-free* if $X - A$ is free under the action of G. Particularly, if $X \in CG'$ and $A = \{*\}$, we have the notion of *relatively G-free based space* or simply *G-free based space.*

1.2 Suppose that we are given a sequence of spaces in CG:

$$X_0 \subset X_1 \subset \; . \; . \; . \; . \subset X_k \subset \ldots$$

where X_k is embedded in X_{k+1} as a closed subspace for each k. Let $X = \bigcup_{k \geq 0} X_k$ have the topology of the union. Then, according to [34; 9.2 and 9.4], we have.

(1.3) *If each* (X_k, X_{k-1}) *is an NDR-pair, then* X *is in CG and each* X_k *is an NDR in* X.

By a *filtered space* X we understand a space X in CG and a sequence of closed subspaces

$$X_0 \subset X_1 \subset \dots \subset X_k \subset \dots$$

of X such that $X = \bigcup_{k \geqslant 0} X_k$ and X has the topology of the union. (Note that a closed subspace of a space in CG is in CG [34 : 2.4], so the definition implies $X_k \in CG$ for each k). In addition, if each (X_k, X_{k-1}) is an NDR-pair, X is said to be *filtered by NDR's*.

1.4 Given the filtered spaces $X = \{ X_k \}$, $Y = \{ Y_k \}$, ... , $Z = \{ Z_k \}$. Their *product* is the space $X \times Y \times \dots \times Z$ filtered by

$$(X \times Y \times \dots \times Z)_k = \bigcup_{l+m+\dots+n=k} X_l \times Y_m \times \dots \times Z_n.$$

By [34; 10.3 and 10.5], $X \times Y \times \dots \times Z$ is then a filtered space. Further, if the filtrations of X, Y, ... Z are by NDR's, then $X \times Y \times \dots \times Z$ is also filtered by NDR's.

1.5 Let (X,A) be an NDR-pair. Consider X as a space filtered by NDR's with $X_0 = A$ and $X_k = X$ for $k \geqslant 1$. Then the n-fold product $X^n = \{ (X^n)_k \}$ is a space filtered by NDR's.

Moreover, let $\mathbf{G}_n$ be the symmetric group of degree n. Let $\mathbf{G}_n$ operate on X^n by permuting the factors. Then X^n is filtered by $\mathbf{G}_n$ - equivariant NDR's according to the proof of [34; 63] see also May [17; A.4]).

For later convenience, we introduce the following notion.

1.6. Definition. Let $\mathcal{J} = \{G_k\}$ be a sequence of finite groups

$$G_0 \subset G_1 \subset \dots \subset G_k \subset \dots$$

A filtered space $X = \{X_k\}$ is called a *$\mathcal{J}$-space* if the following conditions (i) and (ii) hold.

(i) X_k is a G_k-space for $k \geqslant 0$.

(ii) Set $X^{\wedge}_{k-1} = \bigcup_{g \in G_k} g X_{k-1}$. Then the composition of maps

$$X_{k-1}/G_{k-1} \to X^{\wedge}_{k-1}/G_{k-1} \to X^{\wedge}_{k-1}/G_k$$

is a homeomorphism for $k \geqslant 1$.

In addition, if X satisfies the condition

(iii) $(X_k, X^{\wedge}_{k-1})$ is a G_k - equivariant NDR - pair for $k \geqslant 1$, X is said to be a *$\mathcal{J}$-space filtered by NDR's*. In particular, if $(X_k, X^{\wedge}_{k-1})$ are relatively G_k - free, we say that X is *ralatively $\mathcal{J}$-free*.

Given a $\mathcal{J}$ space $X = \{X_k\}$. We write

$$X/\mathcal{J} = X_k/G_k.$$

Here X_{k-1}/G_{k-1} are imbedded in X_k/G_k (as closed subsets) by use of the homeomorphisms $X_{k-1}/G_{k-1} \cong X^{\wedge}_{k-1}/G_k$ in (ii). An immediate consequence of the definition and (1. 3) is the following.

1. 7. Proposition. *If X is a $\mathcal{J}$-space filtered by NDR's then $X/\mathcal{J}$ is in CG and filtered by NDR's.*

Now we recall the notion of symmetric products.

2.8. Let **G** denote the sequence of symmetric groups

$$\mathbf{G}_0 \subset \mathbf{G}_1 \subset \dots \subset \mathbf{G}_k \subset \dots$$

where $\mathbf{G}_0 = \mathbf{G}_1$ and $\mathbf{G}_k$ is the symmetric group on the set $\{1, 2, \dots, k\}$ for each $k \geqslant 1$. Let $(X, *)$ be a space in CG' and let X^∞ the filtered space

$$X^\infty = \{X^k\}$$

where $X^0 = \{*\}$ and X^{k-1} is imbedded in X^k by the injection $(x_1, \dots, x_{k-1}) \to (x_1, \dots, x_{k-1}, *)$ for each $(x_1 \dots x_{k-1}) \in X^{k-1}$. Then we have

$$(X^k)^\frown = (X^k)^\frown_{k-1} = \bigcup_{i=1}^{k} X \times \dots \times \overset{i}{\{*\}} \times \dots \times X.$$

So we have $X^{k-1}/\mathbf{G}_{k-1} = (X)^\frown_{k-1}/\mathbf{G}_k$. From 1.5, we observe that $X^\infty = \{X^k\}$ is an **G**-space filtered by *NDR*'s.

Define

$SP^\infty X = X^\infty/\mathbf{G} = \{SP^k X\}$

with $SP^0X = \{*\}$, $SP^kX = X^k/\mathbf{G}_k$, $k \geqslant 1$. SP^kX and $SP^\infty X$ are called the *k-fold symmetric product* and the *infinite symmetric product* of X respectively. The point $*$ of SP^0X will be considered as their base point. According to Proposition 1.7, $SP^\infty X = \{SP^kX\}$ is in CG' and it is a space filtered by *NDR*'s (see May [17; §3]).

For k finite or infinite. SP^k is clearly a natural functor in CG', Further, it is also a homotopy and limit preserving functor (see Spanier [30; §6]. If $f(X, *) \to (X', *)$ is a map in CG', then we denote by $SP^kf : SP^kX \to SP^kX'$ the map induced by f. In particular, if $i : A \subset X$, then $SP^ki : SC^kA \to SP^kX$ is an injection. Via this map, SP^kA is regarded as a subspace of SP^kX. From 1.5, if A is an *NDR* in X, SP^kA is an *NDR* in SP^kX.

1.9 Let us recall the iterated suspension functor.

Given a nonnegative integer q. Let $\mathbf{S}^q$ be the q-sphere obtained by the one point compactification of the euclidean space $\mathbf{R}^q$ of dimension q. Let the point of compactifcation be its base point. By the standard injection $i : \mathbf{S}^q \to \mathbf{S}^{q+1}$, that is the base point preserving map given by $i(a^1, \dots, a^q) = (a^1, \dots a^q, 0)$, $ai \in \mathbf{R}$, we regard $\mathbf{S}^q$ as a subspace of $\mathbf{S}^{q+1}$.

Let $X \in CG'$. We define $\Sigma^q X$, the q—*iterated suspension* of X, to be the smash product

$$X \wedge \mathbf{S}^q = X \times \mathbf{S}^q / X \times \{*\} \cup \{*\} \times \mathbf{S}^q .$$

If $q = 1$, $\Sigma^1 X$ is denoted simply by ΣX.

By means of the injection

$$1_X \wedge i : \Sigma^q X = X \wedge \mathbf{S}^q \to \Sigma^{q+1} X = X \wedge \mathbf{S}^{q+1} .$$

we regard $\Sigma^q X$ as a subspace of $\Sigma^{q+1} X$. $(\mathbf{S}^{q+1}, \mathbf{S}^q)$ is an *NDR*-pair, so it is easily seen that $(\Sigma^{q+1}X, \Sigma^q X)$ is an *NDR*-pair. Hence define $\Sigma^\infty X = \{\Sigma^q X\}$, $\Sigma^\infty X$ is a space filtered by *NDR*'s according to 1.3, $\Sigma^q X$ is also in CG'.

Given a map $f : X \to X'$ in CG', we have the map $\Sigma^q f = f \wedge 1_{S^q} : \Sigma^q X \to \Sigma^q X'$. It is easily seen that Σ^q is a natural functor in CG', and further it is a homotopy and limit preserving functor for q finite or infinite.

§ 2. INSIDE OF THE SYMMETRIC PRODUCTS

Let $(X, *)$ be a space in CG'.

We first consider the k-fold symmetric product $SP^k\Sigma^q X$ of the q-iterated suspension $\Sigma^q X$. By definition, we have

$$SP^k\Sigma^q X/SP^{k-1}\Sigma^q X \cong (\Sigma^q X)^{[k]}/G_k \cong X^{[k]} \wedge_{G_k} (S^q)^{[k]}.$$

Thus, we have

$$SP^k\Sigma^q X/SP^{k-1}\Sigma^q X \cong X^{[k]} \wedge_{G_k} (\times_k R^q)^{\cdot}.$$

Here and in what follows, $X^{[k]}$ denote the k-fold smash product $X \wedge \dots \wedge X$ (k times), and $(\times_k R^q)^{\cdot}$ the one point compactification of $\times_k R^q$ on which G_k operates by fixing the point of compactification $*$, and by permuting the factors of $\times_k R^q$. (Remind that for $k = 0$, $\times_k R^q = \phi$ and $(\times_k R^q)^{\cdot} = \{ * \}$.) This homeomorphism leads us to the following review of the space $SP^k\Sigma^q X$.

Let $p_{k,i} : (\times_k R^q)^{\cdot} \to (\times_{k-1} R^q)^{\cdot}$; $1 \leqslant i \leqslant k$ denote the base point preserving maps given by

$$p_{k,i}(a_1, \dots, a_k) = (a_1, \dots, \check{a}_i, \dots, a_k). \tag{2.1}$$

Let $\approx$ denote the equivalence relation on $X^k \times (\times_k R^q)^{\cdot}$ generated by

$$((x_1, \dots, \overset{i}{\check{*}}, \dots, x_k), a) \approx ((x_1, \dots, \overset{i}{\check{*}}, \dots, x_k), a') \tag{2.2}$$

whenever $p_{k,i}(a) = p_{k,i}(a')$ with $1 \leqslant i \leqslant k$. Then we define the spaces

$$\widetilde{SP}(X, q, 0) = *, \quad \widetilde{SP}(X, q, k) = X^k \times (\times R^q)^{\cdot}/\approx, \quad k \geqslant 1 \tag{2.3}$$

with quotient topology. Via the embedding of $X^{k-1} \times (\times_{k-1} R^q)^{\cdot}$ in $X^k \times (\times_k R^q)^{\cdot}$ by the inclusion $X^{k-1} \cong X^{k-1} \times \{ * \} \subset X^k$ and the base point preserving map

$$i_{k-1} : (\times_{k-1} R^q)^{\cdot} \to (\times_k R^q)^{\cdot} \tag{2.4}$$

$$i_{k-1}(a_1, \dots, a_{k-1}) = (a_1, \dots, a_{k-1}, a_{k-1}),$$

we consider the space $\widetilde{SP}(X, q, k-1)$ as a subspace of $\widetilde{SP}(X, q, k)$. The single point in $\widetilde{SP}(X, q, 0)$ is to be taken as the base point of $\widetilde{SP}(X, q, k)$. Observing that $\widetilde{SP}(X, q, k-1)$ is a closed subset of $\widetilde{SP}(X, q, k)$, we define

$$\widetilde{SP}(X,q)=\bigcup_{k\geqslant 0}\widetilde{SP}(X,q,k)$$

with the topology of the union. $\widetilde{SP}(X,q)$ is sometimes written by $\widetilde{SP}(X,q,\infty)$ for convenience.

Via the diagonal action of G_k on $X^k\times(\times_k \mathbf{R}^q)^{\cdot}$: $g(x,a)=(gx,ga)$ for each $g\in G_k$, $x\in X^k$, $a\in(\times_k \mathbf{R}^q)^{\cdot}$, the space $\widetilde{SP}(X,q,k)$ becomes visibly an G_k-space, From this, $\widetilde{SP}(X,q)$ is an G-space. It is obviouly an G-space filtered by NDR's (see also the discussion below Definition 2.9). We define

$$SP(X,q)=\widetilde{SP}(X,q)/G=\{SP(X,q,k)\}$$

with $SP(X,q,k)=\widetilde{SP}(X,q,k)/G_k$. Then, according to 1.7, $SP(X,q,\infty)=SP(X,q)$ is a space filtered by NDR's. Now, by means of the maps $\Psi: X^k\times(\times_k \mathbf{R}^q)^{\cdot}\to(\Sigma^q X)^k$ given by the formula

$$\Psi((x_1,\ldots,x_k),(a_1,\ldots,a_k))=([x_1,a_1],\ldots,[x_k,a_k])$$

where $[x_i,a_i]$ denotes the equivalence class represented by $(x_i\ a_i)$ in $\Sigma^q X$, we obtain immediately from definitions the homeomorphisms

$$SP(X,q,k)\cong SP^k\,\Sigma^q X,\quad 0\leqslant k\leqslant\infty. \tag{2.5}$$

Let $\mathcal{R}(q)^{\cdot}=(\times_k \mathbf{R}^q)^{\cdot}$ denote the sequence of G_k-spaces

$$\{*\}=(\times_0 \mathbf{R}^q)^{\cdot}\subset(\times_1 \mathbf{R}^q)^{\cdot}\subset\ldots\subset(\times_k \mathbf{R}^q)^{\cdot}\subset\ldots$$

where the embeddings are the maps i_k given in 2.4. Visibly $\mathcal{R}(q)^{\cdot}$ is an G. space filtered by NDR's. Now, to generalize the construction of $SP(X,q)$, we introduce the following

2.6 Definition. Let $\mathcal{E}=\{E_k\}$ be an G-space filtered by NDR's with $E_0=*$, $E_k\subset(\times_k \mathbf{R}^q)^{\cdot}$ and $i_k|E_k:E_k\subset E_{k+1}$, $k\geqslant 0$. Then $\mathcal{E}$ is said to be an **G-NDR** in $\mathcal{R}(q)^{\cdot}$ if $((\times_k \mathbf{R}^q)^{\cdot}, E_k)$, $k\geqslant 0$ are G-equivariant NDR pairs. In addition, if $\mathcal{E}$ satisfies the condition

$$p_{k,j}(E^c_k)\doteq E^c_{k-1},\ 1\leqslant j\leqslant k,\ k\geqslant 1$$

where E^c_k is the complement of E_k in $(\times_k \mathbf{R}^q)^{\cdot}$, then we say that $\mathcal{E}$ is an *allowable* **G-NDR in $\mathcal{R}(q)^{\cdot}$**

Clearly $\mathcal{R}(q)^{\cdot}$ is itself an allowable G-NDR. Another trivial example of allowable G-NDR is the G-space $*(q)=\{E_k\}$ with $E_k=\{*\}$ for every $k\geqslant 0$.

Let $A^{\cdot}$ (resp. A^+) denote the one point compactification (resp. one point disjoint union) of A. Unless otherwise specified, these added points will be regarded as their base points. If A is an open set of $(\times_k \mathbf{R}^q)^\cdot$, then we have the homeomorphism

$$A^\cdot \cong (\times_k \mathbf{R}^q)^\cdot / A^c \text{ with } A^c = (\times_k \mathbf{R}^q)^\cdot - A . \tag{2.7}$$

by which we identify them from now on.

Suppose we are given an allowable $\mathbf{G}$-NDR $\mathscr{E} = \{E_k\}$ in $\mathscr{R}(q)^\cdot$. Let $\mathscr{L} = \{L_k\}$ denote one of the following sequences of $\mathbf{G}_k$-spaces.

$$\mathscr{E}^{c\cdot} = \{E_k^{c\cdot}\} \text{ or } \mathscr{E}^{c+} = \{E_k^{c+}\}.$$

By 2.6, let $\overline{p}_{k,i}: L_k \to L_{k-1}$ $(1 \leq i \leq k)$ be the base point preserving maps induced canonically from $p_{k,i}: (\times_K \mathbf{R}^q)^\cdot \to (\times_{k-1} \mathbf{R}^q)^\cdot$ given in 2.1. Then for each X in CG', we define the $\mathbf{G}_k$—spaces.

$$\widetilde{L}(X, \mathscr{L}, O) = *, \ \widetilde{L}(X, \mathscr{L}, k) = X^k \times L_k / \approx, \ k \geqslant 1 \tag{2.8}$$

with the quotient topology. Here $\approx$ is the equivalence relation on $X^k \times L_k$ given (similarly as in 2.2) by

$$((x_1, \ldots, \overset{i}{\underset{*}{\vee}}, \ldots, x_k), a) \approx (x_1, \ldots, \overset{i}{\underset{*}{\vee}}, \ldots, x_k), b)$$

with $x_j \in X$, $a, b \in L_k$ whenever $\overline{p}_{k,i}(a) = \overline{p}_{k,i}(b)$. For each (x, a) in $X^k \times L_k$, we let $[x, a]$ denote its equivalence class in $\widetilde{L}(X, \mathscr{L}, k)$. The action of $\mathbf{G}_k$ on $\widetilde{L}(X, \mathscr{L}, k)$ is given by the relation $g[x, a] = [gx, ga]$, $g \in \mathbf{G}_k$.

$\widetilde{L}(X, \mathscr{L}, k-1)$ will be regarded as a subspace of $\widetilde{L}(X, \mathscr{L}, k)$ by identifying each point $[(x_1, \ldots, x_{k-1}), a]$ in $\widetilde{L}(X, \mathscr{L}, k-1)$ with $[(x_1, \ldots, x_{k-1}, *), a']$ $(a = \overline{p}_{k,k}(a'))$ in $\widetilde{L}(X, \mathscr{L}, k)$. The single point in $\widetilde{L}(X, \mathscr{L}, 0)$ will be taken as the base point of $\widetilde{L}(X, \mathscr{L}, k)$. Obviously from the relation $\overline{p}_{k,i}(L_k) = L_{k-1}$, we have

$$\widetilde{L}(X, \mathscr{L}, k-1)^\frown = g \in \bigcup_{\mathbf{G}_k} g \widetilde{L}(X, q, k-1) = ((X^k)_{k-1} \times L_k))/\approx$$

and the projection

$$p: (X^k \times L_k, (X^k)_{k-1} \times L_k) \to (\widetilde{L}(X, \mathscr{L}, k), \widetilde{L}(X, \mathscr{L}, k-1)^\frown) \tag{2.9}$$

is a relative $\mathbf{G}_k$-equivariantly homeomorphism. On the other hand, by 1.5, $(X^k \times L_k, (X^k)_{k-1} \times L_k)$ is an $\mathbf{G}_k$-equivariant NDR-pair. Hence, according to Steenrod [34; 8.4], $(\widetilde{L}(X, \mathscr{L} k), \widetilde{L}(X, \mathscr{L}, k-1)^\frown)$ is an $\mathbf{G}_k$-equivariant NDR-pair. Now the following is evident.

2.10 Proposition.

(i) The space $\widetilde{L}(X, \mathscr{L}) = \widetilde{L}(X, \mathscr{L}, k)$ is an $\mathbf{G}$-space filtered by NDR's. Therefore $L(X, \mathscr{L}) = \{L(X, \mathscr{L}, k)\}$ defined by

$$L(X, \mathscr{L}) = \widetilde{L}(X, \mathscr{L})/\mathbf{G} = \{\widetilde{L}(X, \mathbf{G}, k)/\mathbf{G}_k\}$$

is a space filtered by NDR's (see 1.7).

