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p-ADIC INTERPOLATION AND THE MELLIN–MAZUR TRANSFORM

HÀ HUY KHOÁI
Institute of Mathematics, Hanoi, Vietnam

Introduction. One of basic problems of p-adic analysis and number theory is to construct p-adic analogs of Archimedean concepts. Here p-adic L-functions are of special interest. The construction of p-adic analogs of L-functions uses basically two methods, which are actually closely related : the method of p-adic interpolation and the p-adic Mellin-Mazur transform. There is a conjecture of Mazur and Swinnerton-Dyer concerning the Mellin-Mazur transform corresponding to a Weil elliptic curve, which says that the p-adic L-function is not identically zero.

This paper studies the analytic properties of certain large classes of p-adic analytic functions, in particular, their interpolation properties and their integral representations, and then applies these results to give a partial confirmation of the conjecture of Mazur and Swinnnerton-Dyer.

Essential differences between p-adic analytic and complex analytic functions arise, of course, from having a non-Archimedean ground field. One such difference is that the modulus of a p-adic analytic function only depends on the modulus of the argument, except of a discrete set of values of the modulus of the argument. Hence, the graph of the modulus of a p-adic analytic function is a polygonal line, known as the Newton polygon of the function.

The Newton polygon determines other interesting properties of a p-adic analytic function, facilitates the interpolation process, and highlights the dependence of the function on its zeros. These properties allow us, in a rather simple way, to extend a function defined on a sequence of points to a p-adic analytic function. It was in this way that many interesting p-adic analogs of arithmetic functions were obtained ([2], [3],...).

In the present paper the concept of a Newton polygon is generalized and then applied to study the interpolation and continuation properties of p-adic analytic functions. We obtain some well-known results, of p-adic analytic functions. We obtain some well-known results, as well as new results, about interpolation. In §1 we introduce the concept of the sequence of Newton polygons of a p-apic analytic function $f(z)$ defined in the unit disc in C_p. These are the Newton polygons of the functions $\widetilde{f}_k(z)$ if the p-adic analytic function $f(z)$ is written in the form $f(z) = \sum_{k=0}^{\infty} \widetilde{f}_k(z)$, where only leading terms (see §1.2) occur in the expansion of $\widetilde{f}_k(z)$. Theorem 1.1 gives the basic properties of the Newton sequence.

In §2 we consider the problem of p-adic interpolation. The results of this section show that a key role in interpolation is played by the relation between the rate of growth of a function and the «number» of points between which the function is being interpolated. Namely, if u is a discrete sequence of points in the unit disc in C_p, and Φ_u is an analytic function in this disc for which the number of zeros (counting multiplicity) in every subregion is equal to the number of points of u in the same subregion (such a function is constructed in the proof of Theorem 1.1), then u is an interpolation sequence for $f(z)$, if and only if $f(z)$ belongs to the class $o(\Phi_u)$. This theorem is used to obtain some well-known results on interpolation of the Mellin-Mazur transform corresponding to modular forms. The interpolation theorem also gives conditions for a function given on a discrete sequence of points to be extendible to p-adic analytic function in the unit disc in C_p. Another consequence of the theorem is a uniqueness theorem (Theorem 4.1) for p-adic analytic functions, which concerns the question of the extent to which a p-adic analytic function is determined by its zeros.

In §5 we give a partial confirmation of the conjecture of Mazur and Swinnerton-Dyer on the non-vanishing of the p-adic L-function associated to an elliptic curve ([9]). The following theorem is proved : Let $N = l^n$, where l is a prime number and n is any positive integer, let p be a primitive root mod N. Then for every finite abelian extension A of Q of conductor $m = p^r$ and every Weil elliptic curve E of conductor N, the p-adic L-function L_p $(E/A, s)$ is not identically zero.

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§1. p-ADIC ANALYTIC FUNCTIONS

1. The Newton polygon. For completeness we shall recall the concept of the Newton polygon.

In what follows p will always denote a fixed prime number. Q_p is the field of p-adic number, and C_p is the p-adic completion of the algebraic closure of Q_p. We let T denote the disc in C_p : $T = \{z \in C_p, |z| < 1\}$. The absolute value is normalized as follows : $|p| = \frac{1}{p}$. We also use the notation $v(z)$ for the additive valuation on C_p, which extends ord_p.