(ii) $\widetilde{L}(\ ,\mathscr{L},k)$ and $L(\ ,\mathscr{L},k)$ are natural, and homotopy and limit preserving functors in CG'.

(iii) $\widetilde{L}(X,\mathscr{L},k)/\widetilde{L}(X,\mathscr{L},k-1)^{\wedge}\cong X^{[k]}\wedge L_k$
$L(X,\mathscr{L},k)/L(X,\mathscr{L},k-1)\cong X^k\underset{G_k}{\wedge}L_k.$

For convenience, $\widetilde{L}(X,\mathscr{L})$ and $L(X,\mathscr{L})$ are sometimes written by $\widetilde{L}(X,\mathscr{L},\infty)$ and $L(X,\mathscr{L},\infty)$ respectively. Later we also use the convention $L(X,\mathscr{L},-1)=\phi$.

2.11. Definition. Suppose that we are given an allowable $\mathbf{G}$-NDR $\mathscr{E}=\{E_k\}$ in $\mathscr{R}(q)^{\circ}$. Then we define

$$\widetilde{N}(X,\mathscr{E})=\{\widetilde{N}(X,\mathscr{E},k)\}$$

to be the $\mathbf{G}$-subspace of $\widetilde{SP}(X,q)=\{\widetilde{SP}(X,q,k)\}$ with $\widetilde{N}(X,\mathscr{E},k)$ the $\mathbf{G}_k$-subspace of $\widetilde{SP}(X,q,k)$ consisting of all elements represented by points in $X^k\times(\underset{k}{\times}\mathbf{R}^q)$ of the form $g(x,a)$ with $g\in\mathbf{G}_k$ and $x=(x_1,\dots,x_n,*,\dots,*)\in X^k$, $x_i\in X-\{*\}$ for $1\leqslant i\leqslant n$ if $n\geq 1$ $a=(a_1,\dots,a_n,a_n,\dots,a_n)$, $(a_1,\dots,a_n)\in E_k$, $1\leqslant n\leqslant k$.

We prove

2.12. Proposition.

(i) $\widetilde{N}(X,\mathscr{E})$ is an $\mathbf{G}$-space filtered by NDR's. Therefore $N(X,\mathscr{E})=\{N(X,\mathscr{E},k)\}$ defined by

$$N(X,\mathscr{E})=\widetilde{N}(X,\mathscr{E})/\mathbf{G}=\{\widetilde{N}(X,\mathscr{E},k)/\mathbf{G}_k\}$$

is a space filtered by NDR's.

(ii) $(\widetilde{SP}(X,q,k),\widetilde{N}(X,\mathscr{E},k))$ is an $\mathbf{G}_k$-equivariant NDR pair. Therefore $(SP(X,q,k),N(X,\mathscr{E},k))$ is an NDR pair.

(iii) $\widetilde{L}(X,\mathscr{E}^{c\cdot},k)\cong\widetilde{SP}(X,q,k)/\widetilde{N}(X,\mathscr{E},k)$
$L(X,\mathscr{E}^{c\cdot},k)\cong SP(X,q,k)/N(X,\mathscr{E},k).$

Proof. According to 1.7, to prove (i), we need only to prove its first part. By definition, $\mathscr{E}$ is an $\mathbf{G}$-space filtered by NDR's. Immediately we have $\widetilde{N}(X,\mathscr{E},k-1)\doteq N(X,\mathscr{E},k-1)^{\wedge}/\mathbf{G}_k$. So it remains to show that $(\widetilde{N}(X,\mathscr{E},k),\widetilde{N}(X,\mathscr{E},k-1)^{\wedge})$ is an $\mathbf{G}_k$-equivariant NDR pair.

$$\text{Let } \widetilde{N}'=\widetilde{N}(X,\mathscr{E},k-1)^{\wedge}\cup\{[x,a];x\in(X-*)^k,a\in\widehat{E_{k-1}}\}$$

and let (u_k,h_k) be a representation of $(E_k,\widehat{E_{k-1}})$ as an $\mathbf{G}_k$-equivariant NDR pair. Then define the maps $u:\widetilde{N}(X,\mathscr{E},k)\to I$ and $h:I\times\widetilde{N}(X,\mathscr{E},k)\to\widetilde{N}(X,\mathscr{E},k)$ by

$$u([x,a])=u_k(a),\ h(t,[x,a])=[x,h_k(t,a)].$$

Obviously, (u,h) is a representation of $(\widetilde{N}(X,\mathscr{E},k),\widetilde{N}')$ as an $\mathbf{G}_k$-equivariant NDR pair. Similarly, by use of a representation of $(X^k,\widehat{(X^k)_{k-1}})$, we obtain easily a representation of $(\widetilde{N}',\widetilde{N}(X,\mathscr{E},k-1)^{\wedge})$. Now, from the proof of [34 : 7,2] $(\widetilde{N}(X,\mathscr{E},k),\widetilde{N}(X,\mathscr{E},k-1)^{\wedge})$ is an $\mathbf{G}_k$-equivariant NDR pair.

To prove (ii), again we need only to prove the first part. Let $\widetilde{N}'' = \widetilde{N}(X, \mathscr{G}, k) \cup \{[x, a]; x \in X^k, a \in E_k\}$. By use of the above argument, we have the G_k-equivariant *NDR* pairs $(\widetilde{SP}(X, q, k), \widetilde{N}'')$ and $(\widetilde{N}'', \widetilde{N}(X, \mathscr{G}, k))$. Thus $(\widetilde{SP}(X, q, k), \widetilde{N}(X, \mathscr{G}, k))$ is an G_k-equivariant *NDR* pair.

The assertion (iii) is by definition. The proposition follows.

Apply the above constructions to the obvious allowable G-*NDR*s $\mathscr{R}(q)^{\cdot}$ and $*(q)$ in $R(q)^{\cdot}$ (given below 2.6): we observe that

$$SP(X, q) = L(X, *(q)^{c\cdot}) = N(X, \mathscr{R}(q)^{\cdot}).$$

Thus $SP(X, q)$ is obtained as a special case. It appears quite likely many interesting filtered spaces may be constructed by this way.

Now we define the filtered spaces $B(X, q)$, and $C(X, q)$ and $D(X, q)$. Note that $C(X, q)$ is (homotopically equivalent to) a space introduced by P. May by use of the little cubes operads [17]. Let $F(A, k)$ denote as usual the k-th configuration space of a, space A, that is

$$F(A, k) = \{(a_1, \ldots, a_k); a_i \in A, a_i \neq a_j \text{ for } i \neq j\}. \tag{2.13}$$

Let $\mathscr{T}(q) = \{T(\mathbf{R}^q, k)\}$ with $T(\mathbf{R}^q, k) = (\times_k \mathbf{R}^q)^{\cdot} - F(\mathbf{R}^q, q)$. Then we have the G_k-equivariant *NDR* pairs $((\times_k \mathbf{R}^q)^{\cdot}, T(\mathbf{R}^q, k))$, and $(T(\mathbf{R}^q, k), T(\mathbf{R}^q, k-1))$ for each k (see the *CW*-decomposition of $(\times_k \mathbf{R}^q)^{\cdot}$ given by Nakamura [26] which will be recalled later in Section (§ 6). From this fact, it is easily observed that $\mathscr{T}(q)$ is an G-*NDR* in $R(q)^{\cdot}$. We define

$$\begin{aligned} B(X, q, k) &= L(X, \mathscr{T}(q)^{c\cdot}, k) \\ C(X, q, k) &= L(X, \mathscr{T}(q)^{c+}, k) \\ D(X, q, k) &= N(N, \mathscr{T}(q), k) \end{aligned} \tag{2.14}$$

Thus, we have the filtered spaces $B(X, q), = \{B(X, q, k)\}$, $C(X, q) = \{C(X, q, k)\}$ and $D(X, q) = \{D(X, q, k)\}$. Correspondingly we shall use the notation $\widetilde{B}(X, q, k)$, $\widetilde{C}(X, q, k)$ and $\widetilde{D}(X, q, k)$, etc.

For each k, the map

$$\widetilde{C}(X, q, k) \to \widetilde{SP}(X, q, k)$$

defined canonically by means of the base point preserving map $F(\mathbf{R}^q, k)^+ \to (\times_k \mathbf{R}^q)^{\cdot}$ is not an injection of spaces, because not so is $F(\mathbf{R}^q, k)^+ \to (\times_k \mathbf{R}^q)^{\cdot}$.

Let $\mathbf{R}'^q = (-1,1) \times \cdots \times (-1,1)$ (q times). Let $f: \mathbf{R}^q \to \mathbf{R}'^q$ denote the map $(a^1, \ldots, a^q) \to (\tanh a^1, \ldots, \tanh a^q)$. Then we have a homeomorphism (denoted also by) $f: \times_k \mathbf{R}^q \to \times_k \mathbf{R}'^q$ with $f(a_1, \ldots, a_k) = (f(a_1), \ldots, f(a_k))$. Let

$$T'(\mathbf{R}^q, k) = f(T(\mathbf{R}^q, k) - \{\bullet\}) \cup ((\times_k \mathbf{R}^q)^{\cdot} - (\times_k \mathbf{R}'^q)).$$

Obviously, by a suitable G_k - equivariant CW - decomposition on $(\underset{k}{\times} \mathbf{R}^q)^{\cdot}$, $T'(\mathbf{R}^q, k)$ is an G_k-equivariant NDR in $(\times \mathbf{R}^q)^{\cdot}$. Further, we have $i_{k-1}(T'(\mathbf{R}^q, k-1)) \subset T'(\mathbf{R}^q, k)$, and $(T'(\mathbf{R}^q, k), T'(R^q, k-1)^{\frown})$ is an G_k-equivariant NDR pair. Consequently, we have an allowable G-NDR $\mathcal{C}'(q) = \{T'(\mathbf{R}^q, k)\}$ in $\mathcal{R}(q)$. Now, let

$$B'(X, q) = L(X, \mathcal{C}'(q)^c),\ C'(X, q) = L(X, \mathcal{C}'(q)^c)$$
$$D'(X, q) = L(X, \mathcal{C}'(q)).$$

We have evidently the following

2. 15. Proposition.

(i) $B(X, q) \cong B'(X, q)$, $C(X, q) \simeq C'(X, q)$, and $D(X, q) \simeq D'(X, q)$.

(ii) The canonical map $C'(X, q) \to SP(X, q)$ is an injection of spaces (by which we consider $C'(X, q)$ as a subspace of $SP(X. q)$), and we have

$$SP(X, q) = C'(X, q) \dot{\cup} D'(X, q)$$
$$C'(X, q) \cap D'(X, q) = \{*\}.$$

§ 3. DECOMPOSITION FORMULA

Let $(X, *)$ be in CG'. $S_*(X)$ denote the (integral) singular chain complex of X. Let $\overline{S}_*(X) = \operatorname{Ker} \varepsilon$ where $\varepsilon : S_*(X) \to S_*(\{*\})$ is the augmentation given by the surjection $X \to *$. In this section, we shall analyse the chain complexes $S_*(\boldsymbol{B}(A, q))$ and $S_*(C(X, q))$.

As it is well known, under the language of complete semi - simplicial complexes, Steenrod has stated in his lecture [33] the decomposition formula

$$S_*(SP^k X) \simeq \bigoplus_{n=0}^{k} \overline{S}_*(SP^n X / SP^{n-1} X) \tag{3.1}$$

for each CW - complex X. Th[illegible] we have the corresponding formula for $SP(X, q, k) \cong SP^k \Sigma X$ in this case. Also we have

3. 2. Theorem. If X is a CW—complex with base point, then

$$S_*(L(X, \mathcal{L}, k) \simeq \bigoplus_{n=0}^{k} \overline{S}_*(L(X, \mathcal{L}, n / \mathcal{L}(X, \mathcal{L}, n-1))$$

for $\mathcal{L} = \zeta^{c+}$ *or* $\zeta^{c'}$ *with* ζ *an allowable* G - NDR *in* $\mathcal{R}(q)^{\cdot}$.

Proof. (Sketch) The theorem can be proved by the same argument used in Dold [8 ; § 9] or Spanier [31 ; 6.7] in the proof of 3. 1 :

— Construct the c. s. s. complexes $L(\boldsymbol{K}, \mathcal{L}, k)$ for each $C.S.S.$ complex $\boldsymbol{K}$;

— Show that there is a natural weak homotopy equivalence $L(K, \mathcal{L}, k) \to L(K, \mathcal{L}, k)$ where K denotes the geometric realization of K as a CW - complex ;

— Prove the decomposition formula for $L(\boldsymbol{K}, \mathcal{L}, k)$.

If X is a CW-complex, then there is a weak homotopy equivalence $|S_*(X)| \to X$ according to Milnor [22], and $|S_*(X)| \to X$ also is a homotopy equivalence as it is well known (e. g. Spanier [26 ; 7.6.24]). So we have

$$L(X, \mathcal{L}, k) \simeq L(|S_*(X)|, \mathcal{L}, k) \simeq |L(S_* \mathcal{L}, (X), k)|. \tag{2.10}$$

The theorem follows.

We shall prove the following

3. 3. Theorem. Let $\zeta = \{E_k\}$ be an allowable $\mathbf{G}$-*NDR* in $\mathcal{R}(q)^{\cdot}$. Suppose that $((\underset{k}{\times}\mathbf{R}^q)^{\cdot}, E_k)$ is relatively $\mathbf{G}_k$-free for each k. Then we have

$$S_*(L(X, \mathcal{L}, k)) \simeq \bigoplus_{n=0}^{k} \overline{S}_*(L(X, \mathcal{L}, n)/L(X, \mathcal{L}, n-1))$$

for each $X \in CG'$ where $\mathcal{L} = \zeta^{c+}$ or $\mathcal{L} = \zeta^{c\cdot}$.

3.4. Corollary. If $X \in CG'$, then we have

$$S_*(B(X,q,k)) \simeq \bigoplus_{n=0}^{k} \overline{S}_*(B(X, q, n)/B, (X, q, n-1)) ,$$

$$S_*(C(X,q,k)) \simeq \bigoplus_{n=0}^{k} \overline{S}_*(C(X, q, n)/C(X,q, n-1)).$$

The proof of Theorem 3.3 is divided into a number of lemmata.

3.5 Lemma.

$$S_*(\widetilde{L}(X, \mathcal{L}, k)) \simeq S_*(\widetilde{L}(X, \mathcal{L}, k-1)^{\frown}) \oplus \overline{S}_*(\widetilde{L}(X, \mathcal{L}, k)/\widetilde{L}(X, \mathcal{L}, k-1)^{\frown})$$

Proof. We use the notation given in §2. According to 2.9, we have the following commutative diagram

$$\begin{array}{ccccc} (X^k)_{k-1} \times L_k & \to & X^k \times L_k & \to & X^k \times L_k/(X^k)_{k-1} \times L_k \\ \downarrow & & \downarrow & & \wr\| \\ \widetilde{L}(X, \mathcal{L}, k-1)^{\frown} & \to & \widetilde{L}(X, \mathcal{L}, k) & \to & \widetilde{L}(X, \mathcal{L}, k)/\widetilde{L}(X, \mathcal{L}, k-1)^{\frown}. \end{array}$$

By the Eilenberg—Zilber theorem, we have $S_*(X^k \times L_k) \simeq S_*(X)^k \otimes S_*(L_k)$ From tkis we obtain easily

$$S_*(X^k \times L_k) \simeq S_*((X^k)_{k-1} \times L_k)) \oplus (S_*(X)^k \otimes \overline{S}_*(L_k)). \tag{3.6}$$

In the other words, we have the splitting

$$H_*(X^k \times L_k;\ Z) \cong H_*((X^k)_{k-1} \times L_k;\ Z) \oplus H_*(X^k \times L_k;\ (X^k)_{k-1} \times L_k;\ Z).$$

Since the last vertical map of the above diagram is a homeomorphism, this splitting implies the splitting

$$H_*(\widetilde{L}(X, \mathcal{L}, k); Z) \cong H_*(\widetilde{L}(X, \mathcal{L}, k-1)^{\frown}; Z) \oplus H_*(\widetilde{L}(X, \mathcal{L}, k), \widetilde{L}(X, \mathcal{L}, k-1)^{\frown}; Z)$$

by means of the homology sequences for the two horizontal sequences of the diagram. Now, according to Proposition 2.10 (i), $(\widetilde{L}(X, \mathcal{L}, k), \widetilde{L}(X, \mathcal{L}, k-1)^{\frown}$ is an *NDR* pair, we have

$$H_*(\widetilde{L}(X, \mathcal{L}, k); Z) \cong H_*(\widetilde{L}(X, \mathcal{L}, k-1)^{\frown}; Z) + H_*(\widetilde{L}(X, \mathcal{L}, k)/\widetilde{L}(X, \mathcal{L}, k-1)^{\frown}; Z).$$

The isomorphism of the integral homology groups of two free complexes is equivalent with the homotopy equivalence of the two complexes. Consequently, the lemma is proved.

3.7 Lemma. If (X, A) is a relative G-free NDR pair, then we have the homotopy equivalence

$$S_*(X, A) \underset{G}{\otimes} Z = S_*(X/G, A/G).$$

Therefore we have $S_*(X/G) \underset{G}{\otimes} Z = S_*((X/A)/G)$.

Here as usual $S_*(X, A) = S_*(X)/S_*(A)$ for each pair of spaces (X, A).

Proof. First we let (X, A) be an NDR pair. Let U denote an open DR neighborhood of A in X. Then we have the open covering $\mathcal{U} = \{X - A, U\}$ of X. Let $S_*(\mathcal{U}) = S_*(X - A) + S_*(U)$, the subcomplex of $S_*(X)$ generated by $S_*(X - A)$ and $S_*(U)$. Then the inclusion $S_*(\mathcal{U}) \subset S_*(X)$ is a homotopy equivalence (refer to Spanier [31; 4.4.14]. Let $\tau: S_*(X) \to S_*(\mathcal{U})$ denote the homotopy equivalence given in the proof of [31; 4.4.14] from there we observe that $\tau \mid S_*(\mathcal{U}) = 1_{S_*(\mathcal{U})}$. Thus $\tau(S_*(A)) \subset S_*(U)$ and τ induces the chain map

$$S_*(X, A) \to S_*^U(X, A) = S_*(\mathcal{U})/S_*(U).$$

Now we have the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & S_*(A) & \to & S_*(X) & \to & S_*(X, A) & \to & 0 \\ & & \cap & & \downarrow \tau & & \downarrow \bar{\tau} & & \\ 0 & \to & S_*(U) & \to & S_*(\mathcal{U}) & \to & S_*^U(X, A) & \to & 0. \end{array}$$

The two horizontal sequences are exact, The two first vertical maps are chain equivalences. By means of the homology sequences for these exact sequences, it is easily seen that

$$\bar{\tau}: S_*(X, A) = S_*^U(X, A). \qquad (3:8)$$

Now let (X, A) be a relative G-free NDR pair, and U is a G-equivariant open DR neighborhood of A in X. Let $p: (X, A) \to (X/G, A/G)$ denote the projection. Then we have the chain map $p_*: S_*(X, A) \underset{G}{\otimes} Z \to S_*^{pU}(X/G, A/G)$ given by $p_*(\sigma \otimes 1) = p\sigma$ for $\sigma \in S_*(X-A)$. p_* is clearly well-defined and injective. On the other hand, $X-A$ is G-free by assumption, the surjectivity of p_* is a direct consequence of the well known relation $S_*(X-A) \underset{G}{\otimes} Z \to S_*((X-A)/G)$ (see MacLane [16; IV.11.3]. Hence p_* is an isomorphism. Thus from 3.8 follows the lemma.

Combining Lemmata 3.5, 3.7 and Proposition 2.10 (i), we complete the proof of Theorem 3.3. Here are some direct consequences of this theorem.

3.9 Corollary. Under the assumption of Theorem 3.3, we have

$$S_*(L(X, \mathcal{L}, k)) \simeq Z \oplus \left(\bigoplus_{n=1}^{k} \bar{S}_*(X)^n \underset{G_n}{\otimes} \bar{S}_*(L_n)\right).$$

Proof Use the relation 3.6.

3.10 Corollary. Under the assumption of Theorem 3.3, we have

$$S_*(L(X, \mathcal{L}, k)) \simeq Z \oplus \left(\bigoplus_{n=1}^{k} \bar{C}_*(X)^n \underset{G_n}{\otimes} \bar{C}_*(L_n)\right)$$

for any chain complex $C_*(X) \simeq S_*(X)$ and any $\mathbf{G}_n$-free chain complexes $C_*(L_n)$ which are $\mathbf{G}_n$-equivariantly homotopic to $S_*(L_n)$, $n \geqslant 1$.

Proof. According to the Steenrod's fundamental theorem on extended tensor products of chain complexes [32;5.2], we have $\overline{S_*}(X)^n \otimes \overline{S_*}(L_n) \simeq \overline{C_*}(X)^n \otimes \overline{C}_*(L_n)$. The assertion follows from 3.10.

Apply this result to the case where $\mathcal{L} = \mathcal{C}(q)^{c\cdot}$ or $\mathcal{C}(q)^{c+}$ i.e. to the case $L_n = F(\mathbf{R}^q,n)^{\cdot}$ or $F(\mathbf{R}^q,n)^+$, we shall construct in the later sections the chain complexes $V(M,q) \simeq S_*(B(X,q))$ and $W(M,q) \simeq S_*(C(X,q))$ for $M \simeq S_*(X)$ by which the further computations become effective.

§ 4. mDGA-ALGEBRAS

Suppose that we are given a filtered space $X = \{X_k\}$. Suppose further that it satisfies the decomposition theorem, i.e.

$$S_*(X_k) \simeq \bigoplus_{n=0}^{k} S_*(X_n, X_{n-1})$$

for each k $(0 \leqslant k \leqslant \infty)$ where $X_\infty = X$. For simplicity, all homology groups will have coefficients in a fixed commutative ring Λ with unit which will be usually deleted from the notation from now on. In practice, $\Lambda = \mathbf{Z}$, $\mathbf{Z}_p$ (the prime field with p elements). Saying that elements of $H_*(X_n, X_{n-1})$ are of rank n, we obtain then a module $H_*(X)$ bigraded by dimension and by rank. If $H_*(X_k)$ is determined for each $k \geqslant 0$, so is obviously $H_*(X)$. However, as usually seen, there exists a mathematical phenomenon that the infinite may be well understood before we know about the finite. Thus suppose that $H_*(X)$ has been computed, then there exists a suitable bigrading by dimension and by rank on $H_*(X)$ from which $H_*(X_k)$ are determined.

This general approach is due to Steenrod [33; 22]. He has called attention that this is an effective method to compute homology groups of the symmetric products SP^kX for a connected CW-complex X. In fact, by a theorem of Dold-Thom, $SP^\infty X$ is weakly homotopy equivalent to $\times_{n \geq 1} K(H_n(X;\mathbf{Z}), n)$. It follows that $H_*(SP^\infty X)$ can be computed by the method of the Cartan construction [6]. In [26], Nakamura has indicated how to give a bigrading on $H_*(K(\mathbf{Z}, n))$ and on $H_*(K(\mathbf{Z}_p, n))$ by which one can determine the homology groups of the symmetric products of the n-sphere and the Moore space respectively; hence one knows how to compute $H_*(SP^kX)$ for each connected CW-complex X. Our further studies are substantially based on the above described approach.

Assuming of familiarity with elementary differential homological algebra theory, we first fix some notation and terminology which will be used later: Let Λ be a commutative ring with unit fixed as the ring of coefficients. All D(differential) G(graded) modules are differential modules $M = (M, \partial_M)$ over Λ with positive gradation, i.e. $M = \bigoplus_{i \geq 0} M_i$, unless otherwise specified, and with differential ∂_M having degree -1. An element $x \in M_i$ is said to have dimension dim $x = i$. We also write $|x| = \dim x$. For each morphism of DC-modules $f: M \to N$, we write $f_i = f | M_i$. Λ itself is considered sometimes as a DG-module with trivial grading $\Lambda_0 = \Lambda$, $\Lambda_i = 0$ for $i > 0$, and differential $\partial_\Lambda = 0$.

A DGA-module M is a DG-module (M, ∂_M) together with a morphism of DG-modules $\varepsilon_M: N \to \Lambda$ called the augmentation of M. A DGA-module M is said to be n-connected if $(\varepsilon_M)_i : M_i \cong \Lambda_i$ for $i \leq n$. The 0-connected DGA-modules are called connected, and the I-connected DGA-modules are also called simply connected. For each DGA-module M, the kernel Ker ε_M of ε_M will be denoted by IM.

By a DG-algebra A, we mean a DG-module (A, ∂_A) together with a multiplication $\Phi_A : A \otimes A \to A$ (which is assumed to be associative) and a unit $\eta_A : \Lambda \to A$ for Φ_A. Dually, a DG-coalgebra C means a DG-module (C, ∂_C) together with a comultiplication $\Delta_C: C \to C \otimes C$ (which is assumed to be associative) and a counit $\eta_C : C \to \Lambda$ for Δ_C. A DG-algebra A (resp. a DG-coalgebra C) is called n-connected if $(\varepsilon_A)_i : A_i \cong \Lambda_i$ (resp. $(\eta_C)_i : \Lambda_i \cong C_i$) for $i \leq n$.

A DG-algebra A equipped with an augmentation, i.e. a morphism of DG-algebras $\varepsilon_A : A \to \Lambda$ is called a DGA-algebra. Here we consider Λ as a DG-algebra in a canonical way. If A is a DGA-algebra, we have $\varepsilon_A \eta_A = 1_\Lambda$. From this, as a DG-module, A may be identified with the direct sum $A = \Lambda \oplus IA$.

A DG-coalgebra C equipped with a coaugmentation, i.e. a morphism of DG-coalgebras $\eta_C : \Lambda \to C$ is called a DCA-coalgebra. Here we consider Λ as a DG-coalgebra in a canonical way. If C is a DGA-coalgebra, we have $\varepsilon_C \eta_C = 1_\Lambda$. From this, as a DG-module, C may be identified with the direct sum $C = \Lambda \oplus JC$. Here we let JC denote the cokernel Coker η_C of η_C.

A differential Hopf algebra is a DG-module A equipped with the structure morphisms $\Phi_A : A \otimes A \to A$, $\Delta_A : A \to A \otimes A$, $\varepsilon_A : A \to \Lambda$, $\eta_A : \Lambda \to A$ such that $(A, \partial_A, \Phi_A, \varepsilon_A, \eta_A)$ is a DGA-coalgebra and either Δ_A is a morphism of DGA-algebras or Φ_A is a morphism of DGA-coalgebras.

Forgetting the differentials, we have the notion of algebras, augmented algebras, coalgebras, coaugmented coalgebras, Hopf algebras. Now we recall the following definition.

4.1. Definition (Nakamura [26; 1.2]). Let A be a DGA-algebra. A is called a *DGA-algebra with multiplicity*, briefly an *$mDGA$-algebra*, if A satisfies the following properties.

(i) $A = \bigoplus_{n \geq 0} {}_nA$, a direct sum of DG-modules such that

$$ {}_nA = \bigoplus_{i \geq 0} {}_nA_i, \quad {}_nA_i = {}_nA \cap A_i, \quad n,\ i \geq 0. $$

(ii) $\Phi_A({}_lA \otimes {}_mA) \subset {}_{l+m}A,\ l, m \geq 0.$

(iii) $\varepsilon_A|_0A = {}_0A \cong \Lambda$ (therefore $\varepsilon_A({}_nA) = 0,\ n \geq 1$).

Forgetting the differential, we have the notion of *algebra with multiplicity*, briefly *m-algebra*. For each *mDGA*-algebra A, $H_*(A)$ is eventually an m-algebra, and $H_*(A) = \bigoplus_{n \geq 0} {}_nH_*(A) = \bigoplus_{n \geq 0} H_*({}_nA)$.

If $a \in A$, we define $\mu_A(a) = n$ and say that a is a homogeneous element of *multiplicity* n. We shall denote $A(k) = \bigoplus_{n=0}^{k} {}_nA,\ k \geq 0$. A *morphism of mDGA-algebras* $f: A \to A'$ is a morphism of *DGA*-algebras preserving the multiplicity, i.e. $f({}_nA) \subset {}_nA',\ n \geq 0$.

Given two *mDGA*-algebras A and A'. The *tensor product* of A and A' is the *DGA*-algebra $A \otimes A'$ with the multiplicity given by ${}_n(A \otimes A') = \bigoplus_{l+m=n} {}_lA \otimes {}_mA'$. Remind that A is a commutative *mDGA*-algebra, if and only if $\Phi_A : A \otimes A \to A$ is a morphism of *mDGA*-algebras.

An *mDGA*-algebra A is said to have *trivial multiplicity* if $IA = {}_1A$ or equivalently ${}_iA = 0$ for $i > 1$. Note that if A has trivial multiplicity, then A is clearly an algebra with trivial multiplication i.e. $ab = 0$ for every $a, b \in IA$.

Here are some examples of *mDGA*-algebras.

4.2. mDGA-algebras $\mathcal{S}^q$, $q \geqslant 0$. For each non-negative integer q, $\mathcal{S}^q$ is an *mDGA*-algebra with trivial multiplicity such that ${}_1\mathcal{S}^q = \Lambda\sigma^q$, the free Λ-module generated by a single element σ^q of dimension q. Necessarily we must have $\mathcal{S}^q = \Lambda \oplus \Lambda\sigma^q$ $\partial\sigma^q = 0$, $(\sigma^q)^2 = 0$.

4.3. Iterated suspension. Let q be a non-negative integer and M a *mDGA*-module. Then we define $\mathcal{S}^qM$ to be the *mDGA*-algebra such that as a *DG*-module, $\mathcal{S}^q M = \Lambda \oplus (IM \otimes I\mathcal{S}^q)$ and it is given a trivial multiplicity by ${}_1\mathcal{S}^qM = IM \otimes I\mathcal{S}^q$

If $q = 1$, $\mathcal{S}^1M$ will be written simply by as $\mathcal{S}M$. Immediately from the definition, we have $\mathcal{S}^q M \cong \mathcal{S}\mathcal{S}^{q-1} M$ for $q > 0$, so we call $\mathcal{S}^q M$ the *q-iterated suspension* of M.

Let $\sigma^q : IM \to \mathcal{S}^q M$ denote the map given by $\sigma^q x = (-1)^{q|x|} x \otimes \sigma^q$ for $x \in IM$. Then obviously we have

$$\partial(\sigma^q x) = (-1)^q \sigma^q \partial x \text{ and } (\sigma^q x)(\sigma^q y) = 0 \text{ for } x, y \in IM.$$

In particular, we suppose that M is a *DGA*-algebra A. As easily seen, we have $\mathcal{S}^qA \cong (A \otimes \mathcal{S}^q)/(\Lambda \otimes I\mathcal{S}^q \oplus IA \otimes \Lambda)$ as *DGA*-algebras. Remark that $\mathcal{S}^0A \cong A$ as *DGA*-modules but generally $\mathcal{S}^0A \neq A$ as *DGA*-algebras. Since $(\sigma^0 a)(\sigma^0 b) = 0$ for $a, b \in IA$. $\mathcal{S}^0A$ can be seen as a trivialization of the multiplication of A. Further, if A is an

$mDGA$-algebra, we define $\mathcal{S}^q A$ by neglecting the multiplicity on A. When $q=0$, $\mathcal{S}^0 A$ can be seen also as a *trivialization of the multiplicity* of A.

Now, we suppose that M is an n-connected DGA-module. For each negative integer q such that $n+q \geqslant -1$, we define the $mDGA$-algebra $\mathcal{S}^q M$ with trivial multiplicity-called the q-suspension ($(-q)$-desuspension) of M by the conditions

$$(I\mathcal{S}^q M)_i = (IM)_{i-q} \text{ and } \partial_{\mathcal{S}^q M} = (-1)^q \partial_M.$$

We also have the map $\sigma^q : IM \to \mathcal{S}^q M$ which is the identity in each dimension.

Note that if M is n-connected, we have by definition $\mathcal{S}^{q+q'} M \simeq \mathcal{S}^{q'} \mathcal{S}^q M$ if $n+q \geqslant -1$ and $n+q+q' \geqslant -1$.

4.4. Bar construction. Let A be a DGA-algebra. We recall that the (reduced normalized) bar construction of A is the DGA-coalgebra BA defined as follows.

(i) As a graded module, $BA = \bigoplus_{t \geq 0} (I\mathcal{S}A)^t$ where $(I\mathcal{S}A)^0 = \Lambda$.

If $t>0$, the element of BA corresponding to each element $\sigma a_1 \otimes \dots \otimes \sigma a_t \in (I\mathcal{S}A)^t$ will be denoted by $[a_1|\dots|a_t]$; if $t=0$, $[\ ]$ indicates the unit $1 \in (I\mathcal{S}A)^0$.

By definition, we have $\dim [a_1|\dots|a_t] = \sum_{i=1}^{t} (|a_i|+1)$

(ii) The boundary formula :

$$\partial_{BA} [a_1|\dots|a_t] = -\sum_{i=1}^{t} (-1)^{e_{i-1}} [a_1|\dots|\partial_A a_i|\dots|a_t] + \sum_{i=1}^{t} (-1)^{e_i} [a_1|\dots|a_i a_{i+1}|\dots|a_t]$$

where $e_i = \dim [a_1|\dots|a_i]$ for $1 \leq i \leq t$.

(iii) $\eta_{BA}(1) = [\ ]$,

$$\varepsilon_{BA}([a_1|\dots|a_t]) = \begin{cases} 1 & \text{if } t=0 \\ 0 & \text{if } t>0. \end{cases}$$

(iv) $\Delta_{BA} [a_1|\dots|a_t] = \sum_{i=0}^{t} [a_1|\dots|a_i] \otimes [a_{i+1}|\dots|a_t]$

(4.5) Suppose that A is in addition a commutative DGA-algebra, then the multiplication $\Phi_A : A \otimes A \to A$ is by definition a morphism of DGA-algebras. Thus, in this case, we have a natural morphism of DGA-coalgebras. $B(A\otimes A) \to BA$. Composing this with the natural morphism of coalgebras $BA \otimes BA \to B(A \otimes A)$, we have the commutative multiplication $\Phi_{BA}: BA \otimes BA \to BA$ (see e.g. Cartan [6 ; Exposé 4]).Thus BA is a Hopf algebra. Now, from [26 ; 1. 2], we have an explicit formula for Φ_{BA} given as follows:

$[a_1|\dots|a_t]\,[a_{t+1}|\dots|a_{t+u}] = [\sum_{\pi} (-1)^{e_\pi} [a_{\pi-1(1)}|\dots|a_{\pi-1(t+u)}]$. Here $\sum_{\pi}$ is the summation running over the set of all permutations $\pi \in G_{t+u}$ satisfying the condition.

$$\pi(1) < \dots < \pi(t),\ \pi(t+1) < \dots < \pi(t+u)$$

and e_π is the exponent given for each π by

$$e_\pi = \Sigma' (|a_i| + 1)(|a_j| + 1)$$

where Σ' runs over the set of all couples (i, j) such that $1 \leq i \leq t$, $t+1 \leq j \leq t+u$ and $\pi(i) > \pi(j)$.

Now let A be a commutative $mDGA$-algebra. Considered as an DGA-algebra; BA is defined to be an $mDGA$-algebra with the multiplicity extended from that given by the relation

$$\mu_{BA}([a_1|\dots|a_t]) = \sum_{i=1}^{t} \mu_A(a_i).$$

By iteration, $B^qA = BB^{q-1}A$, $q \geq 1$ are $mDGA$.algebras. Here, by convention, we let $B^0A = A$.

4.6. $A(tN, q)$, $t>0$, $q \geq 0$. Let C be an abelian semigroup and $Z(G)$ its semigroup ring. With the usual augmentation, we regard $Z(G)$ as a DGA-algebra where $Z(G)_i = 0$ for $i > 0$. Then we define

$$A(G, 0) = Z(G),\ A(G, q) = BA(G, q-1),\ q \geq 1.$$

In particular. we consider the case where G is the semigroup of natural number $N = \{a^n ;\ n \geq 0\}$. We have then $Z(N) = \bigoplus_{n \geq 0} Z(a-1)^n$. Now, for each positive integer t, we define $A(tN, 0)$, to be the $mDGA$-algebra such that $A(tN,0) = Z(N)$ and

$${}_nA(tN, 0) = \begin{cases} Z(a-1)^n & \text{if } n = tm,\ m \geqslant 0 \\ 0 & \text{otherwise.} \end{cases}$$

By iteration, we define the $mDGA$-algebras $A(tN, q) = BA(tN, q-1)$, $q \geqslant 1$. If $t = 1$, we write $A(N, q) = A(1N, q)$. This is the DGA-algebra with the multiplicity defined in [26; 1§2].

Remind that $A(Z, q)$ is not an $mDGA$-algebra. However, the morphism of DGA-algebras $A(N, q) \to A(Z, q)$ induced by the natural injection $N \to Z$ is a chain equivalence see [26; I. 1.11]). So $H_*(Z, q; \Lambda) = H_*(A(Z, q); \Lambda)$ is an m-algebra with respect to the multiplicity given on $A(N, q)$.

A topological version to the notion of $mDGA$-algebras appears in the following

4.7. Definition. A *filtered monoid* is a filtered space $X = \{X_k\}$ with $X_0 = *$ (see the definition below 1.3) together with a multiplication $\Phi: X \times X \to X$ such that X is also a topological monoid and Φ is a morphism of filtered spaces, that is $\Phi(X_l \times X_m) \subset X_{l+m}$ for every l, $m \geqslant 0$.

As a direct consequence of Definitions 4.1 and 4.6, we have the following

4.8. Proposition If $X = \{X_k\}$ is a filtered monoid satisfying the decomposition theorem, i.e.

$$S_*(X_k) \simeq \bigoplus_{n=0}^{k} S_*(X_n, X_{n-1}),\ k \geqslant 0,$$

then $S_*(X)$ is homotopic to an $mDGA$-algebra.

Let $X \in CG'$. Then the filtered space $\widetilde{SP}(X,q)$ equipped with the multiplication given by the relation

$$[(x_1,.., x_l),\ (a_1,.., a_l)]\ [(x_{l+1},.., x_{l+m}),\ (a_{l+1},.., a_{l+m})] = \tag{4.9}$$

$$= [(x_1,..., x_{l+m}),\ (a_1,...., a_{l+m})]$$

is a filtered monoid. Also $\widetilde{B}(X, q)$ is a filtered monoid with the multiplication induced from that of $\widetilde{SP}(X, q)$ via the projection $\widetilde{SP}(X, q) \to \widetilde{B}(X, q)$. Further, as readily seen, $SP(X, q)$ and $B(X,q)$ are abelian filtered monoids.