Now let $f(z)$ be a p-adic analytic function on T, represented by the power series:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n .$$

For each n we draw the graph Γ_n which depicts $v(a_n z^n)$ as a function of $v(z)$. This is a straight line with slope n. Since $\lim_{n\to\infty} \{ v(a_n) + nt \} = \infty$ for all $t > 0$, there exists an n for which $v(a_n) + nt$ reaches its minimum. Let $v(f, t)$ be the boundary of the intersection of the half-planes under all the Γ_n. Then in every segment $t \in [r, s]$, $0 < r < s < \infty$, only finitely many of the Γ_n appear in the graph of $v(f, t)$. Hence, $v(f, t)$ is a continuous polygonal line, called the Newton polygon of the function $f(z)$. The points $t > 0$ corresponding to the vertices of the graph of $v(f, t)$ are called the critical points of $f(z)$. There are finitely many of them in any finite segment $[r, s]$. Obviously, if t is a critical point, then the number $\{ v(a_n) + nt \}$ reaches a minimum at at least two values of n. For other values of t we have:

$$v(f(z)) = v(a_n) + nt = \min_m \{ v(a_m) + mt \},$$

for all z, $v(z) = t$. Thus, for non-critical t,

$$|f(z)| = p^{-v(f,t)} \text{ if } v(z) = t.$$

The Newton polygon gives complete information about the number of zeros of the function. Namely, $f(z)$ has zeros for $v(z) = t_i$, where $t_o > t_1 > t_2 > \ldots$ are the critical points of the function, and the number of zeros of $f(z)$ on $v(z) = t_i$ (counting multiplicity) is equal to the difference $n_{i+1} - n_i$ between the slopes of $v(f, t)$ at $t_i - 0$ and $t_i + 0$. It is easy to see that n_i and n_{i+1} are the least n and the greatest n at which $\{ v(a_n) + nt \}$ reaches its minimum. These terms $a_n z^n$ for which the graph of $v(a_n z^n)$ takes part in forming $v(f, t)$ are called the leading terms in the expansion of $f(z)$.

Example. Consider the function

$$\log(1 + z) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{z^n}{n}$$

For every $t > 0$ we have:

$$v((-1)^{n-1}/n) + nt \begin{cases} = nt - \log n/\log p, & \text{if } n = p^k \\ > nt - \log n/\log p, & \text{otherwise.} \end{cases}$$

Hence, only the graphs Γ_{p^k} $(k = 0, 1, 2, \ldots)$ take part in forming $v(f, t)$, and the leading terms are:

$$p^{-k} z^{p^k}, \qquad k = 0, 1, 2, \ldots$$

It hence follows that the function $\log(1 + z)$ has the following critical points:

$$t_k = \frac{1}{p^k - p^{k-1}} = \frac{1}{\varphi(p^k)}, \quad k = 1, 2, \ldots$$

At each t_k, $\log(1 + z)$ has $\varphi(p^k)$ zeros, and

$$v(f, t_k) = -k + \frac{p}{p-1}.$$

2. The Newton sequence of an analytic function.

Let $\mathcal{H}$ denote the space of functions analytic in T with the topology of uniform convergence on the sets $\{z \in C_p : v(z) \geqslant t > 0\}$. Suppose we have a function $f(z) \in \mathcal{H}$, represented by a convergent series

$$f(z) = \sum_{n=o}^{\infty} a_n z^n.$$

Let $\{a_{n_k} z^{n_k}, k = 1,2,\dots\}$ be the set of leading terms of the séries (1). We set

$$\widetilde{f}(z) = \sum_{k=1}^{\infty} a_{n_k} z^{n_k}.$$

We define the sequence

$$\widetilde{f}_o(z), \ \widetilde{f}_1(z), \dots, \widetilde{f}_m(z), \dots$$

inductively, by setting

$$\widetilde{f}_o(z) = \widetilde{f}(z), \ \widetilde{f}_m(z) = f(z) - \sum_{i=o}^{m-i} \widetilde{f}_i(z).$$

It is clear that

$$f(z) = \sum_{k=o}^{\infty} \widetilde{f}_k(z) \tag{2}$$

(where we use the fact that convergence of series in C_p is always absolute convergence).

We note that the expansion of each $\widetilde{f}_k(z)$ $(k = 0,1,\dots)$ has only leading terms (unless $\widetilde{f}_k(z) \equiv 0$ for k sufficiently large).