The relation 4.8 does not induce canonically a multiplication on $\widetilde{C}(X, q)$. However, $\widetilde{C}(X,q)$ (then $C(X,q)$ has a filtered monoid structure Φ ; $\widetilde{C}(X,q) \times \widetilde{C}(X,q) \to \widetilde{C}(X, q)$ with

$\Phi([x, a], [x', a']) = [x \times x', \Phi_{l,m}(a, a')]$ where $x \in X^l$, $x' \in X^m$ and $\Phi_{l,m}: F(\mathbf{R}^q, l) \times F(\mathbf{R}^q, m) \to F(\mathbf{R}^q, \chi + m)$, the map given by the formula:

$$\Phi_{l,m}((a_1,.., a_l),\ (a_{l+1},.., a_{l+m})) \tag{4.10}$$

$$= \left(a_1,.., a_l, \left(1 + \frac{d+1}{|a_{l+1}|+1}\right) a_{l+1},.. \left(1 + \frac{d+1}{|a_{l+m}|+1}\right) a_{l+m}\right)$$

Here $|a_i|$ = the distance between $a_i \in \mathbf{R}^q$ and the origin of $\mathbf{R}^q$, and $d = \max(|a_1|,..., |a_l|)$. Obviously $(\Phi_{l,m} \times 1)\ \Phi_{l+m,n} = \Phi_{l,m+n}\ (1 \times \Phi_{m,n})$. Hence Φ is associative, and thus $\widetilde{C}(X,q)$ (then $C(X, q)$) is a monoid.

5.1. Definition. Let C be a DGA-coalgebra. C is called a *DGA-coalgebra with multiplicity*, briefly an *mDGA-coalgebra*, if C satisfies the following properties.

(i) $C = \bigoplus_{n \geq 0} {}_nC$, a direct sum of DG — modules such that

$$_nC = \bigoplus_{i \geq 0} {}_nC_i\,,\ {}_nC_i = {}_nC \cap C_i\,,\ n, i \geq 0$$

(ii) $\Delta_C({}_nC) \subset \bigoplus_{l+m=n} {}_lC \otimes {}_mC,\ n \geqslant 0.$

(iii) $\eta_C : \Lambda \cong {}_oC$ (therefore $\varepsilon_C \mid {}_nC = 0,\ n \geqslant 1$).

Forgetting the differential, we have the notion of *coalgebra with multiplicity*, briefly *m-coalgebra*. For each $mDGA$-algebra $mDGA$-coalgebra C, $H_*(C)$ is eventually an m-coalgebra.

The multiplicity of a certain element in an $mDGA$-coalgebra, morphisms of $mDGA$-algebras, the tensor product of two $mDGA$-coalgebras are defined by a similar way as in the case of $mDGA$-algebras.

An $mDGA$-coalgebra C is said to have *trivial multiplicity* if $JC = {}_1C$, or equivalently ${}_iC = 0$ for $i > 1$. Note that if C has trivial comultiplicity, then C is clearly an coalgebra such that every element $c \in JC$ is primitive, i.e. $\Delta_C(c) = 1 \otimes c + c \otimes 1$.

A Hopf algebra A is called a *Hopf mDGA-algebra* either if A is an $mDGA$-algebra and $\Delta_A : A \to A \otimes A$ is a morphism of $mDGA$-algebras, or if A is an $mDGA$-coalgebra and $\Phi_A : A \otimes A \to A$ is a morphism of $mDGA$-coalgebras.

Here are some examples of $mDGA$-coalgebras.

5.2. mDGA-coalgebras $\mathcal{S}^q$, $q \geqslant 0$. We use the same notation as in 4.2. For each non-negative integer q, $\mathcal{S}^q$ is an $mDGA$-coalgebra with trivial multiplicity such that as DG-modules it is $\mathcal{S}^q$ of 4.2. It is easily seen that $\mathcal{S}^q$ together with its algebra structure is a Hopf $mDGA$-algebra if and only if q is odd or $2=0$ in Λ.

5.3. q-iterated suspension. Let M be an n-connected DGA-module. For each integer q such that $n + q \geqslant -1$, we introduce an $mDGA$-coalgebra structure on the DGA-module $\mathcal{S}^qM$ defined in 4.3 by giving a trivial multiplicity. Again, we denote this $mDGA$-coalgebra by $\mathcal{S}^qM$ and call it the *q-iterated suspension* of M. For suitable n, q, q', we have $\mathcal{S}^{q+q'}M \cong \mathcal{S}^q\mathcal{S}^{q'}M$.

In particular, we suppose that M is a DGA-coalgebra. Then, we have $\mathcal{S}^qC \cong (C \otimes \mathcal{S}^q)/(\Lambda \otimes J\mathcal{S}^q \oplus JC \otimes \Lambda)$ as DGA-coalgebras. for $q \geq 0$. Here the right side has the structure induced (uniquely) from that of $C \otimes \mathcal{S}^q$ such that the projection from $C \otimes \mathcal{S}^q$, is a morphism of DGA-coalgebras.

Remind that $\mathcal{S}^{\circ}C \simeq C$ as DGA-modules but generally $\mathcal{S}^{\circ}C \otimes C$ as DGA coalgebras. Since $\Delta_{\mathcal{S}^{\circ}C}(\sigma^{\circ}c) = 1 \otimes \sigma^{\circ}c + \sigma^{\circ}c \otimes 1$ for $c \in JC$, $\mathcal{S}^{\circ}C$ can be seen as a trivialization of the comultiplication of C. Further, if C is an $mDGA$-coalgebra, we define $\mathcal{S}^{q}C$ by neglecting the comultiplicity on C. When $q = 0$, $\mathcal{S}^{\circ}C$ can be seen also as a trivialization of the multiplicity of C.

5.4. Cobar construction. Let C be a connected DGA—coalgebra. We recall that the (reduced normalized) cobar construction of C if the DGA – algebra FC defined as follows

(i) As a graded module, $FC = \bigoplus_{t \geqslant 0} (J\mathcal{S}^{-1}C)^{t}$.

If $t > 0$, the element of FC corresponding to each element $\sigma^{-1}c_1 \otimes \dots \otimes \sigma^{-1}c_t \in (J\mathcal{S}^{-1}C)^{t}$ will be denote by $\langle c_1 \mid \dots \mid c_t \rangle$; if $t = 0$, $\langle\ \rangle$ indicates the unit $1 \in (J.\mathcal{S}^{-1}C)^{0} = \Lambda$. By definition we have $\dim \langle c_1 \mid \dots \mid c_t \rangle = \sum_{i=1}^{t} (\mid c_i \mid - 1)$.

(ii) The boundary formula:

$$\partial_{FC} < c_1 \mid \dots \mid c_t > = - \sum_{i=1}^{t} (-1)^{e_{i-1}} < c_1 \mid .. \mid \partial_C c_i \mid .. \mid c_t >$$

$$+ \sum_{i=1}^{t} (-1)^{e_{i-1}} < c_1 \mid .. \mid \Delta_C c_i \mid .. \mid c_t >$$

where $e_i = \sum_{j=1}^{t} (\mid c_j \mid - 1) = \dim < c_1 \mid \dots \mid c_i >$ mod 2 for $1 \leqslant i \leqslant t$.

Here if $\Delta_C c = \sum_c c' \otimes c''$.(the Swedler's notation), we write

$$\Delta_C c = \sum_c (-1)^{|c'|} c' \otimes c''.$$

(iii) $\eta_{FC}(1) = < >$, $\varepsilon_{FC} < c_1 \mid \dots \mid c_t > = \begin{cases} 1 & if\ t = 0 \\ 0 & if\ t > 0 \end{cases}$

(iv) $\Phi_{FC}(< c_1 \mid .. \mid c_t > \otimes < c_{t+1} \mid .. \mid c_{t+u} >) = < c_1 \mid \dots \mid c_{t+u} >$

(5.5) In addition, we suppose that C is a commutative DGA-coalgebra. Then, by definition, $\Delta_C : C \to C \otimes C$ is a morphism of DGA-coalgebras. Thus, in this case, we have the natural morphism of DGA-algebras $FC \to F(C \otimes C)$. Composing this with the natural morphism of DGA-algebras $F(C \otimes C) \to FC \otimes FC$, we have the commutative comultiplication $\Delta_{FC} : FC \to FC \otimes FC$. Thus FC is a Hopf algebra. An explicit formula for Δ_{FC} is obtained as follows (refer to [26; II § 3]).

$$\Delta_{FC} < c_1 \mid \dots \mid c_t > = \sum_{\pi} (-1)^{e_\pi} < c_{\pi(1)} \mid .. \mid c_{\pi(u)} > \otimes < c_{\pi(u+1)} \mid .. \mid c_{\pi(t)} >$$

Here $\sum_{\pi}$ is the summation running over the set of all permutations $\pi \in \sigma_t$ satisfying the condition

$$\pi(1) < \dots < \pi(u), \pi(u+1) < \dots < \pi(t), \ 0 \leqslant u \leqslant t$$

and e_π is the exponent given for each by

$$e_\pi = \Sigma' (|c_i| + 1)(|c_j| + 1)$$

where Σ' runs over the set of all couples (i, j) such that $1 \leqslant i \leqslant u$, $u + 1 \leqslant j \leqslant t$ and $\pi(i) > \pi(j)$.

Now let C be a commutative connected $mDGA$-coalgebra. Considered as a BGA-coalgebra, FC is defined to be an $mDGA$-coalgebra with the multiplicity extended from that of C by setting

$$\mu(< c_1 | \dots | c_t >) = \sum_{i=1}^{t} \mu(c_i).$$

Further, assume that C is n-connected, FC is then a commutative $(n\text{-}1)$-connected $mDGA$-coalgebra. Thus, for $0 \leqslant q \leqslant n + 1$, by iteration, we define

$$F^0C = C, \ F^qC = FF^{q-1}C.$$

F^qC are commutative $(n-q)$-connected $mDGA$-coalgebras.

Given a DGA-module M, $\mathcal{S}^n M$ is then by definition an $(n-1)$-connected commutative mDA-coalgebra. Thus, for $0 \leqslant q \leqslant n$, the q-iterated cobarconstruction $F^q\mathcal{S}^n M$ is an $(n-q)$-connected commutative $mDGA$-coalgebra. If $M = \mathcal{S}^0$, $F^q\mathcal{S}^n$, $0 \leqslant q \leqslant n$ are $mDGA$-algebras of particular interest.

Examples of Hopf $mDGA$-algebras can be found in the following proposition which is ready seen in 4.5 and 5.5.

5.6. Proposition. Let $q \geqslant 1$. Let A be a commutative $mDGA$-algebra and C an $(n-1)$-connected commutative $mDGA$-coalgebra. Then B^qA is a q-connected Hopf $mDGA$-algebras with commutative multiplication, and F^qC with $q \leqslant n$ is an $(n-q)$-connected commutative Hopf $mDGA$-algebra with commutative comultiplication.

Let A be a DGA-algebra and C a connected DGA-coalgebra. Let $p_A : BA \to \mathcal{S}A$ and $r_C : FC \to \mathcal{S}^{-1} C$ be the maps given by

$$p_A([a_1 | \dots | a_t]) = \begin{cases} \sigma a_1 & \text{if } t = 1 \\ 0 & \text{if } t > 1, \end{cases}$$

$$r_C(< a_1 | \dots | a_t >) = \begin{cases} \sigma^{-1} c_1 & \text{if } t = 1 \\ 0 & \text{if } t > 1. \end{cases}$$

Then obviously p_A is a morphism of DGA-coalgebras and r_C is a morphism of DGA-algebras. Thus, we have the morphism of DGA-algebras

$$\alpha_A = r_{\mathcal{S}_A} F(p_A) : FBA \to F\mathcal{S}A \to \mathcal{S}^\circ A.$$

On the other hand, let

$$\beta_C : \mathcal{S}^\circ C \to BFC$$

be the map given by $\beta_C(\sigma^\circ c) = [<c>]$ for $c \in JC$. Then obviously, β_C is a morphism of DGA-coalgebras.

Now, according to Drachman [10], Husemoller-Moore-Stasheff [14; II.4], and Munkholm [23; 2.15], we have the following

5.7. Theorem. *$\alpha_A : FBA \to \mathcal{S}^\circ A$ and $\beta_C : \mathcal{S}^\circ C \to BFC$ are chain equivalences.*

§6. NAKAMURA' S CW — DECOMPOSITION OF $(\underset{n}{\times}\mathbf{R}^q)$

In this section, we recall a decomposition of $\underset{n}{\times}\mathbf{R}^q$ given by Nakamura in [26; II.1] from which we obtain an $\mathbf{G}_n$-equivariant CW-decomposition of $(\underset{n}{\times}\mathbf{R}^q)^\cdot$. Since this will be a main tool of our further study, we shall attempt to present the material in details for the convenience of the readers.

Consider $\underset{n}{\times}\mathbf{R}^q = (\underset{n}{\times}\mathbf{R}^q)^\cdot - \{*\}$. Let $\mathbf{R}^q$ be equipped with the lexicographic order by agreeing that $a > b$ with $a = (a^1,..., a^q)$, $b = (b^1,..., b^q) \in \mathbf{R}^q$ if and only if there exists a non-negative integer ρ such that $a^\sigma = b^\sigma$ if $\sigma \leqslant \rho$ and $a^{\rho+1} > b^{\rho+1}$. Then an arbitrary point of $\underset{n}{\times}\mathbf{R}^q$ can be written in the form $g(a_1,..., a_n) = (a_{g^{-1}(1)},..., a_{g^{-1}(n)})$ where $g \in \mathbf{G}_n$, $a_\nu \in \mathbf{R}^q$ for $1 \leqslant \nu \leqslant n$ satisfying the relation

$$a_1 \geqslant ... \geqslant a_n.$$

Obviously, each point $(a_1,..., a_n)$ satisfying this relation determines a sequence of integer $r_1,..., r_n$ with $r_1 = q$ and $0 \leqslant r_\nu \leqslant q$ by the condition

$$a^\sigma_{\nu-1} = a^\sigma_\nu \qquad 1 \leqslant \sigma \leqslant \rho_\nu \tag{6.1}$$

$$a^{\rho_\nu+1}_{\nu-1} > a^{\rho_\nu+1}_\nu \quad \text{if } \rho_\nu < q$$

where $\rho_\nu = a - r_\nu$. Conversely, given a sequence of integers $r_1,..., r_n$ with $r_1 = q$ and $0 \leqslant r_\nu \leqslant q$, we can determine a subset of $\underset{n}{\times}\mathbf{R}^q$ consisting of all points $(a_1,..., a_n)$

satisfying (6.1). Let $\alpha = (r_1, \ldots, r_n)$ denote this subset and let $g\alpha$ the subset $\{gP; P \in \alpha\}$ of $\underset{n}{\times} \mathbf{R}^q$ for each $g \in G_n$. Then the collection $\mathbf{R}(q, n) = \{g\alpha\,;\, g \in G_n,\ \alpha = (r_1, \ldots, r_n),\ r_1 = q,\ 0 \leqslant r_\nu \leqslant q\}$ evidently covers $\underset{n}{\times} \mathbf{R}^q$.

Set $[\ ] = \{*\}$ and

$$R(q, n)' = \{[\ ]\} \cup R(q, n).$$

$R(q, n)'$ is an G_n-equivariant finite covering of $(\underset{n}{\times} \mathbf{R}^q)^{\cdot}$ by pairwise disjoint subsets. Every $g\alpha$ in $R(q, n)^{\cdot}$ is homeomorphic to a disk of dimension

$$|g\alpha| = \sum_{\nu=1}^{n} r_\nu$$

if $\alpha = (r_1, \ldots, r_n)$, gG_n or 0 if $\alpha = [\ ]$. Clearly, $|g\alpha| \geqslant q$ if $\alpha \neq [\ ]$. Call $g\alpha$ a cell of dimension i or an i-cell if $|g\alpha| = i$. By the condition (6.1), the boundary of every i-cell in $R(q, n)^{\cdot}$ is contained in an union of cells with lower dimensions; it is homeomorphic to an (i-1)-sphere if $i > q$, and it is just the point $*$ for any q-cell ($q \neq 0$). From the above remarks, we observe that

(6.2) $R(q, n)^{\cdot}$ is a cell decomposition of $(\underset{n}{\times} \mathbf{R}^q)^{\cdot}$ as an G_n-equivariant **CW**-complex.

Suppose that we are given a cell $\alpha = (r_1, \ldots, r_n)$ in $R(q, n)$. In order to express the faces of α, we first introduce some notations.

Let r be an integer such that $0 \leqslant r \leqslant q$. Then we define the monotone increasing sequence

$$\nu(r, 1), \ldots, \nu(r, t(r)) \tag{6.3}$$

consisting of all indices ν's such that $r_\nu \geqslant r$. This sequence defines the *partition*

$$n_\alpha(r) = (n(r, 1), \ldots, n(r, t(r))) \tag{6.4}$$

of n by putting $n(r, i) = \nu(r, i) = \nu(r, i+1) - \nu(r, i)$ where $\nu(r, t(r) + 1) = n + 1$. Here a partition of a non-negative integer n means an ordered sequence $(n_1, \ldots, n_t)$ such that $n_1 + \ldots + n_t = n$.

Let $\phi = (n_1, \ldots, n_t)$ be a partition of n and let π be permutation of degree t. Then we define their associated element $g(\phi, \pi)$ in G_n by the formula

$$g(\phi, \pi)\left(\sum_{j=1}^{i} n_j + \lambda\right) = \sum_{j=1}^{\pi(i)-1} n_{\pi^{-1}(j)} + \lambda,\quad 1 \leqslant \lambda \leqslant n_i \tag{6.5}$$

with $1 \leqslant i \leqslant t$. In particular, we set

$$g_\alpha(r, \pi) = g(n_\alpha(r), \pi).$$

Note that for each ϕ and π given above, the signature $e(\phi, \pi)$ of $g(\phi, \pi)$ is the integer

$$e(\phi, \pi) = \Sigma' n_i n_j \tag{6.6}$$

where Σ' is the summation running over the set of all couples (i, j) with $i < j$ satisfying $\pi(i) > \pi(j)$. The sign of $g(\phi, \pi)$ is sign $g(\phi, \pi) = (-1)^{e(\phi, \pi)}$.

Now we consider the boundary of $\alpha = (r_1, \ldots, r_n)$. If α is of the least dimension q (i.e. the cell of the form $(q, 0, \ldots, 0)$ then [ ] is the only cell in its boundary. Hence it has no faces unless $q = 1$.

Suppose $|\alpha| > q$. A cell $g\beta$ is a face of α if and only if it is a connected component of the set of $\underset{n}{\times} R^q$ consisting of all points $(a_1, \ldots, a_n)$ satisfying all relations in (6.1) except that

$$a^\sigma_{\mu-1} = a^\sigma_\mu \quad 1 \leqslant \sigma \leqslant \rho_\mu + 1$$

$$a^{\rho_\mu+2}_{\mu-1} \neq a^{\rho_\mu+2}_\mu$$

for a certain index μ with $\mu \neq 1$ and $r_\mu = q - \rho_\mu > 0$. From this remark, the faces of α can be expressed as follows.

(6.7) Let μ be an index such that $r_\mu > 0$ and $\mu \neq 1$. Put $r = r_\mu$, and let i, u, v, w denote the integers defined by the relations

$$\begin{aligned} \mu &= \nu(r, i) \\ \nu(r, i-1) &= \nu(r-1, u) \\ \nu(r, 1) &= \nu(r-1, v) \\ \nu(r, i+1) &= \nu(r-1, w) \end{aligned}$$

Then a face $g\beta$ of α is a cell of the following form:
(i) $g^{-1} = g_\alpha(r-1, \pi)$ where $\pi \in G_t$, $t = t(r-1)$ such that

$$\pi(u) < \ldots < \pi(v-1), \ \pi(v) < \ldots < \pi(w-1)$$

and that fixes every k with $1 \leqslant k < u$, $w \leqslant k \leqslant t$;
(ii) $\beta = (r_{g(1)}, \ldots, r_{g(v)} - 1, \ldots, r_{g(n)})$ where v is the index such that $g(v) = \mu$.

Let us define an orientation of a cell $g\alpha$ with $\alpha = (r_1, \ldots, r_n)$ in $R(q, n)$ for every $g \in G_n$ by the coordinate system

$$a^1_1 = a^{\rho_1+1}_1, \ldots, a^q_1, \ a^{\rho_2+1}_2, \ldots, a^{\rho_n+1}_n, \ldots, a^q_n. \tag{☆}$$

For a face $\beta = (r_1, ..., r_\mu - 1, ..., r_n)$ of α, we obtain its coordinate system by delecting the $\left(1 + \sum_{\lambda=1}^{\mu-1} r_\lambda\right)$-th coordinate function $a_\mu^{\rho_\mu + 1}$ from the coordinate system of α. So, as in the theory of simplicial complexes, we define the incidence relation $[\alpha : \beta]$ by the formula

$$[\alpha : \beta] = (-1)^{1 + \sum_{\lambda=1}^{\mu-1} r_\lambda}$$

Let $g\beta$ be a face of α as mentioned in 6.7. By definition, the coordinate system (☆) is a reorientation $\delta\alpha'$ of the cell $\alpha' = (r_{g(1)}, ..., r_{g(n)})$, and we have

$$[\alpha' : g\beta] = (-1)^{1 + \sum_{\lambda=1}^{\mu-1} r_\lambda}.$$

Further, we have

$$[\alpha : g\beta] = \delta[\alpha' : g\beta] = \delta . (-1)^{1 + \sum_{\lambda=1}^{\mu-1} r_\lambda}$$

To consider

$$\delta = \operatorname{sgn} \begin{pmatrix} a_1^{\rho_1+1} & \ldots & a_n^q \\ a_{g(1)}^{\rho_{g(1)}} & \ldots & a_{g(n)}^q \end{pmatrix},$$

we set

$$s_{r,j} = r + \sum_{\nu(r,\, j) < \lambda < \nu(r,\, j+1)} r_\lambda .$$

Then, from 6.6, we have

$$\delta = (-1)^{\Sigma' s_{r-1,j}\, s_{r-1,k}}$$

where Σ' is the summation running over the set of all couples (j, k) such that $j < k$ and $\pi(j) > \pi(k)$.

Summarizing, we have

$$[\alpha : g\beta] = (-1)^{1 + \sum_{\lambda=1}^{\mu-1} r_\lambda + \Sigma' s_{r-1,j}\, s_{r-1,k}} \qquad (6.8)$$

6.2, 6.7 and 6.8 determine completely the $\mathbf{G}_n$-chain complex $C_*(R(q, n)^{\cdot})$. Concerning the structure of this complex we have the following fundamental result.

6.9. Theorem (Nakamura). *There exists an isomorphism of DG-modules*

$$\overline{C}_*(R(q, n)^{\cdot}) \underset{\mathbf{G}_n}{\otimes} Z \cong {}_nA(\mathbf{N}, q)$$

for $n, q \geqslant 0$.

Here $\overline{C}_*(R(q, n)^{\cdot})$ is the kernel of the augmentation $\varepsilon : C_*(R(q; n)^{\cdot}) \to Z$ given by $\varepsilon([\,]) = 1$, and ${}_nA(\mathbf{N}, q)$ is defined in 4.5. As we have noted in the beginning of Section §2, $SP^n\mathbf{S}^q/SP^{n-1}\mathbf{S}^q \cong (\underset{n}{\times} \mathbf{R}^q)^{\cdot}/\mathbf{G}_n$. Hence, according to the decomposition theorem, an important consequence of Therem 6.9 can be found in the following from which we obtain the assertion of Dold—Thom on the infinite symmetric products of the spheres.

6.10. Theorem. The singular chain complex $S_*(SP(\mathbf{S}^\circ, q))$ of the filtered monoid $SP(\mathbf{S}^\circ, q)$ (see 4.7) is homotopic to the *mDGA*-algebra $A(\mathbf{N}, q)$ for $q \geqslant 0$.

For the proof of Theorem 6.9, refer to [26 : II.1], or to the proof of Theorem 7.6 in the next section.

§7. THE MDGA—ALGEBRAS V (q)

We have observed in 4.9 that $S_*(B(X,q))$ is homotopic to the *mDGA*-algebra

$S'_*(B(X,q)) = \underset{n\geqslant 0}{\oplus} S_*(B(X,q,n), B(X, q, n-1))$ for each X in *CG'*. The purpose

of this section and the following is to construct an *mDGA*-algebra $V(M, q)$

for each *DGA*-module M such that $V(M,q) \simeq S'_* B(X,q)$ by a morphism of *mDGA*-

algebras if $M \simeq S_*(X)$. Here, we deal first with the construction of the *mDGA*-algebra $V(q) = V(S^\circ,q)$ following the argument used by Nakamura in the proof of Theorem 6.9.

Let $T(q,n)$ denote the $\mathbf{G}_n$-invariant subcomplex of $R(q,n)^{\cdot}$ consisting of all cells $g\alpha$ with $g \in \mathbf{G}_n$, $\alpha = [\,]$ or $(r_1, \dots, r_n)$ such that there exists an index $\nu \neq 1$ with $r_\nu = 0$;

and let $F(q,n)^{\cdot} = R(q,n)^{\cdot}/T(q,n)$ Remind that we have then $F(q,n)^{\cdot} = \{[\ ]\} \cup F(q,n)$ as sets where $F(q,n) = R(q,n)^{\cdot} - T(q,n)$. Remind further that if $q = 0$, we have by definition

$$F(0,n)^{\cdot} = \begin{cases} \{[\ ], (0)\} & \text{if } n=1 \\ [\ ] & \text{if } n\geqslant 1. \end{cases}$$

Obviously $T(q,n)$ is an G_n-equivariant CW-decomposition of the space $T(\mathbf{R}^q,n)$, an $F(q,n)^{\cdot}$ is an G_n-equivariant CW-decomposition of $F(\mathbf{R}^q, n)$ (see 2.13).

Now we define the $mDGA$-algebra $\widetilde{V}(q)$ which will be by definition homotopic to $S_*(\widetilde{B}(S^o. q))$. We set

$$_n\widetilde{V}(q) = \begin{cases} Z & n=0 \\ \overline{C}_*(F(q,n)^{\cdot}) & n \geqslant 1. \end{cases}$$

Here $\overline{C}_*(F(q,n)^{\cdot}) \cong C_*(R(q,n)^{\cdot})/C_*(T(q,n))$. First we define $\widetilde{V}(q)$ as DGA-module to be the direct sum of DG-modules

$$\widetilde{V}(q) = \bigoplus_{n \geqslant 0} {}_n\widetilde{V}(q) \quad ,$$

and to be augmented by the projection $\varepsilon : \widetilde{V}(q) \to {}_n\widetilde{V}(q) = Z$.
For $\alpha \in {}_n\widetilde{V}(q)$, we let $\mu(\alpha) = n$.

If $q = 0$, we have

$$_n\widetilde{V}(0) = \begin{cases} Z\sigma^0 & n = 1 \\ 0 & n \geq 1 \end{cases} \tag{7.1}$$

Here and in what follows, σ^0 denotes the cell $(0) \in F(0,1)^{\cdot}$

Suppose $q \geq 1$. Let α be a cell of the form $(r_1,\dots,r_n)$ in $F(q,n)^{\cdot}$ Let $v(i) = v(q,i)$ and $n_i = v(i+1) - v(i)$ for $1 \leq i \lesssim t = t(q)$ (see 6.3 and 6.4). Then we obtain the cells

$$\alpha_i = (q-1, r_{v(i)+1},\dots, r_{v(i+1)-1})$$

in $F(q-1,n_i)$, $1 \leqslant i \leq t$. Conversely, a such sequence $\alpha_1,\dots, \alpha_t$ determines a cell α. Note that we have the corresponding formula on dimensions

$$|\alpha| = \sum_{i=1}^{t} (|\alpha_i| + 1).$$

Thus we write formally

$$\alpha = |\alpha_1| \dots |\alpha_t| .$$

As easily seen, this correspondence yields an isomorphism

$$_n\widetilde{V}(q) \cong \bigoplus_{n_1+\dots+n_t=n} Z(G_n) \underset{G_{n_1}\times\dots\times G_{n_t}}{\otimes} ({}_{n_1}\widetilde{V}(q-1) \otimes \dots \otimes {}_{n_t}\widetilde{V}(q-1)) \tag{7.2}$$

of G_n-modules. In particular, from 7.1, we have

$$_n\widetilde{V}(1) = Z(G_n) \otimes Z\,\Theta_n \tag{7.3}$$

where we denote $\Theta_n = [\sigma^0 | \dots | \sigma^0] = (1,\dots,1) \in F(1,n)^{\cdot}$

Next, we define the morphism of graded modules $\widetilde{V}(q) \otimes \widetilde{V}(q) \to \widetilde{V}(q)$ by the morphisms of graded $G_l \times G_m$-modules

$$ {}_lV(q) \otimes {}_m\widetilde{V}(q) \to {}_{l+m}\widetilde{V}(q) \qquad l.m \geq 0 $$

given as follows. Later, by the boundary formula, it will be automatically verified that this morphism is a multiplication of $\widetilde{V}(q)$.

If $q = 0$, it is necessary from 7.1 that $(\sigma^o)^2 = \theta$.

Suppose $q \geq 1$. Let $\alpha = [\alpha_1 \mid \dots \mid \alpha_t]$, $\Theta = [\alpha_{t+1} \mid \dots \mid \alpha_{t+1}]$ be in ${}_l\widetilde{V}(q)$ and ${}_m\widetilde{V}(q)$ respectively. Then we define

$$ \alpha\beta = \sum_{\pi} (-1)^{e_\pi(\alpha,\beta)} g(\pi)\,[\alpha_{\pi^{-1}(1)} \mid \dots \mid \alpha_{\pi^{-1}(t+u)}] \tag{7.4} $$

where the summation runs over the set of all premutations $\pi \in G_{t+u}$ satisfying $\pi(1) < \dots < \pi\langle (t)$, $\pi(t+1) < \dots < (t+u)$; $g(\pi) = g(\phi,\pi) \in G_{l+m}$ with $\phi = (\mu(\alpha_1),\dots, \mu(\alpha_{t+u}))$ (see 6.5); and $e_\pi(\alpha,\beta)$ is the exponent given by

$$ e_\pi(\alpha, \beta) = \Sigma'(\,|\alpha_i| + 1)(\,|\alpha_j| + 1) $$

with Σ' running over the set of all couples (i, j) such that $1 \leq i, \leq t$, $t+1 \leq j \leq t+u$ and $\pi(i) > \pi(j)$.

From the incidence relation in $R(q, n)$. given in 6.7 and 6.8, and the above notion of multiplication, we have the following boundary formula for $\widetilde{V}(q)$:

$$ \partial[\alpha_1 \mid \dots \mid \alpha_t] = -\sum_{i=1}^{t} (-1)^{e_{i-1}} [\alpha_1 \mid \dots \mid \alpha_i \mid \dots \mid \alpha_t] \tag{7.5} $$

$$ + \sum_{i=1}^{t-1} (-1)^{e_i} [\alpha_1 \mid \dots \mid \alpha_i\alpha_{i+1} \mid \dots \mid \alpha_t]. $$

where $e_i = \sum_{j=1}^{i} (\,|\alpha_j| + 1)$ for $0 \leq i \leq t$.

This has the same form as the boundary formula of the bar construction. The construction of the $mDGA$-algebra $\widetilde{V}(q)$ is completed.

The $mDGA$-algebra $\widetilde{V}(q)$ defines the $mDGA$-algebra $\widetilde{V}(q)$ by setting

$$ V(q) = \bigoplus_{n \geq 0} {}_nV(q),\ {}_nV(q) = Z \otimes_{G_n} V(q) \text{ for } n \geq 0. $$

Clearly by definition $V(0) \simeq \mathcal{S}^o$, and $V(q) = BV(q-1)$ as $mDGA$—algebras. From this fact, we have the following

7.6 Theorem. We have a canonical isomorphism of $mDGA$-algebras

$$V(q) \cong B^q \mathcal{S}^o$$

for every $q \geq 0$. In particular, we have ${}_nV(q) \cong {}_nB^q\mathcal{S}^o$, $n,q \geq 0$.

To determine the homology algebra of $V(q)$, we consider $V(1) = B\mathcal{S}^o$. By use the notation in 7.3, we have

$$V(1) = \bigoplus_{n \geq 0} Z\theta_n, \ \theta_n = [\sigma^o | \dots | \sigma^o] \in F(1, n)^{\cdot}/\mathbf{G}_n.$$

An easy computation shows that

$$\partial\theta_n = 0, \ \theta_1^2 = 0, \ \theta_1\theta_{2n} = \theta_{2n+1}, \ \theta_{2l}\,\theta_{2m} = (l, m)\,\theta_{2(l+m)} \tag{7.7}$$

for l, m, $n \geq 0$. Here as usual, $(l, m) = (l+m)!/l!\,m!$.

Let $P(x;\ 2i)$ denote the divided polynomial algebra with a generator x of even dimension $2i$ over Z, and $E(y;\ 2i+1)$ the exterior algebra with a genetator y of odd dimension $2i+1$ over Z. Then, from 7.7, we have

$$V(1) = E(\theta_1;\ 1) \otimes P(\theta_2;\ 2), \ (\partial\theta_n = 0, \ n = 1, 2)$$

as $mDGA$-algebras. Remind here that $\mu(\theta_n) = n$. Using the notation given in 4.5, it is well known that, by the method of Cartan's construction, we have the homotopy equivalences of $mDGA$-argebras:

$$E(\theta_1;\ 1) \simeq A(\mathrm{N},\ 1) \qquad (\theta_1 \to [a-1]),$$

$$P(\theta_2;\ 2) \simeq A(2\mathrm{N},\ 2) \qquad (\theta_2 \to [[a-1]]).$$

Consequently, we have $V(1) \simeq A(N,\ 1) \otimes A(2N,\ 2)$.

Now, by means of the homotopy equivalence of $mDGA$-algebras $\mathrm{B}(A \otimes A') \simeq BA \otimes BA'$ for $mDGA$-algebras A and A', we reach the following

7.8. Theorem. *We have the homotopy equivalence of mDGA—algebras*

$$V(q) \simeq A(\mathrm{N},\ q) \otimes A(2\mathrm{N},\ q+l).$$

According to the decomposition theorem, a direct consequence of this theorem is the determination of the homology groups of the spaces $F(R^q, n)^{\cdot}/\mathbf{G}_n$.

7.9. Corollary.

$$H_*(F(\mathbf{R}^q,\ n)^{\cdot}/\mathbf{G}_n);\ \Lambda) \simeq {}_nH_*(A(\mathrm{N},\ q) \otimes A(2\mathrm{N},\ q+1);\ \Lambda))$$

Let $\mathcal{B}_n$ be the braid group of n strings. Then $F(\mathbf{R}^2, n)/\mathbf{G}_n$ is a space $K(\mathcal{B}_n, 1)$ as proved in Fox—Neuwirth [12]. Thus we have $H_*(\mathcal{B}_n;\ \Lambda) \cong H_*(F(\mathbf{R}^2, n)/\mathbf{G}_n);\ \Lambda)$. On the other hand, according to the Poincaré — Lefschetz duality, we have $H_i(F(\mathbf{R}^2, n)/\mathbf{G}_n;\ \Lambda) \cong \widetilde{H}^{2n-i}(F(\mathbf{R}^2, n)^{\cdot}/\mathbf{G}_n;\ \Lambda)$, (see 10.1). Thus, from 7,9, we have the following

7.10. Theorem. $H_i(\mathcal{B}_n; \Lambda) \cong \begin{cases} {}_nH^{2n-i}(A(N, 2) \otimes A(2N, 3); \Lambda) & 0 \leq i < 2n \\ 0 & i \geqslant 2n. \end{cases}$

Remark that $H_i(\mathbf{G}_n;)$ is trivial for $i > n$. (This fact is obvious from 10.3 and the definition of ${}_n\widetilde{W}(2, +)$ in (§9). A related result to the homology groups of $\mathbf{G}_n$ will be found in §10.

7.11. Remark. By means of the method of Cartan construction, the m-algebras $H^*(V(q); Z_p)$, hence the groups $H_*(\mathcal{B}_n; Z_p)$, are easily determined. In 1970, Fucks [13] has computed labourously $H^*(F(\mathbf{R}^2, n)^\cdot/\mathbf{G}_n; Z_2)$ to determine $H_*(\mathcal{B}_n; Z_2)$ by use of the $\mathbf{G}_n$-equivariant CW-decomposition for $(F(\mathbf{R}^2, n)$ which is in fact the decomposition denoted by $F(2, n)^\cdot$ here.

8. THE mDGA-ALGEBRAS V (M, q)

Let M be a DGA-module over $\mathbf{Z}$, We define now the $mDGA$-algebra $\widetilde{V}(M, q)$ for each $q \geqslant 0$. First, as a DC-module, it is the direct sum of DC-modules

$$\widetilde{V}(M, q) = \bigoplus_{n \geqslant 0} {}_n\widetilde{V}(M, q), \quad {}_n\widetilde{V}(M, q) = (IM)^n \otimes {}_n\widetilde{V}(q), \ n \geqslant 0.$$

The unit will be an identification $\mathbf{Z} \cong {}_o\widetilde{V}(M, q)$, and the augmentation will be the canonical projection $\widetilde{V}(M, q) \to \mathbf{Z}$. For every $\alpha \in {}_n\widetilde{V}(q)$ and $x \in (IM)^n$, we write

$$(\alpha, x) = (-1)^{|\alpha||x|} x \otimes \alpha.$$

Then, by definition, we have the boundary formula

$$\partial(\alpha, x) = (\partial\alpha, x) + (-1)^{|\alpha|}(\alpha, \partial x).$$

The multiplication on $\widetilde{V}(M, q)$ is given by the relation

$$(\alpha, x)(\beta, y) = (-1)^{|\alpha||x|}(\alpha\beta, x \otimes y).$$

An easy verification shows that $\widetilde{V}(M, q)$ is an mDGA-algebra with multipliciyt $\mu((\alpha, x)) = \mu(\alpha)$.