For each $f(z) \in \mathcal{H}$ with expansion (2) we set

$$\rho_k^f(t) = v(\widetilde{f}_k, t),$$

where $t > 0$, $k = 0, 1, 2,\dots$ If the function $f(z)$ is fixed, we shall omit the f in the notation $\rho_k^f(t)$. We take

$$\rho_k^f(t) \equiv \infty, \text{ if } \widetilde{f}_k(t) \equiv 0.$$

Definition 1.1. The sequence

$$\rho^f(t) = \left(\rho_o^f(t), \quad \rho_1^f(t), \dots\right)$$

is called the Newton sequence of the function $f(z)$.

The basic properties of the Newton sequence are given in the following theorem.

Theorem 1.1. Let $\rho = (\rho_0(t), \rho_1(t), \ldots)$ be the Newton sequence of an analytic function $f(z) \in \mathcal{H}$. Then ρ has the following properties:

1. $\rho_{k+1}(t) \geqslant \rho_k(t)$ for all $k = 0, 1, 2, \ldots$ and $t > 0$

2. The functions $\rho_k(t)$ are continuous and left differentiable, and their derivatives are monotonic decreasing, piecewise linear functions which take non-negative integer values.

3. Let $\frac{d\rho_k}{dt}$ denote the left derivative of $\rho_k(t)$ at t. Then for each k: either there exists $M_k > -\infty$ such that $\rho_k(t) > M_k$ for $t > 0$, or else, for any $t_0 > 0$,

$$\lim_{t \to 0} \left\{ \rho_k(t) + (t_0 - t) \frac{d\rho_k}{dt} \right\} = \infty.$$

4. For any pair $i \neq k$, $d\rho_i/dt$ and $d\rho_k/dt$ have disjoint sets of values, except for the case $\rho_k(t) \equiv \rho_i(t) \equiv \infty$.

Conversely, for any sequence $\rho(t)$ satisfying 1) — 4), there exists an analytic function $f(z)$ such that $\rho^f(t) \equiv \rho(t)$.

Proof. By the definition of the operation $f \to \tilde{f}$ and the properties of the Newton polygon, we have: $v(f, t) = v(\tilde{f}, t)$ for all $f(z) \in \mathcal{H}$ and $t > 0$. This implies that:

$$v(\tilde{f}_{k+1}, t) = v(f - \tilde{f}_0 - \ldots - \tilde{f}_k, t) \geqslant \min \{ v(f - \tilde{f}_0 - \ldots - \tilde{f}_{k-1}, t), v(\tilde{f}_k, t) \} =$$
$$= \min \{ v(\tilde{f}_k, t), v(\tilde{f}_k, t) \} = v(\tilde{f}_k, t).$$

This proves 1).

Properties 2) and 4) are direct consequences of the definitions of Newton polygons and sequences.

We now prove 3). Suppose that for k we have: $\lim_{t \to 0} \rho_k(t) = -\infty$. Note that $\rho_k(t) + \frac{d\rho_k}{dt}(t_0 - t)$ is the ordinate of the point of intersection of the line supporting the graph $\rho_k(t)$ and the line $t = t_0$. Since $d\rho_k/dt$ is non-increasing, this ordinate does not decrease as $t \to 0$. Suppose there exists M such that

$$\rho_k(t) + \frac{d\rho_k}{dt}(t_0 - t) \leqslant M.$$

Consider the points $A = (t_1, \rho_k(t_1))$ and $B = (t_0, M)$, where $t_1 < t_0$. Then for $t < t_1$ the entire line $\rho_k(t)$ lies above the line AB, and this contradicts the assumption that $\lim_{t \to 0} \rho_k(t) = -\infty$.

We now prove the second part of the theorem. Suppose that the sequence $\rho(t) = (\rho_0(t), \rho_1(t), \ldots)$ satisfies 1) — 4). Set: $\rho'_k(t) = \frac{d\rho_k}{dt}$. Let $t_1^k > t_2^k > \ldots > t_n^k > \ldots$

be the sequence of points of discontinuity of the function $\rho'_k(t)$, and let $\{a_n^k\}$ be a sequence of numbers such that

$$v(a_n^k) = \rho_k(t_n^k) - t_n^k \rho'_k(t_n^k).$$

Recalling that $\rho'_k(t)$ takes integer values, we set

$$P_k(z) = \sum_{n=1}^{\infty} a_n^k z^{\rho'_k(t_n^k)}, \tag{3}$$

$$f(z) = \sum_{k=0}^{\infty} P_k(z) \tag{4}$$

We prove that (3) and (4) converge in the topology of $\mathscr{H}$, and that $f(z)$ is an analytic function in T satisfying the relations:

$$\widetilde{f}_k(z) = P_k(z), \ v(P_k, t) = \rho_k(t)$$

for $t > 0$, in other words, $\rho(t)$ is the Newton sequence for $f(z)$.