The $mDGA$-algebra $\widetilde{V}(M, q)$ defines the $mDGA$-algebra $V(M, q)$ with

$$V(M, q) = \bigotimes_{n \geqslant 0} {}_nV(M, q), \quad {}_nV(M, q) = \mathbf{Z} \otimes_{\mathbf{G}_n} {}_n\widetilde{V}(M, q).$$

Since $V(0) \cong \mathcal{S}^o$, we have obviously $V(M, 0) \cong \mathcal{S}^o M$. In general, we have the following

8.1 Theorem. There exists a canonical isomorphism of $mBDA$-algebras

$$V(M, q) \cong B^q(\mathcal{S}^o M) \text{ for } q \geqslant 0.$$

In particular, we have ${}_nV(M, q) \cong {}_nB^q(\mathcal{S}^o M)$.

This theorem is a direct consequence of our below consideration of the $mDGA$-algebra $\widetilde{V}(M, q)$.

If $q = 0$, we have already $\widetilde{V}(M, 0) = \mathcal{S}^0 M$.

Suppose $q \geqslant 1$. Let $(\alpha, x) \in {}_n\widetilde{V}(M, q)$ with $\alpha = [\alpha_1 | \ldots | \alpha_t]$ and $x = x_1 \otimes \ldots \otimes x_n$ $x_k \in IM$. If $\mu(\alpha_i) = n_i$, $1 \leqslant i \leqslant t$, we write

$$\mathbf{x}_i = x_{n_1 + \ldots + n_{i+1} + 1} \otimes \ldots \otimes x_{n_1 + \ldots + n_i}, \; 1 \leqslant i \leqslant t.$$

Then we write

$$(\alpha, x) = (-1)^{d(\alpha, x)} [(\alpha_1, \mathbf{x}_1) | \ldots | (\alpha_t, \mathbf{x}_t)]$$

where $d(\alpha, x)$ is the signature given by the interchanges of $[\alpha_i]$ with $\mathbf{x}_j$ when $i > j$:

$$d(\alpha, x) = \sum_{1 \leqslant j < i \leqslant t} (|\alpha_i| + 1) |\mathbf{x}_j|. \tag{8.2}$$

Here we note that we have the corresponding formula on the dimensions and the multiplicities as follow:

$$|(\alpha, x)| = \sum_{i=1}^{t} (|\alpha_i, \mathbf{x}_i)| + 1)$$

$$(\alpha, x) = \sum_{i=1}^{t} (\alpha_i, \mathbf{x}_i).$$

This correspondence yields an isomorphism of G_n—modules

$${}_n\widetilde{V}(M, q) \cong \bigoplus_{n_1 + \ldots + n_t = n} Z(G_n) \bigotimes_{G_{n_1} \times \ldots \times G_{n_t}} ({}_{n_1}\widetilde{V}(M, q-1) \otimes \ldots \otimes_{n_t} \widetilde{V}(M, q-1) \quad (8.3)$$

according to 7.2.

Now we prove the following formula for the multiplication.

8.4 Lemma. Let $a = [(\alpha_1, \mathbf{x}_1) | \ldots | (\alpha_t, \mathbf{x}_t)] \in {}_l V(M, q)$ and

$b = [(\alpha_{t+1}, \mathbf{x}_{t+1}) | \ldots | (\alpha_{t+u}, \mathbf{x}_{t+u})] \in {}_m\widetilde{V}(M, q)$. Then we have

$$ab = \Sigma (-1)^{e_\pi(a,b)} g(\pi) [(\alpha_{\pi^{-1}(1)}, x_{\pi^{-1}(1)}) | \ldots | (\alpha_{\pi^{-1}(t+u)}, \mathbf{x}_{\pi^{-1}(t+u)})]$$

Here the summation, $g(\pi)$ are as in 7.4, and the exponents $e_\pi(a,b)$ are given by

$$e_\pi(a,b) = \Sigma' (|\alpha_i| + |\mathbf{x}_i| + 1)(|\alpha_j| + |x_j + 1)$$

where Σ' is again as in 7.4.

Proof. Let $\alpha = [\alpha_1 | \dots | \alpha_t]$, $\beta = [\alpha_{t+1} | \dots | \beta_{t+u}]$ and let $x = x_1 \otimes \dots \otimes x_t$, $y = x_{t+1} \otimes \dots \otimes x_{t+u}$. Then, by definition, we have $(-1)^{|\beta||x|}(\alpha, x)(\beta, y) = (\alpha\beta, x \otimes y)$

$$= \Sigma(-1)^{e_\pi(\alpha, \beta)}(g(\pi)[\alpha_{\pi^{-1}(1)} | \dots | \alpha_{\pi^{-1}(t+u)}], x_1 \otimes \dots \otimes x_{t+u})$$

$$= \Sigma(-1)^{e_\pi(\alpha, \beta) + e_\pi(x, y)} g(\pi)([\alpha_{\pi^{-1}(1)} | \dots | \alpha_{\pi^{-1}(t+u)}], g(\pi)(x_1 \otimes \dots \otimes x_{t+u}))$$

$$= \Sigma(-1)^{e_\pi(\alpha, \beta) + e_\pi(x, y)} g(\pi)([\alpha_{\pi^{-1}(1)} | \dots | \alpha_{\pi^{-1}(t+u)}], x_{\pi^{-1}(1)} \otimes \dots \otimes x_{\pi^{-1}(t+u)}))$$

$$= (-1)^{e_\pi(\alpha, \beta) + e_\pi(x, y) + d_\pi(\alpha, \beta, x, y)} g(\pi) [(\alpha_{\pi^{-1}(1)}, x_{\pi^{-1}(1)}) | \dots | (\alpha_{\pi^{-1}(t+u)}, x_{\pi^{-1}(t+u)})]$$

Here we set

$$e_\pi(x, y) = \Sigma' |x_i||x_j|,$$

$$d_\pi(\alpha, \beta; x, y) = \sum_{1 \leqslant j \leqslant i \leqslant t+u} (|\alpha_{\pi^{-1}(i)}| + |x_{\pi^{-1}(i)}| + 1)(|\alpha_{\pi^{-1}(j)}| + |x_{\pi^{-1}(j)}| + 1).$$

Our purpose is to prove the relation which implies the lemma:

$$|\beta||x| + e_\pi(\alpha, \beta) + e_\pi(x, y) + d_\pi(\alpha, \beta; x, y)$$
$$\equiv d(\alpha, x) + d(\beta, y) + e_\pi(\alpha, \beta; x, y) \mod 2 \qquad (*)$$

where $d(\alpha, x)$ and $d(\beta, x)$ are as in (8.2). First we observe

$$d_\pi(\alpha, \beta; x, y) = (\Sigma_1 + \Sigma_2 + \Sigma_3)(|\alpha_i| + |x_i| + 1)(|\alpha_j| + |x_j| + 1)$$

where Σ_1, Σ_2, Σ_3 are the summations

$$\sum\nolimits_1 = \sum_{1 \leqslant \pi^{-1}(i) < \pi^{-1}(j) \leqslant t,\ i > j}$$

$$\sum\nolimits_2 = \sum_{t+1 \leqslant \pi^{-1}(i) < \pi^{-1}(j) \leqslant t+u,\ i > j}$$

$$\sum\nolimits_3 = \sum_{1 \leqslant \pi^{-1}(i) \leqslant t,\ t+1 \leqslant \pi^{-1}(j) \leqslant t+u,\ i > j.}$$

Since $\pi(1) < \dots < \pi(t)$, $\pi(t+1) < \dots < \pi(t+u)$, we have $d_\pi(\alpha, \beta; x, y)$

$$= \Big(\sum_{1 \leqslant j < i \leqslant t} + \sum_{t+1 \leqslant j < i \leqslant t-u} + \sum{}' \Big)(|\alpha_i| + |x_i| + 1)(|\alpha_j| + |x_j| + 1)$$

$$= d(\alpha, x) + d(\beta, y) + \Sigma'((|\alpha_i| + 1)(|\alpha_j| + 1)|x_i||x_j| + (|\alpha_i| + 1)|x_j| - (|\alpha_j| + 1)|x_i|)$$

$$= d(\alpha, x) + d(\beta, y) + e_\pi(\alpha, \beta) + e_\pi(x, y) + \Sigma'(|\alpha_i| + 1)|x_j| + (|\alpha_j| + 1)|x_i|) \qquad (**)$$

Further, we have

$$|\beta|\,|x| + \Sigma'\left((|\alpha_i| + 1)\,|x_i| + (|\alpha_j| + 1)\,|x_i|\right.$$
$$= 2\,\Sigma'\,(|\alpha_j| + 1)\,|x_i| + \Sigma'\,(|\alpha_i| + |x_i| + 1)\,(\alpha_j| + x_j| + 1)$$
$$\equiv e_\pi\,(\alpha, \beta\,;\, x, y)\ mod\ 2 \qquad (***)$$

Obviously, from (**) and (***), we obtain (*). The lemma follows.

8.5 Lemma. In $\widetilde{V}\,(M, q)$ we have a boundary formula of the same form as that on the bar construction, i. e.

$$\partial\,[(\alpha_1, x_1)\,|\ldots|\,(\alpha_t, x_t)]$$

$$= -\sum_{i=1}^{t}(-1)^{e_{i-1}}\,[(\alpha_1, x_1)\,|\ldots|\,\partial\,(\alpha_i, x_i)\,|\ldots|\,(\alpha_t, x_t)]$$

$$-\sum_{i=1}^{t-1}(-1)^{e_i}\,[(\alpha_1, x_1)\,|\ldots|\,(\alpha_i, x_i)\,(\alpha_{i+1}, x_{i+1})\,|\ldots|\,(\alpha_t, x_t)]$$

where $e_i = \dim\,[(\alpha_1, x_1)\,|\ldots|\,(\alpha_i, x_i)]$ for $1 \leqslant i \leqslant t$.

We omit the proof since it is merely a comparision of the signatures as in 8.4. The proof of Theorem 8.1 is completed.

An important application of the algebras $V\,(M, q)$ can be found in the following

8.6 Theorem. Let X be in CG' and let $C_*\,(X)$ be a DGA - module such that $C_*\,(X) \simeq S_*\,(X)$. Then the singular chain complex of the filtered monoid $B\,(X,q)$ is homotopic to the $mDGA$ - algebra $B^q\,(\mathcal{S}^o C_*\,(X))$ for $q \geqslant 0$.

Proof. According to the definition of the DGA - module $\widetilde{V}\,(M, q)$ and the proof of 3.10, we have

$$\widetilde{V}\,(C_*\,(X), q) \simeq S'_*\,(\widetilde{B}\,(X, q)) = \bigoplus_{n\geqslant 0}\widetilde{S}_*\,(\widetilde{B}\,(X, q, n), \widetilde{B}\,(X, q, n-1))$$

for any $X \in CG'$ and any DGA - module $C_*\,(X) \simeq S_*\,(X)$. Thus what we need to prove is that the multiplication defined on $\widetilde{V}\,(C_*\,(X), q)$ is an approximation for the monoid structure $\widetilde{B}\,(X, q) \times \widetilde{B}\,(X, q) \to \widetilde{B}\,(X,q)$ given in 4.9. Again by 3.10, we have the homotopy equivalence $S_*\,(\widetilde{B}\,(|\,S_*\,(X)|, q)) \simeq S_*\,(\widetilde{B}\,(X, q))$ which in obviously a morphism of $mDGA$-algebra. So, without loss of generality, we can assume that X is a CW - complex. Thus $\widetilde{B}\,(X, q)$ becomes then a CW complex according to Proposition 2.10. Let $C_*\,(A)$ denote the cellular chain complex of a CW - complex A. Then we have $C_*\,(\widetilde{B}\,(X, q)) = \widetilde{V}\,(C_*\,(X), q)$ by definition

Let $g: F(\mathbf{R}^q, l)^{\cdot} \times F(\mathbf{R}^q, m)^{\cdot} \to F(\mathbf{R}^q, l+m)^{\cdot}$ be the based point preserving nap given by $((a_1, \ldots, a_l), (a_{l+1}, \ldots, a_{l+m})) \to ((a_1, \ldots, a_{l+m}))$. Then we have

$$f(x \times \alpha, y \times \beta) = (x \times y, g(\alpha \times \beta))$$

where $\alpha \in F(q, l)^{\cdot}$, $\beta \in F(q, m)^{\cdot}$ and x and y are cells in X^l and X^m respectively. Hence f in a cellular map.

Because the multiplication on $\widetilde{V}(C_*(X), q)$ is given by the formula $(\alpha, x)(\beta, y) = (-1)^{|\beta|\,|x|}(\alpha\beta, xy)$, it remains to shows that the formula 7.4 for the definition of $\alpha\beta$ can be obtained by $\alpha\beta = g_*(\alpha \otimes \beta)$ where g_* is the induced chain map of g.

Let α, β be as in 7.4. Then we observe that the cells having dimension $|\alpha| + |\beta|$ contained in $g(\alpha \times \beta)$ are just those which appear in the summation $\sum_{\pi}$ of 7.4. Moreover, the signatures in this summation have been given in concordence with respect to the orientation of cells defined in terms of coordinate system as in §6. These facts show $\alpha\beta = g_*(\alpha \otimes \beta)$. The proof of the theorem is completed.

§ 9. THE G_n—FREE COMPLEX ${}_n\widetilde{W}(q)$ AND A FREE RESOLUTION FOR $Z(G_n)$

In this section we review the structrure of the dual complex ${}_n\widetilde{W}(n)$ of ${}_n\widetilde{V}(q)$ which has been formulated in Nakamura [26; II § 2]. By definition, it is the G_n-equivariant complex given as follows.

In each dimension i, the i-chain group of ${}_n\widetilde{W}(q)$ is the $(nq\text{-}i)$-cochain group of ${}_n\widetilde{V}(q) = \mathrm{Hom}\,({}_n\widetilde{V}(q), Z)$. If we let $\eta: {}_n\widetilde{W}(q)_i = {}_n\widetilde{V}(q)^*_{nq-i}$ denote this identification, the boundary ∂ of ${}_n\widetilde{W}(q)$ is given by the relation $\partial = \eta^{-}\delta\eta$ wheve δ is the coboundary of ${}_n\widetilde{V}(q)^*$, and the action of G_n on ${}_n\widetilde{W}(q)$ is introduced so that η is G_n-equivariant.

For each $\alpha \in {}_n\widetilde{V}(q)$, we shall denote by $\omega = \omega(\alpha)$ the corresponding element of α in ${}_n\widetilde{W}(q)$, i.e. $\omega(\alpha) = \eta(\alpha^*)$ with the dual α^* of α in ${}_n\widetilde{W}(q)$. Then we have $\omega(g\alpha) = g\omega(\alpha)$ for $g \in G_n$, and the incidence relation in ${}_n\widetilde{W}(q)$ obtained easily by the formula $[\omega(\beta) : g\omega(\alpha)] = [\alpha : g^{-1}\beta]$.

Suppose that we are given a cell $\alpha = (r_1, \ldots, r_n)$ in ${}_n\widetilde{V}(q)$. Let us associate to $\omega(\alpha)$ the sequence of integers $< \rho_1, \ldots, \rho_n >$ with $\rho_\nu = q - r_\nu$, $1 \leq \nu \leq n$. We shall denote this sequence also by $\omega(\alpha)$. We note that this notation is compatible with respect to the dimension of α and $\omega(\alpha)$ since we have

$$|\omega(\alpha)| = \sum_{i=1}^{n} \equiv nq - |\alpha|.$$

From the structure of ${}_n\widetilde{V}(q)$, ${}_n\widetilde{W}(q)$ is obviously the G_n-free module having a G_n-basis consisting of the elements $<\rho_1, \dots, \rho_n>$ with $\rho_1=0$, $0 \le \rho_\nu < q$ $(1<\nu\le n)$.

The following is an inductive study on q of the structure of ${}_n\widetilde{W}(q)$. If $q=0$, we have from 7.1.

$${}_n\widetilde{W}(q) = \begin{cases} Z & n=1 \\ 0 & n>1. \end{cases}$$

Suppose $q \geq 1$. Suppose we are given an element $\omega = \omega(\alpha)$ in ${}_n\widetilde{W}(q)$ where $\alpha = (r_1, \dots, r_n) = [\alpha_1 | \dots | \alpha_t]$ with $\alpha_i \in {}_{n_i}\widetilde{V}(q-1)$, $1 \le i \le t$. Then ω determines and is determined uniquely by the sequence $\omega_1, \dots, \omega_t$ with $\omega_i = \omega(\alpha_i) \in {}_{n_i}\widetilde{W}(q-1)$ for $1 \le i \le t$. From this we write formally

$$\omega = <\omega_1, \dots, \omega_t>.$$

As easily seen, we have

$$\omega_i = <0, \rho_{\nu(i)-1}, \dots, \rho_{\nu(i+1)-1}>$$

for $1 \le i \le t$ if $\omega = <\rho_1, \dots, \rho_n>$, where $\nu(1), \dots, \nu(t)$ is the monotone increasing sequence consisting of all indices ν's such that $\rho_\nu = 0$. We have also the relations

$$|\omega| = \sum_{i=1}^{t} (|\omega_i| + \mu(\omega_i) - 1),$$

$$\mu(\omega) = \sum_{i=1}^{t} \mu(\omega_i).$$

Here and in what follows we write $\mu(\omega) = n$ if $\omega \in {}_n\widetilde{W}(q)$.

The above correspondence yields an isomorphism

$${}_n\widetilde{W}(q) \cong \bigoplus_{n_1+\dots+n_t=n} Z(G_n) \underset{G_{n_1}+\dots+G_{n_t}}{\otimes} ({}_{n_1}\widetilde{W}(q-1) \otimes \dots \otimes {}_{n_t}\widetilde{W}(q-1)) \qquad (9.1)$$

of G_n-modules. This is an induction formula for ${}_n\widetilde{W}(q)$.

To express the boundary formula we define the morphism of DG G_n-modules

$$\Delta(q) = \bigotimes_{l+m=n} {}_{l,m}(q) : {}_n\widetilde{W}(q) \to \bigotimes_{l+m=n} Z(G_n) \underset{G_l \times G_m}{\otimes} ({}_l\widetilde{W}(q) \otimes_m \widetilde{W}(q)) \qquad (9.2)$$

where ${}_{l,m}\Delta(q)$ is a morphism of *DG* $\mathbf{G}_n$-modules given for each pair of nonnegative integers (l, m) with $l + m = n$ as follows. Let $\omega = <\omega_1, \ldots, \omega_t> \in {}_n\widetilde{W}(q)$ with $\omega_i \in {}_{n_i}\widetilde{W}(q-1)$. Then we define

$$
{}_{l,m}\Delta(q) = \sum (-1)^{e_\pi(\omega\,;\,q)} g(\pi) <\omega_{\pi(1)}, \ldots, \omega_{\pi(u)}> \otimes <\omega_{\pi(u+1)}, \ldots, \omega_{\pi(t)}>
$$

Here: (i) the summation runs over the set of all permutations $\pi \in \mathbf{G}_t$ such that $\pi(1) < \ldots < \pi(u)$, $\pi(u+1) < \ldots < \pi(t)$ for some u satisfying the conditions

$$
\sum_{i=1}^{u} n_{\pi(i)} = l, \qquad \sum_{i=u+1}^{t} n_{\pi(i)} = m\,;
$$

(ii) $g(\pi) = g(\Phi, \pi)$ with $\Phi = (n_1, \ldots, n_t)$

(iii) $e_\pi(\omega\,;\,q) = \Sigma'(|\omega_i|) + n_i(q-1)+1)(|\omega_j| + n_j(q-1)+1)$

where Σ' runs over the set of all couples (i, j) such that $1 \leqslant i \leqslant u$, $u+1 \leqslant j \leqslant t$ and $\pi(i) > \pi(j)$.

For later convenience, we define

$$
\underline{\Delta}(q) = \bigoplus_{l+m=n} {}_{l,m}\underline{\Delta}(q) \tag{9,3}
$$

where

$$
{}_{l,m}\underline{\Delta}(q)\,\omega = \Sigma(-1)^{\mathrm{e}_\pi(\omega\,;\,q)} g(\pi) <\omega_{\pi(1)} \ldots \omega_{\pi(u)}> \otimes <\omega_{\pi(u+1)}, \ldots, \omega_{\pi(t)}>
$$

with $\mathrm{e}_\pi(\omega\,;\,q)$ is the signature given by

$$
\mathrm{e}_\pi(\omega\,;\,q) = e_\pi(\omega\,;\,q) + \sum_{i=1}^{u} (|\omega_{\pi(i)}| + n_{\pi(i)}(q-1) + 1).
$$

Now, under the above notations, we have a formula for the boundary $\partial(q)$ on ${}_n\widetilde{W}(q)$ as follows

$$
\begin{aligned}
\partial(q) <\omega_1, \ldots, \omega_t> = &- \sum_{i=1}^{t} (-1)^{e_{i-1}(\omega\,;\,q)} <\omega_1, \ldots, \partial(q-1)\omega_i, \ldots, \omega_t> \\
&+ \sum_{i=1}^{t} (-1)^{e_{i-1}(\omega,\,q)} <\omega_1, \ldots, \underline{\Delta}(q-1)\omega_i, \ldots, \omega_t>
\end{aligned} \tag{9.4}
$$

where $e_i(\omega\,;\,q) = \sum_{j=1}^{i} (|\omega_j| + n_j(q-1)+1) (\equiv \sum_{j=1}^{i} (n_j q - |<\omega_j>|) \bmod 2)$

The formulation of the chain complex ${}_n\widetilde{W}(q)$ is completed.

Let $\Lambda(q)$ denote the G-module Λ on which $\mathbf{G}_n$ operates trivially if q is even, and by sign if q is odd. Then, from the above boundary formula, we observe that ${}_n\widetilde{W}(q)$ is an DG $\mathbf{G}_n$-module augmented by $Z(q)$. Further, if q and q' are two non-negative integers such that $q \leqslant q'$ and $q-q'$ is even, then there exists a canonical immedding of DG $\mathbf{G}_n$-modules

$$ {}_n\widetilde{W}(q) \subset {}_n\widetilde{W}(q'). $$

Hence we have the limits

$$ \widetilde{W}(\mathbf{G}_n, +) = \lim_{q\to\infty} {}_n\widetilde{W}(2q) $$

$$ \widetilde{W}(\mathbf{G}_n, -) = \lim_{q\to\infty} {}_n\widetilde{W}(2q+1) $$

Let $\mathbf{Z}_+ = \mathbf{Z}(q)$ if q is even, and $\mathbf{Z}_- = \mathbf{Z}(q)$ if q is odd. We have the following

9.5 Theorem (Nakamura). $\widetilde{W}(\mathbf{G}_n, \pm)$ *in a free resolution of* $Z_\pm$ *over the group ring* $Z(\mathbf{G}_n)$.

Proof. By definition we have

$$ \bigoplus_{0<i<q} {}_n\widetilde{W}(q)_i \cong \bigoplus_{0\leqslant i<q} \mathbf{C}^{nq-i}(R(q, n)^\cdot). $$

Since $H^i(R(q, n)^\cdot ; Z) = H^i(\mathbf{S}^{nq} ; Z) = 0$ for $0 < i < nq$, we have $H_i({}_n\widetilde{W}(q) ; Z) = 0$ for $0 < i < q-1$. Letting $q \to \infty$, we obtain the theorem.

We take this opportunity to derive some consequences of this theorem on the homology groups of the symmetric groups. First we have

9.6 Theorem (Steenrod). *If* $i < q$, *the we have*

$$ H_i(\mathbf{G}_n ; \Lambda(q) \cong H^{nq-i}(SP^n\mathbf{S}^q ; \Lambda). $$

This result has stated without proof by Steenrod in his lecture note [33] for q even, A proof can be found in Nakaoka [25].

Proof. If $0 \leqslant i < q$, we have

$$ H^{nq-i}(SP^n\mathbf{S}^q ; \Lambda) \cong H^{nq-i}(SP^n\mathbf{S}^q/SP^{n-1}\mathbf{S}^q ; \Lambda) $$

according to the Steenrod's decomposition theorem 3.1, because $H^{nq-i}(SP^{n-1}\mathbf{S}^q ; \Lambda) = 0$ in this case. Further, by the relation $SP^n\mathbf{S}^q/SP^{n-1}\mathbf{S}^q \cong (\times_n \mathbf{R}^q)^\cdot/\mathbf{G}_n$, and the isomorphism in the proof of 1.5, we have

$$H^{nq-i}(SP^nS^q\,;\,\Lambda) \cong H^{nq-i}((\underset{n}{\times}\,\mathbf{R}^q)^{\cdot}/\mathbf{G}_n\,;\,\Lambda)$$

$$\cong H_i\,(\Lambda \underset{\mathbf{G}_n}{\otimes} {}_n\widetilde{W}(q)).$$

if $0 \leqslant i < q$. On the other hand, let $q' = q - 2$, we observe then that ${}_n\widetilde{W}(q')$ (resp. ${}_n\widetilde{W}(q)$) is an $\mathbf{G}_n$-free and acyclic in dimensions $< q'$ (resp. in dimensions $< q$). Consequently, in dimensions $i \leqslant q$, the canonical imbedding ${}_n\widetilde{W}(q) \subset {}_n\widetilde{W}(q')$ is an $\mathbf{G}_n$-equivariant chain equivalence. From this, we have

$$H_i(\mathbf{G}_n\,;\,\Lambda(q)) \cong H_i(\Lambda \underset{\mathbf{G}_n}{\otimes} {}_n\widetilde{W}(q)), \qquad 0 \leqslant i < q\,;$$

according to 9.5. The theorem follows.

Now, combine Theorems 6.10, 9.5 and 9.6 we obtain the Nakamura's formulation an idea due to Steenrod in [33; 22] for the computation of the homology groups of symmetric groups in the following

9.7 Theorem, *If* $0 \leqslant i < q$, *we have*

$$H_i(\mathbf{G}_n\,;\,\Lambda(q)) \cong {}_nH^{nq-i}(Z,\,q\,;\,\Lambda).$$

Here $H^*(Z,\,q\,;\,\Lambda)$ is the algebra with the multiplicity given via the isomorphism $H^*(Z,\,q\,;\,\Lambda) \cong H^*(A(\mathbf{N},\,q)\,;\,\Lambda)$ (sse 4.5). In the case where q is even and $\Lambda = Z_p$, we reach easily the result of Nakaoka [22; 6.3] on the homology groups of the symmetric groups.

10. THE mDCA-COALGEBRA W (M, q)

Let $Z(q,\,n)$ denote the graded $\mathbf{G}_n$-module with $Z(q,\,n) \cong Z(q)$ as $\mathbf{G}_n$-modules generated by a single element of dimension nq. Obviously we have then

$$H_{nq}\,((\underset{n}{\times}\,\mathbf{R}^q)^{\cdot}\,;\,Z) \cong Z\,(q,\,n).$$

Considering $Z\,(q,\,n)$ as a LG $\mathbf{G}_n$-module with the trivial boundary operator, the Poincaré—Lefschetz duality theorem applied to the spaces $((\underset{n}{\times}\,\mathbf{R}^q)^{\cdot}.\,T\,(\mathbf{R}^q,\,n))$ gives rise to the canonical isomorphism of DG $\mathbf{G}_n$-modules

$$Z\,(q.\,n) \otimes \overline{S}^*\,(F\,(\mathbf{R}^q.\,n)^{\cdot}) \cong \overline{S_*}\,(F\,(R^q,\,n)+). \tag{10.1}$$

On the other hand, from the definition of ${}_n\widetilde{W}\,(q)$, we have

$$\mathrm{Z}\,(q) \otimes {}_n\widetilde{W}\,(q) \cong \mathrm{Z}\,(q,\,n) \otimes {}_n\widetilde{V}\,(q)^*.$$

So, via the $\mathbf{G}_n$-equivariant chain equivalence ${}_n\widetilde{V}(q)^* \cong \overline{S^*}(F(\mathbf{R}^q, n)^{\cdot})$, we obtain the $\mathbf{G}_n$-equivariant chain equivalence

$$Z(q) \otimes {}_n\widetilde{W}(q) \simeq \overline{S_*}(F(R^q, n)^+). \tag{10.2}$$

For later convenience, instead of $Z(q) \otimes_n \widetilde{W}(q)$, we shall use the DG $\mathbf{G}_n$-module ${}_n\widetilde{W}(q, +)$ defined in the following

10.3 Definition Let ${}_n\widetilde{W}'(q)$ denote the DG $\mathbf{G}_n$-submodule of ${}_n\widetilde{W}(q+1)$ with a $\mathbf{G}_n$-basis consisting of the sequences $< \rho_1, \ldots, \rho_n >$, $\rho_1 = 0, 0 \leqslant \rho_i < q$. Then we define:

$${}_n\widetilde{W}(q, +) = {}_n\widetilde{W}(q) \text{ and } {}_n\widetilde{W}(q, -) = {}_n\widetilde{W}'(q) \text{ if } q \text{ is even,}$$

$${}_n\widetilde{W}(q, +) = {}_n\widetilde{W}'(q) \text{ and } {}_n\widetilde{W}(q, -) = {}_n\widetilde{W}(q) \text{ if } q \text{ is odd.}$$

We prove

10.4 Lemma. ${}_n\widetilde{W}(q, +) \cong Z(q) \otimes {}_n\widetilde{W}(q) \simeq S_*(F(R^q, n))$.

Proof. The lemma is clear when q is even. We suppose that q is odd. Let ${}_n\widetilde{V}'(q+1)$ denote the $\mathbf{G}_n$-submodule of ${}_n\widetilde{V}(q+1)$ with a $\mathbf{G}_n$-basis consisting of the sequences $(r_1, \ldots, r_n)$, $r_1 = q+1$ and $1 < r_i \leqslant q+1$. Then, according to 7.5, the map ${}_n\widetilde{V}(q) \to {}_n\widetilde{V}'(q+1)$ given by $\alpha \to (-1)^{n+1}[\alpha]$ is clearly an isomorphism of DG $\mathbf{G}_n$ — modules of degree n. From this and Definition 10.3, we have

$$\begin{aligned} {}_n\widetilde{W}(q, +) = {}_n\widetilde{W}'(q) &\cong Z(q+1, n) \otimes {}_n\widetilde{V}'(q-1)^* \\ &\cong Z(q) \otimes (Z(q, n) \otimes {}_n\widetilde{V}(q)^*) \\ &\cong Z(q) \otimes {}_n\widetilde{W}(q). \end{aligned}$$

The lemma follows.

Remark. We can prove that ${}_n\widetilde{W}(q, +) \simeq S_*(F(\mathbf{R}^q, n))$ as follows. Suppose q odd. Let

$$F_n = \bigcup \{g\alpha;\ g \in \mathbf{G}_n, \alpha = (r_1, \ldots, r_n), r_1 = q+1, 1 < r_i \leqslant q+1\}.$$

Then we have $F(\mathbf{R}^q, n) \subset F_n \subset (\underset{n}{\times} \mathbf{R}^{q+1})$. Let $p_t : \mathbf{R}^{q+1} \to \mathbf{R}^{q+1}$ denote the map $(a^1, \ldots, a^q, a^{q+1}) \to (a^1, \ldots a^q, (1-t)\, a^{q+1})$ for each $0 \leqslant t \leqslant 1$. Then we obtain by a natural way a $\mathbf{G}_n$-equivariant retracting deformation of R_n onto $F(\mathbf{R}^q, n)$. Thus we have $S_*(F_n) \simeq S_*(F(\mathbf{R}^q, n))$.

On the other hand, by use of the Poincaré — Lefschetz duality to the spaces $((\underset{n}{\times} \mathbf{R}^{q+1})^{\cdot}, (\times \mathbf{R}^{q+1})^{\cdot} - F_n)$ and a similar argument in proving 10.2, we obtain the relation ${}_n\widetilde{W}(q, +) \simeq S_*(F_n)$ if q is odd. This is what we need to prove.

Let M be a DGA-module. The above lemma and the assertion 3.10 lead us to the DG-module $\widetilde{W}(M, q, \pm)$ defined to be the direct sum of DG-modules

$$W(M, q, \pm) = \bigoplus_{n \geqslant 0} {}_n\widetilde{W}(M, q, \pm), \; {}_n\widetilde{W}(M\, q, \pm) = (IM)^n \otimes {}_nW(q, \pm).$$

For every $\omega \in {}_n\widetilde{W}(q,)$ and $x \in (IM)^n$, we write

$$(\omega, x) = (-1)^{|\omega||x|}\, x \otimes \omega.$$

Then, we have by definition the boundary formula

$$\partial(\omega, x) = (\partial\omega, x) + (-1)^{|\omega|}(\omega, \partial x).$$

Further, we define the morphism of DG $\mathbf{G}_n$—modules (10.5)

$${}_n\Delta(M, q, \pm) = \bigoplus_{l+m=n} {}_{l,m}\Delta(M\, q, \pm)$$

where, for each pair of non-negative integers (l, m) with $l + m = n$, ${}_{l,m}\Delta(M, q, \pm)$ is the following composition:

$$ {}_n\widetilde{W}(M, q, \pm) \xrightarrow{\mathrm{id}\otimes {}_{l,m}\Delta(q, \pm)} (IM)^n \otimes (Z(\mathbf{G}_n) \otimes_{\mathbf{G}_l \times \mathbf{G}_m} ({}_l\widetilde{W}(q, \pm) \otimes {}_m\widetilde{W}(q, \pm)) $$

$$ \xrightarrow{\cong} Z(\mathbf{G}_n) \otimes_{\mathbf{G}_l \times \mathbf{G}_m} (((IM)^l \otimes {}_l\widetilde{W}(q, \pm)) \otimes ((IM)^m \otimes {}_m\widetilde{W}(q, \pm))) $$

$$ \xrightarrow{\cong} Z(\mathbf{G}_n) \otimes_{\mathbf{G}_l \times \mathbf{G}_m} ({}_l\widetilde{W}(M, q, \pm) \otimes {}_m\widetilde{W}(M, q, \pm)). $$

Here, ${}_{l,m}\Delta(q, \pm) : {}_n\widetilde{W}(q, \pm) \to Z(\mathbf{G}_n) \otimes_{\mathbf{G}_l \times \mathbf{G}_m} ({}_l\widetilde{W}(q, \pm) \otimes_m \widetilde{W}(q, \pm))$ is the $\mathbf{G}_n$-morphism of DG $\mathbf{G}_n$-modules obtained from ${}_{l,m}\Delta(q)$ defined in 9.2.

From the DG-module $\widetilde{W}(M, q, \pm)$ and the morphisms ${}_n\Delta(M, q, \pm)$, we define the $mDGA$-coalgebra $W(M, q, \pm)$ with

$$W(M, q, \pm) = \bigoplus_{n \geqslant 0} {}_nW(M, q, \pm), \; {}_nW(M, q, \pm) = Z \otimes_{\mathbf{G}_n} {}_n\widetilde{W}(M, q, \pm)$$

as a DG-module, and with the comultiplication $\Delta = \bigoplus_{n \geqslant 0} \mathrm{id} \otimes_{\mathbf{G}_n} {}_n\Delta(M, q, \pm)$. Further, the coaugmentation of $W(M, q, \pm)$ is an identification $Z \cong {}_0W(M, q, \pm)$, the counit is the canonical projection $W(M, q, \pm) \to {}_0W(M, q, \pm) = Z$, and the multiplicity is given by the relation $\mu((\omega, x)) = n$ for $(\omega, x) \in {}_nW(M, q, \pm)$.

We shall use also the notations $\widetilde{W}(M, q) = \widetilde{W}(M, q, +)$ and $W(M, q) = W(M, q, +)$ for the sake of simplicity.

10.6 Theorem. *There exists a canonical isomorphism of mDGA-algebras*

$$W(M, q) \cong F^q \mathcal{S}^q M$$

for each $q \geqslant 0$. In particular, we have ${}_nW(M, q) \cong {}_nF^q \mathcal{S}^q M$.

Of particular importance is the case where $M = \mathcal{S}^o$. Here we have

$$\widetilde{W}(\mathcal{S}^o q) \cong \bigoplus_{n \geqslant 0} {}_n\widetilde{W}(q, +) \text{ and } W(\mathcal{S}^o, q) \cong \bigoplus_{n \geqslant 0} {}_nW(q, +).$$

From the above theorem, we obtain

10.7 Corollary. There exists a canonical isomorphism of $mDGA$-algebras $W(\mathcal{S}^o, q) \cong F^q \mathcal{S}^q$. Particularly, we have

$${}_nW(q, +) = Z(q) \underset{\mathbf{G}_n}{\otimes} {}_n\widetilde{W}(q) \cong {}_nF^q \mathcal{S}^q$$

Further, according to the relation 10.4, we have

10.8 Corollary. $H_*(F(\mathbf{R}^q, n)/\mathbf{G}_n;\ \Lambda) \cong {}_nH_*(F^q \mathcal{S}^q;\ \Lambda)$.

This assertion permits another method to determine the homology groups of the spaces $F(\mathbf{R}^q, n)/\mathbf{G}_n$ by the method of the Cartan construction which is effective in the case where the coefficients are taken in the field $\mathbf{Z}_p$ (compare with 7.9).

Now we prove Theorem 10.6. This will be a consequence of our below inductive consideration of the DGA-modules $\widetilde{W}(M, q, \pm)$.

If $q = 0$, we have $\widetilde{W}(M, 0, \pm) = \mathcal{S}^o M$ from the structure of ${}_n\widetilde{W}(0)$. We suppose $q \geqslant 1$. Let $(\omega, x) \in {}_n\widetilde{W}(M, q, \pm)$ where $\omega = < \omega_1, ..., \omega_t >$ with $\omega_i \in \widetilde{W}(q-1, \mp)$ and $x = x_1 \otimes ... \otimes x_n$ with $x_k \in IM$. If $\mu(\omega_i) = n_i$, $1 \leqslant i \leqslant t$, we write

$$x_i = x_{n_1 + \cdots + n_{i+1}+1} \otimes \ldots \otimes x_{n_1 + \cdots + n_1}, \ 1 \leqslant i \leqslant t$$

as in §8. Then we write

$$(\omega, x)' = (-1)^{d(\omega, x)} < (\omega_1, x_1), \ldots, (\omega_t, x_t) > \tag{10.9}$$

where $d(\omega, x)$ is the signature given by the interchanges of $< \omega_i >$ with x_j, $i > j$:

$$d(\omega, x) = \sum_{1 \leqslant j < i \leqslant t} (|\omega_i| + n_i + 1)\,|x_j|.$$

Here we note that we have the corresponding formula on dimensions and multiplicities as follows.

$$|(\omega, x)| = \sum_{i=1}^{t} (|(\omega_i, x_i)| + \mu(\omega_i, x_i) - 1)$$

$$\mu((\omega, x)) = \sum_{i=1}^{t} \mu((\omega_i, x_i)).$$

From the induction formula for ${}_n\widetilde{W}(q)$, the above correspondence yields an isomorphism of G_n-modules

$$ {}_n\widetilde{W}(M, q, \pm) \cong \sum_{n_1+\dots+n_t=n} Z(G_n) \underset{G_{n_1}\times\dots\times G_{n_t}}{\otimes} ({}_{n_1}\widetilde{W}(M, q-1, \mp) \otimes \dots \otimes {}_{n_t}\widetilde{W}(M, q-1, \mp) \qquad (10.10) $$

This is an induction formula for ${}_n\widetilde{W}(M, q, \pm)$.