We note that if $\rho_k(t) \equiv \infty$, then we take $P_k(z) = 0$ and if $\rho'_k(t)$ only has finitely many points of discontinuity, then the series (3) is a finite sum.

It suffices to consider the case when $\rho'_k(t)$ has infinitely many points of discontinuity. In this case it is easy to see that $\lim\limits_{n\to\infty} t_n^k = 0$. Hence, by property 3), we have

$$\lim_{n\to\infty} \{\rho'_k(t_n^k)(t - t_n^k) + \rho_k(t_n^k)\} = \infty.$$

This shows that (3) converges

We show that only leading terms occur in the expansion of each $P_k(z)$, and that $v(P_k, t) = \rho(t)$. We set:

$$\varphi_k^n(t) = \rho'_k(t_n^k)(t - t_n^k) + \rho_k(t_n^k).$$

It is easy to verify the equality:

$$\varphi_k^{n+1}(t_{n+1}^k) = \varphi_k^n(t_{n+1}^k) = \rho_k(t_{n+1}^k).$$

Since $\varphi_k^{n+1}(t)$ and $\varphi_k^n(t)$ are linear and $\rho'_k(t_{n+1}^k) > \rho'_k(t_n^k)$, we have

$$\varphi_k^{n+1}(t) > \varphi_k^n(t) \text{ (for } t > t_{n+1}^k).$$

Similarly,

$$\varphi_k^{n+r}(t) > \varphi_k^{n+r-1}(t) \text{ (for } t > t_{n+r}^k) > \varphi_k^{n+r-2}(t) \text{ (for } t > t_{n+r-1}^k)\ldots$$

Since $t^k_{n+1} > t^k_{n+2} > \dots > t^k_{n+r}$, we have $\varphi^{n+r}_k(t) > \varphi^n_k(t)$ for $t > t^k_{n+1}$. Thus

$$\varphi^n_k(t) = \inf_{m \geqslant n} \varphi^m_k(t) \text{ for } t \geqslant t^k_{n+1}$$

A similar proof gives us

$$\varphi^n_k(t) = \inf_{m < n} \varphi^m_k(t) \quad \text{for} \quad t \leqslant t^k_n.$$

Consequently,

$$\varphi^n_k(t) = \inf_m \varphi^m_k(t) \text{ for } t \in [t^k_{n+1}, t^k_n].$$

This implies that $P_k(z) \equiv \widetilde{P}_k(z)$ and $v(P_k, t) = \rho_k(t)$.

We now prove that the series (4) converges in T. Since $v(P_k, t) = \rho_k(t)$, it suffices to prove that $\lim\limits_{k \to \infty} \rho_k(t) = \infty$ for any fixed $t > 0$. Suppose the contrary, i.e., that there exists $t_0 > 0$, $M > 0$, and a sequence k_n, such that $\lim\limits_{n \to \infty} k_n = \infty$ and $\rho_{k_n}(t_0) < M$ for all $n = 1, 2, \dots$ By properties 1) and 2), we have:

$$\rho_0(t_0/2) \leqslant \rho_{k_n}(t_0/2) \leqslant \rho'_{k_n}(t_0)(t_0/2 - t_0) + \rho_{k_n}(t_0) \leqslant \frac{-\rho'_{k_n}(t_0)}{2} t_0 + M.$$

We obtain a contradiction as $n \to \infty$, because $\lim\limits_{n \to \infty} \rho'_{k_n}(t_0) = \infty$ by property 4).

Since $P_k(z) \equiv \widetilde{P}_k(z)$ for all $k \geqslant 0$ and $v(P_k, t) \geqslant v(P_{k-1}, t)$ for all $t > 0$, it is easy to see that $\widetilde{f}_k(z) \equiv P_k(z)$. Thus, $\rho = (\rho_0(t), \rho_1(t), \dots)$ is the Newton sequence for $f(z)$. The theorem is proved.

3. Analytic functions on the character group.

This section devotes to a very important class of p-adic analytic functions: functions on the analytic group of characters of Z^*_Δ.

Let Δ_0 be an integer, $(\Delta_0, p) = 1$. We set:

$$q = \begin{cases} 4 \text{ if } p = 2 \\ p \text{ otherwise} \end{cases} \quad \Delta = \Delta_0 q \quad \text{and} \quad Z^*_\Delta = \varprojlim (Z/\Delta p^n Z)^*.$$

The p-adic character group is the group of continuous homomorphisms of Z^*_Δ to C^*_p:

$$X(Z^*_\Delta) = \mathrm{Hom}_{\mathrm{cont}}(Z^*_\Delta, C^*_p).$$

Every Dirichlet character χ of conductor Δp^n is an element of the group Hom $((Z/\Delta p^m Z)^*, C_p^*)$ for each $m \geqslant n$, and so gives a unique element of $X(Z_\Delta^*)$, which is also denoted χ.

We set: $U = 1 + q Z_p = \{ z \in Z_p : v(z-1) \geqslant v(q) \}$. Then for any $g \in U$ with $v(g-1) = v(q)$, the map $z \to g^z$ is an isomorphism of Z_p onto U. We call such a g a topological generator of U.

For any generator g of U, the map $\mathrm{Hom}_{\mathrm{cont}} (U, C_p^*) = X(U) \to C_p^*$ which takes a continuous character χ of U to the point $\chi(g) - 1$ is an isomorphism of $X(U)$ onto T. Since $Z_\Delta^* \approx (Z/\Delta_0 Z)^* \times Z_p$ and $Z_p^* \approx (Z/qZ)^* \times U$, it follows that $X(Z_\Delta^*)$ is the product of a finite group and $X(U)$, where the latter group is isomorphic to T. Since T is an open disc in C_p, this isomorphism makes $X(Z_\Delta^*)$ into an analytic manifold, in fact, a disjoint union of open discs. This analytic structure makes $X(Z_\Delta^*)$ into an analytic group.

Analytic functions on $X(Z_\triangle^*)$ are functions whose restriction to each component isomorphic to T is an analytic function on this component.

A function $f(z)$ on $Z_\triangle^*$ is called locally analytic if, for every point $z \in Z_\triangle^*$, there exists a disc $D_z \ni z$ such that $f(z)$ is analytic on D_z. We let Locan $Z_\triangle^*$ denote the space of locally analytic functions on $Z_\triangle^*$, together with its natural topology ([1]).

Definition 1. 2. A continuous linear functional on Locan $Z_\triangle^*$ is called a distribution on $Z_\triangle^*$.

The restriction to $X(Z_\triangle^*)$ of a continuous linear functional on Locan $Z_\triangle^*$ is an analytic function. Letting such a functional correspond to its restriction gives an isomorphism of the space of distributions with the space of analytic functions on $X(Z_\triangle^*)$ ([2]):

$$\text{Distlocan } Z_\triangle^* \approx \text{An } X(Z_\triangle^*).$$

We shall later prove some subtler facts about this correspondence, which relate to certain important constructions in p-adic analysis.

Let μ be a distribution on $Z^*_\triangle$, written symbolically as follows:

$$\mu(\varphi) = \int_{Z^*_\triangle} \varphi \, d\mu.$$

for φ an analytic function on $Z^*_\triangle$. Then restricting to $X(Z^*_\triangle)$ gives a function

$$f(\chi) = \int_{Z^*_\triangle} \chi \, d\mu,$$

which is analytic on $X(Z^*_\triangle)$ and is called the p-adic Mellin — Mazur transform. This is the p-adic analog of L-series.

Definition 1. 3. A distribution on $Z^*_\triangle$ which extends to the space of continuous functions on $Z^*_\triangle$ is called a bounded measure on $Z^*_\triangle$. If a distribution extends to the space of functions which are $h-1$ times differentiable and whose $(h-1)$-st derivative satisfies the Lipschitz condition, then this distribution is called an h-admissible measure.