Now, by a similar way in proving Lemma 8.4, we have the following

10.11 Lemma. Let $c = \langle c_1, \dots, c_n \rangle \in {}_n\widetilde{W}(M, q, \pm)$ with $c_i \in {}_{n_i}\widetilde{W}(M, q-1, \mp)$ $1 \leqslant i \leqslant t$. Then, for every couples of non-negative integers (l, m) with $l+m=n$, we **have**

$$ {}_{l,m}\Delta(M, q, \pm)\, c = \Sigma\, (-1)^{e_\pi(c, \pm)} g(\pi) (\langle c_{\pi(1)}, \dots, c_{\pi(u)} \rangle \otimes \langle c_{\pi(u+1)}, \dots, c_{\pi(t)} \rangle) $$

where the summation, $g(\pi)$ are as in 9.2, and the exponent

$$ e_\pi(c, \pm) = \Sigma'\, (|c_i| + \varepsilon(\pm)\, n_i + 1)\, (|c_j| + \varepsilon(\pm)\, n_j + 1) $$

where Σ' is again as in 9.2. Further, here and in what follows, we let $\varepsilon(+) = 1$, $\varepsilon(-) = 0$.

Now, we define

$$ {}_n\triangle(M, q, \pm) = \bigoplus_{l+m=n} {}_{l,m}\triangle(M, q, \pm) \qquad (10.12) $$

by the formula

$$ {}_{l,m}\triangle(M, q, \pm)\, c = \Sigma(-1)^{e_\pi(c, \pm)} g(\pi) (\langle c_{\pi(1)}, \dots, c_{(u)} \rangle \otimes \langle c_{\pi(u+1)}, \dots, c_{\pi(t)} \rangle) \,. $$

where c and the summation are as in the above lemma, and the exponent

$$ e_\pi(c, \pm) = e_\pi(c, \pm) + \sum_{i=1}^{u} (|c_i| + \varepsilon(\pm) n_i + 1). $$

Then we have the following

10.13 Lemma. The boundary formula on $\widetilde{W}(M, q, \pm)$ is as follows. Let $c = \langle c_1, \dots, c_t \rangle \in \widetilde{W}(M, q, \pm)$ with $c_i \in {}_{n_i}\widetilde{W}(M, q-1, \pm)$. Then

$$ \partial_\pm c = -\sum_{i=1}^{t} (-1)^{e_{i-1}(c, \pm)} \langle c_1, \dots, \partial_\mp c_i, \dots, c_t \rangle + $$

$$ + \sum_{i=1}^{t} (-1)^{e_i(c, \pm)} \langle c_1, \dots, {}_{n_i}\triangle(\pm)\, c_i, \dots, c_t \rangle $$

where $\quad {}_{n_i}\triangle(\pm) = {}_{n_i}\triangle(M, q-1, \pm)$ and

$$e_i(c, \pm) = \sum_{k=1}^{i} (|c_k| + \varepsilon(\pm)\, n_k + 1).$$

To show Lemma 10.11 and Lemma 10.13, we need only to compare the signatures as we have done in proving 8.4. we omit the proofs.

Now, we define the isomorphism of $mDGA$-coalgebras

$$\psi_q : W(M, q) \to FW(\mathcal{S}M, q-1) \tag{10.14}$$

for $q \geqslant 1$. Here, by the relations

$$\psi_q < (\omega_1, x), \ldots, (\omega_t, x_t) > \; = \; < (\omega_1, sx_1) | \ldots | (\omega_t, sx_t) >$$

Here $s: (IM)^n \to (I\mathcal{S}M)^n$ denotes the map given by

$$s(x_1 \otimes \ldots \otimes x_n) = \sigma x_1 \otimes \sigma x_1 \ldots \otimes \sigma x_n .$$

Since $|sx| = |x| + n$ for $x \in (IM)^n$, ψ_q is clearly an isomorphism of graded modules according to 10.10 and (ii) 5. 4. Further, it is a morphism of $mDGA$-coalgebras by the above lemmata and the definition 5.4 of cobar construction.

Theorem 10.6 follows by induction from this isomorphism.

Recall that the cobar construction is defined to be an DGA-algebra. Via the isomorphism $W(M, q) \simeq FW(\mathcal{S}M, q-1)$, $W(M, q)$ is thus equipped with a DGA-algebra structure. Clearly it is also an $mDGA$-algebra with the multiplicity $\mu(\omega, x)) = \mu(\omega)$. Explicitely, we have the multiplication formula on $W(M, q)$:

$$< (\omega_1, x_1), \ldots, (\omega_l, x_l) > < (\omega_{l+1}, x_{l+1}), \ldots, (\omega_{l+u}, x_{t+u}) > =$$
$$= < (\omega_1, x_1), \ldots, (\omega_{l+u}, x_{t+u}) > \tag{10.15}$$

Remark that 10.15 introduces also a multiplication on $\widetilde{W}(M, q)$ by which $\widetilde{W}(M,q)$ (particularly, $\widetilde{W}(q, +)$) becomes an $mDGA$-algebra.

An important application of the algebra $W(M, q)$ can be found in the following

10. 16 Theorem. Let X be in CG' and let $C_*(X)$ be a DGA-module *such that* $C_*(X) \approx S_*(X)$. *Then the singular chain complex of the filtered* $C(X, q)$ *is homotopic to the mDGA-algebra* $F^q \mathcal{S}^q C_*(X)$ *for* $q > 0$.

Proof. In this proof, we let $F_n = F(\mathbf{R}^q, n)$. Then, we have $\Phi : F_l \times F_m \to F_{l+m}$, the map given in 4.10. By a similar argument in proving Theorem 8.6 we observe that the theorem is proved if we can show that 10.15 for $\widetilde{W}(q, +)$ gives rise to an approximation for the map $\Phi : F_l \times F_m \to F_{l+m}$. In the other words, we need to prove the following diagram of morphisms of $DG\ \mathbf{G}_n$ — modules is homotopically commutative.

$$\begin{array}{ccc} {}_l\tilde{\tilde{W}}(M,+)\otimes {}_m\tilde{W}(q,+) & \longrightarrow & {}_{l+m}\tilde{\tilde{W}}(q,+) \\ \downarrow & \Phi_* & \downarrow \\ S_*(F_l)\otimes S_*(F_m) & \longrightarrow & S_*(F_{l+m}) \end{array}$$

where the vertical maps are obtained from Lemma 10.4.

Set $S_{l+m} = (\underset{l+m}{\times} \mathbf{R}^q)^{\cdot}$ and let T_{l+m} and T'_{l+m} denote the complement of F_{l+m} and $\Phi(F_l \times F_m)$ in S_{l+m} respectively. Then we have $S_{l+m}/T_{l+m}=F_{l+m}$ and $S_{l+m}/T'_{l+m}= F_l^{\cdot}\wedge F_m^{\cdot}$. Thus applying to the naturality of the Poincaré — Lefschetz duality with respect to the inclusion to the case $(S_{l+m}, T_{l+m}) \subset (S_{l+m}\, T'_{l+m})$, the following diagram is commutative

$$\begin{array}{ccc} \bar{S}_*(F_m^+ \times F_m^+) & \xrightarrow{i_*} & \bar{S}(F_{l+m}^+) \\ \downarrow & 1\otimes i^* & \downarrow \\ Z(q,l+m)\otimes \bar{S}_*(F_l^{\cdot}\wedge F_m^{\cdot}) & \underset{1\otimes j^*}{\overset{\longleftarrow}{\longrightarrow}} & Z(q,l+m)\otimes \bar{S}_*(F_{l+m}^{\cdot}). \end{array}$$

Here we have identified $F_l\times F_m = \Phi(F_l \times F_m)$, and $i : F_l \times F_m \subset F_{l+m}$. Moreover, let $j : F_{l+m} \to F_l \times F_m$ the map $(a_1,\ldots, a_{l+m}) \to ((a_1,\ldots,a_l), (a_{l+1},\ldots,a_{l+m}))$. Then $ji = id$. Thus $i_* j^* = id^*$, and the above diagram with dotted arrows is commutative

Now, let us consider the diagram

$$\begin{array}{ccccc} S_*(F_l^+)\otimes S_*(F_m^+) & \xrightarrow{\simeq} & S_*(F_l^+\wedge F_m^+) & \xrightarrow{i_*} & S_*(F_{l+m}^+) \\ \downarrow & & & & \downarrow \\ (Z(q,l)\otimes\bar{S}^*(F_l))\otimes(Z(q,m)\otimes(S^*(F_m)) & \simeq & Z(q,l+m)\bar{S}^*(F_l^{\cdot}\wedge F_m^{\cdot}) & \underset{1\otimes j^*}{\overset{1\otimes i^*}{\rightleftarrows}} & Z(q,l+m)\otimes\bar{S}^*(F_{l+m}^{\cdot}) \\ \downarrow & & & & \downarrow \\ (Z(q,l)\otimes {}_l\tilde{V}(q)^*)(Z(q,m)\otimes {}_m\tilde{V}(q)^*) & & \xrightarrow{\Delta^*} & & Z(q,l+m)\otimes {}_{l+m}\tilde{V}(q)^* \\ \downarrow & & & & \downarrow \\ {}_l\tilde{W}(q,+)\otimes {}_m\tilde{W}(q,+) & & \longrightarrow & & {}_{l+m}\tilde{W}(q,+) \end{array}$$

where Δ^* is obtained from the dual of the comultiplication

$$\Delta : {}_n\tilde{V}(q) \to \bigoplus_{l+m=n} {}_l\tilde{V}(q)\otimes {}_m\tilde{V}(q) \qquad \text{with}$$

$$\Delta[\alpha_1 \mid \ldots \mid \alpha_t] = \sum_{i=1}^{t} [\alpha_1 \mid \ldots \mid \alpha_i] \otimes [\alpha_{i+1} \mid \ldots \mid \alpha_t].$$

Obviously, to show that this diagram is homotopically commutative, we need only to show that so is the middle square. To this end, we observe that

$j : F^{\cdot}_{l+m} \to F^{\cdot}_{l} \wedge F^{\cdot}_{m}$ is a cell map. Further, for each $\alpha = (r_1, \ldots, r_{l+m}) \in F(l+m, q)^{\cdot}$, we have if

$$j_*(\alpha) = \begin{cases} (r_1, \ldots, r_l) \otimes (r_{l+1}, \ldots, r_{l+m})/r_{l+1} = q \\ 0 \qquad\qquad \textit{otherwise.} \end{cases}$$

In the other words, we have

$$j_*([\alpha_1 \mid \ldots \mid \alpha_q]) = \begin{cases} [\alpha_1 \mid \ldots \mid \alpha_i] \otimes [\alpha_{i+1} \mid \ldots \mid \alpha_l] \\ \qquad \textit{if } i \textit{ such that } (\alpha_1 \mid \ldots \mid \alpha_i]) = l \\ 0 \textit{ otherwise.} \end{cases}$$

This shows the commutativity of the middle square. The theorem follows.

Let $\mathbf{R}^q$ be imbedded in $\mathbf{R}^{q+1}$ by $(a^1, \ldots, a^q) \to (a^1, \ldots, a^q, 0)$ as usual. Then we have the canonical inclusions $F(\mathbf{R}^q, n) \subset F(\mathbf{R}^{q+1}, n)$ and $C(X, q) \subset C(X, q+1)$. From this, we define

$$F(\mathbf{R}^\infty, n) = \varinjlim_q F(\mathbf{R}^q, n)$$

$$C(X, \infty)) = \varinjlim_q C(X. q)$$

with the topology of union.

By the remark below Lemma 10.4, we can prove easily that the diagram

$$\begin{array}{ccc} {}_n\widetilde{W}(q, +) & \xrightarrow{\subset} & {}_n\widetilde{W}(q+1, +) \\ \wr\| & & \wr\| \\ S_*(F(\mathbf{R}^q, n)) & \longrightarrow & S_*(F(\mathbf{R}^{q+1}, n) \end{array} \qquad (10.17)$$

is homotopically commutative for every $q \geqslant 0$.

On the other hand, by the inclusion $\widetilde{W}(q, +) \subset \widetilde{W}(q+1, +)$, W(M, q) can be considered as a $mDGA$-subalgebra of W(M, $q+1$). Thus, we can define the $mDGA$-algebra

$$W(M, \infty) = \varinjlim_q W(M, q)$$

Further, via the isomorphism $W(M, q) \cong F^q\mathcal{S}^q M$, we consider $F^q\mathcal{S}^q M \subset F^{q+1}\mathcal{S}^{q+1}M$, and then define the $mDGA$-algebra

$$F^\infty\mathcal{S}^\infty M = \varinjlim_q F^q\mathcal{S}^q M.$$

Now, from 10.16 and 10.17, we have the following

10.18 Theorem. *Under the assumption of Theorem 10.16, the singular chain complex of the monoid* $C(X, \infty)$ *is homotopic to the mDGA-algebra* $F^\infty \mathcal{S}^\infty C_*(X)$.

In connection with the results on the homology groups of the symmetric groups and the braid groups given in § 7 and § 9, here are some immediate consequences of the result in this section.

First from 10.8 and the proof of 9.6, we have

10.19 Theorem. If $0 \leqslant i < q$, we have

$$H_i(\mathbf{G}_n ; \Lambda) \cong H_i({}_nF^q\mathcal{S}^q ; \Lambda).$$

Therefore, we have the isomorphism of groups

$$H_*(\mathbf{G}_n ; \Lambda) \cong H_*({}_nF^\infty\Sigma^\infty ; \Lambda).$$

Further, from the proof of 7.10 and 10.8, we have

10.20 Theorem. *We have the isomorphism of groups*

$$H_*(\mathcal{B}_n ; \Lambda) \cong H^*({}_nF^2\mathcal{S}^2 ; \Lambda).$$

Moreover, the diagram

$$\begin{array}{ccc} H_*(\mathcal{B}_n ; \Lambda) & \longrightarrow & H_*(\mathbf{G}_n ; \Lambda) \\ \wr\| & & \|\wr \\ H_*({}_nF^2\mathcal{S}^2 ; \Lambda) & \longrightarrow & H_*({}_nF^\infty\mathcal{S}^\infty ; \Lambda) \end{array}$$

is commutative. Here, the upper horizontal arrow is induced from the canonical epimorphism $\mathcal{B}_n \to \mathbf{G}_n$ (see e.g. [12]).

Note that the above results is true also for $n = \infty$ when we define ${}_\infty F^q\mathcal{S}^q = \varinjlim_n {}_nF^q\mathcal{S}^q$ via the inclusion ${}_nF^q\mathcal{S}^q \subset {}_{n+1}F^q\mathcal{S}^o$ given by $c \to c.\sigma^o$ for $c \in {}_nF^q\mathcal{S}^q$ where $\sigma^o \in {}_1F^q\mathcal{S}^q$.

Remark. Theorem 10.19 is an explanation for the theorem of Barratt — Priddy—Quillen (see [4], [27]) relating to the infinite symmetric group and to the infinite loop space. i.e. $H_*(\mathbf{G}_\infty ; \Lambda) \cong H_*((\Omega^\infty\Sigma^\infty)_o ; \Lambda)$. On the other hand, the first part of Theorem 10.20 is a proof of a Segal's theorem relating to $\mathcal{B}_\infty$ and to $\Omega^2\Sigma^2$, i.e. $H_*(\mathcal{B}_\infty ; \Lambda) \cong H_*((\Omega^2\Sigma^2)_o ; \Lambda)$, (see [5]).

§ 11. AN APPLICATION

For every $X \in CG'$ and $l \leq q < \infty$, let us denote as usual by $\Omega^q \Sigma^q X$ the q-iterated loop space on $\Sigma^q X$, and for $q = \infty$, $\Omega^\infty \Sigma^\infty X = \lim \Omega^q \Sigma^q X$ via the canonical inclusion $\Omega^q \Sigma^q X \to \Omega^{q+1} \Sigma^{q+1} X$. Then we recall from May [17; 5. 2] that there exists uniquely (up to a homotopy) a natural map of H-spaces $\alpha_q : C(X, q) \to \Omega^q \Sigma^q X$ such that the diagram

$$\begin{array}{ccc} & X & \\ \swarrow & & \searrow \\ C(X, q) & \xrightarrow[\alpha_q]{} & \Omega^q \Sigma^q X \end{array}$$

is commutative. Moreover, the diagram of H-maps

$$\begin{array}{ccc} C(X, q) & \xrightarrow{\alpha_q} & \Omega^q \Sigma^q X \\ \downarrow & & \downarrow \\ C(X, q+l) & \xrightarrow[\alpha_{q+l}]{} & \Omega^{q+l} \Sigma^{q+l} X \end{array}$$

is commutative for $q < \infty$, and α_∞ is obtained from the α_q by passage to limits.

We have the following well known theorem.

11.1 Theorem. *If X is a path-wise connected space in CG', then $\alpha_q : C(X, q) \to \Omega^q \Sigma^q X$ is a weak homotopy equivalence for $l \leqslant q \leqslant \infty$.*

This theorem has been the object in the works of many authors. For $q = l$, it is essentially due to James [15]. For $q = \infty$, it was obtained by Barratt [3], Segal (see [1]). For all q, it was proved by May [17]. Further we have

11.2 Theorem (*May* [19], *Segal* [28]). *For every $X \in CG'$ and $l \leqslant q \leqslant \infty$, $\alpha_q : C(X, q) \to \Omega^q \Sigma^q X$ is a group completion.*

For the notion of group completion, we refer to May [18 ; 1.3]. Segal reached to this theorem by a quite sophiscated appreach. For $q = \infty$, May first obtained it by an extensive homological analysis. Later, F. Cohen has followed May to prove the theorem for q finite in [7 ; 3.3]. Now, we apply our homological study of the space $C(X, q)$ to prove the theorem.

Proof of Theorem 11. 2. Let BG denote the classifying space of an H-space G. Then, according to Quillen, the map $G \to \Omega BG$ is a group completion if G is an admissible H-space (see e.g. [19], [20]). Apply this result to the case of $C(X, q)$, the map

$$C(X, q) \to \Omega BC(X, q) \qquad (*)$$

is a group completion. On the other hand, we have from Theorem 5. 7 that $BF^q \mathcal{S}^q S_*(X) \simeq F^{q-1} \mathcal{S}^q S_*(X)$. This implies that

$$BC(X, q) \to C(\Sigma X, q-1)$$

is a weak homotopy equivalence according to Theorem 10. 16. Combine this relation with Theorem 11.1, we have the weak homotopy equivalence

$$BC(X, q) \to \Omega^{q-1} \Sigma^q X \qquad (**)$$

since ΣX is connected, Theorem 11.2 now follows from $(*)$ and $(**)$.

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