Definition 1. 4. Let $f(z)$ and $g(z)$ be two analytic functions in $\mathcal{H}$. We say that $f(z)$ belongs to the class $o(g)$ if

$$\sup_{|z|\leqslant r} |f(z)| = o\,(\sup_{|z|\leqslant r} |g(z)|) \text{ as } r \to 1-0.$$

It has been proved that for a bounded measure the function $f(\chi) = \int_{Z^*_\triangle} \chi d\mu$ bounded, and for an h-admissible measure this function belongs to the class $o(\log^h(1+z))$ ([3]). Here we shall prove that every bounded analytic function $f(\chi)$ on $X(Z^*_\triangle)$ (resp. any analytic function of class $o(\log^h)$) is the Mellin—Mazur transform of a bounded (resp. h-admissible) measure μ. Since $X(Z^*_\triangle)$ is isomorphic to the product of a finite group and $X(U)$, we shall carry out the proof for $X(U)$ and shall identify $X(U)$ and T by means of the isomorphism between them.

Theorem 1. 2. For any function $f(z) \in \mathcal{H}$ with $f(z) \in o(\log^h)$, there exists an h-admissible measure on U such that

$$f(\chi) = \int_U \chi \, d\mu.$$

Proof. In [3] it was proved that, if μ is a linear functional on the space of functions which are locally a polynomial of degree less than h, then μ extends to an h-admissible measure if and only if the following relation holds:

$$\sup_{a\in U} \left| \int_U (z-a)^j \psi_a^{(m)} (z) d\mu \right| = o\,(p^{m(h-i)}) \; . \tag{5}$$

where $j = 0,1,\ldots,h-1$; $\psi_a^{(m)}(z)$ is the characteristic function of the set $a + U_m$, $U_m = \{z \in Z_p : z \equiv 1 \bmod qp^m\}$. Moreover, in this case the p-adic Mellin — Mazur transform is an analytic function in χ and belongs to the class o ($\log^h$). Thus, in order to prove that the Mellin—Mazur transform of μ is equal to $f(\chi)$, it suffices to show that (5) holds for the characters $z^k\chi$, where $0 \leqslant k \leqslant h-1$ and χ is a Dirichlet character modulo p^m. This follows from results on p-adic interpolation in [2], [3], or else from the remark after Corollary 2. 1 in the present paper

The proof that (5) holds for $z^k\chi$ uses several lemmas

Lemma 1. 1. The formula

$$\mu\,(z^k \psi_a^{(m)}) = \frac{1}{\varphi\,(p^m)} \sum_{\chi} \chi^{-1}(a) f(z^k \chi) \tag{6}$$

where $0 \leqslant k \leqslant h-1$ and χ runs through the Dirichlet characters modulo p^m, defines a linear functional on the space of functions which are locally a polynomial in z of degree less than h.

Proof. It suffices to verify : $\mu\,(z^k \psi_a^{(m)}(z)) = \sum_{r=0}^{p-1} \mu\,(z^k \psi_{a+rp^m}^{(m+1)})$. This is proved by computing the right side by formula (6) and noting that, if χ is a primitive character of conductor p^{m+1}, then

$$\sum_{r=0}^{p-1} \chi^{-1}(a + rp^m) = 0.$$

Lemma 1. 2. The linear functional μ defined in Lemma 1. 1. satisfies the conditions:

$$\sup_{a\in U} \left| \int_U (z-a) \psi_a^{(m)}(z)\, d\mu \right| = o\,(p^{m(h-j)}),\; j = 0, 1,\ldots, h-1.$$

Proof. For every $g(z) \in \mathscr{H}$ and every $t_0 > 0$, we set:

$$\| g \|_{t_0} = \sup_{v(z)=t_0} | g(z) |$$

From the example in § 1 we obtain:

$\| \log^h (1+z) \|_{t_m} = p^{mh}$, where $t_m = 1/\varphi(p^m)$, $m = 1, 2,$... this implies that:

$$\| f \|_{t_m} = o\,(p^{mh}) \; (m \to \infty).$$

Now let $S_m(z)$ be the sequence of interpolating polynomial for $f(z)$ between the points $\{g^{i\gamma}-1\}$, $i=0,1,\ldots,h-1$, $\gamma\in\mathcal{M}_{pm}$, $m=1,2,\ldots$, where $\mathcal{M}_{pm}$ is the set of p^m- th roots of unity. $S_m(z)$ is defined by the conditions:
$\deg S_m(z)\leqslant h\,p^m-1$, $S_m(g^{i\gamma}-1)$, $i=0\ldots,h-1$, $\gamma\in\mathcal{M}_{pm}$. By Lazard's lemma ([5]), we have:

$$f(z)=\varphi(z)_{i=0,\ldots,h-1}\,\Pi_{\gamma\in\mathcal{M}_{pm}}\left(1-\frac{z}{g^{i\gamma}-1}\right)+Q_m(z)\text{ where }\deg Q_m(z)\leqslant h\,p^m-1,$$

$v(Q_m,t_m)\geqslant v(f,t_m)$. This implies that $S_m(z)\equiv Q_m(z)$ and $\|S_m\|_{t_m}=o(p^{mh})$.

Write the polynomial $S_m(z)$ in the form $S_m(z)=\sum_{l=0}^{hp^m-1}b_l^{(m)}z^l$. Then $\|S_m\|_{t_m}=$

$$=\max_{0\leqslant l\leqslant hp^m-1}\{|b_l^{(m)}z^l|_{t_m}\}=\max_l\{|b_l^{(m)}|p^{-l/\varphi(p^m)}\}>p^{-hp/p-1}\max_l\{|b_l^{(m)}|\}.$$

Thus, $|b_l^{(m)}|=o(p^{mh})$ for all l. We note that, if we write

$$S_m(z-1)=\sum_{l=0}^{hpm-1}a_l^{(m)}z^l,\text{ then we also obtain:}$$

$|a_l^{(m)}|=o(p^{mh})$ for all l. By the definition of the functional μ, we have:

$$\int_U(z-a)^j\psi_a^{(m)}(z)\,d\mu=\sum_{k=0}^{j}(-a)^{j-k}\binom{j}{k}(1/\varphi(p^m))\sum_\chi\chi^{-1}(a)f(z^k\chi)$$

$$=\quad\text{»}\quad\text{»}\quad\text{»}\ f(g^k\chi(g)-1)$$

$$=\quad\text{»}\quad\text{»}\quad\text{»}\ S_m(g^k\chi(g)-1)$$

$$=\quad\text{»}\quad\text{»}\quad\text{»}\ \Sigma a_l^{(m)}g^{kl}\chi^l(g)$$

$$=\sum_{k=0}^{j}(-a)^{j-k}\binom{j}{k}\sum_{g^l\equiv a\bmod qp^m}^{hp^m-1}a_l^{(m)}g^{kl}$$

$$=\Sigma a_l^{(m)}(g^l-a)^j.$$

Thus,

$$\sup_{a\in U}\left|\int_U(z-a)^j\psi_a^{(m)}(z)\,d\mu\right|=\sup_{a\in U}\left|\Sigma a_l^{(m)}(g^l-a)^j\right|=o(p^{m(h-j)}),\ j=0,\ldots,h-1$$

because $\quad|a_l^{(m)}|=o(p^{mh})$ and $|g^l-a|\leqslant p^{-m}$.

Thus, μ extends to an h-admissible measure, as claimed at the beginning of the proof. Theorem 1.2 will be proved if the equality $f(\chi)=\int_U\chi d\mu$ is verified for all characters $z^k\chi$ with χ a Dirichlet character.

Lemma 1.3. If χ is a Dirichlet character modulo p^m and $0 \leqslant k \leqslant h-1$, then

$$f(z^k\chi) = \int_U z^k \chi d\mu.$$

Proof. By the definition of the measure we have

$$\int_U z^k \chi d\mu = \int_U \Big(\sum_{a \bmod p^m} \chi(a)\, z^k\, \psi_a^{(m)}(z)\, d\mu$$

$$= \sum_{a \bmod p^m} \chi(a) \sum_{\overline{\chi}} \frac{1}{\varphi(p^m)}\, \overline{\chi}^{-1}(a) f(z^k \overline{\chi}) = f(z^k\chi).$$

This proves Lemma 1.3, and hence the theorem.

§ 2. p-ADIC INTERPOLATION

The construction of the p-adic zeta-function by interpolating from a set of integers ([4]) caused many people to become interested in the problem of p-adic interpolation. Amice and other specialists investigated interpolation of functions on a locally compact field. Using p-adic interpolation of an analytic function $T+1$ from the sequence $\{g^k\gamma\}$ (see § 1), Amice and Vélu obtained a p-adic Mellin-Mazur transform associated to modular forms which generalized results of Ju. I. Manin.

In this paper we investigate interpolation of analytic functions on T from an arbitrary discrete sequence of points.

Let $u = \{u_1, u_2 \dots\}$ be a sequence of distinct points in T. Let $N_u(t)$ denote the number of points u_i in the sequence u such that $v(u_i) \geqslant t > 0$. In what follows we shall only consider sequence u for which $N_u(t) < \infty$ for every fixed $t > 0$. We shall always assume that $v(u_i) \geqslant v(u_{i-1})$ $(i = 0, 1, 2, \dots)$ With these assumptions, we may write the sequence u in the form $u = \{u_0, u_1, \dots u_n, u_{n_1-1}, \dots u_{n_2}, \dots\}$ where:

$$v(u_i) = t_k \text{ for } n_{k-1} + 1 \leqslant i \leqslant n_k \; (n_0 = -1), \; \lim_{k\to\infty} t_k = 0.$$

We consider the function:

$$\rho_0(t) = \int_\infty^t N_u(t)\, dt.$$

It is clear that $d\rho_0(t)/dt = n_k$ for $t \in [t_k, t_{k+1}]$, and that $\rho(t) = \{\rho_0(t)\}$ satisfies the conditions of Theorem 1.1. Let $\Phi_u(z)$ be a function for which $\rho(t) = \{\rho_0(t)\}$ is the Newton sequence, we have: $d\rho^{\Phi_u}/dt = N_u(t)$, and, by the property of the Newton polygon, $\Phi_u(z)$ has n_k zeros of ordinal t_k. We shall only consider sequence u for which Φ_u is unbounded, or, equivalently, for which $\lim_{t\to 0} \rho_0(t) = -\infty$.

Definition 2.1. $u = \{u_i\}_{i=0}^{\infty}$ is called an interpolation sequence for $f(z)$ if the sequénce of interpolation polynomials for f on u converges to $f(z)$ in the topology of $\mathcal{H}$.

Theorem 2.1. *The sequence u is an interpolation sequence for $f(z)$ iff $f(z) = o(\Phi_u)$.*

Proof Suppose the sequence u and the function $f(z)$ satisfy the condition $f(z) \in o(\Phi_u)$.

We define a function $\tau : Z \to Z$ by the relation

$$t_{\tau(i)} = v(u_i) \quad (i = 0,1,2,...).$$

From the assumptions concerning u it is clear that τ is a nondecreasing function and $\lim_{i\to\infty} \tau(i) = \infty$.

Using Lazard's result and a proof similar to that of Lemma 1.2, we obtain :

$$v(P_k, t_{\tau(k)}) \geqslant v(f, t_{\tau(k)}).$$

We consider the expression

$$v(S_k, t_{\tau(k)}) - v(f, t_{\tau(k)}),$$

and examine separately the following possible cases:

i) $\tau(k) = \tau(k+1)$. In this case it follows Lazard's lemma that

$$v(P_{k+1}, t_{\tau(k)}) \geqslant v(f, t_{\tau(k)})$$

and, using (26), we obtain :

$$v(S_k, t_{\tau(k)}) - v(f, t_{\tau(k)}) \geqslant 0 \tag{15}$$

ii) $\tau(k) < \tau(k+1)$ and $N_u(t_{\tau(k)}) \geqslant \left.\frac{d\rho^f}{dt}\right|_{t=t_{\tau(k)}}$.

We show that in this case inequality (15) also holds. Suppose that, on the contrary, $v(S_k, t_{\tau(k)}) < v(f, t_{\tau(k)})$. Then the properties of the Newton polygon imply

$$\begin{aligned} v(S_k, t_{\tau(k+1)}) &= v(S_k, t_{\tau(k)}) - N_u(t_{\tau(k)})\,(t_{\tau(k)} - t_{\tau(k+1)}) \\ &\leqslant v(f, t_{\tau(k)}) - N_u(t_{\tau(k)})\,(t_{\tau(k)} - t_{\tau(k+1)}) \\ &\leqslant v(f, t_{\tau(k)}) - \left.\frac{d\rho^f}{dt}\right|_{t=t_{\tau(k)}} (t_{\tau(k)} - t_{\tau(k+1)}) \\ &= v(f, t_{\tau(k)}) \end{aligned} \tag{16}$$

But (14) implies that

$$v(P_{k+1}, t_{\tau(k+1)}) \geqslant v(f, t_{\tau(k+1)}). \tag{17}$$

Using (16) and (17), we obtain

$$v(P_k, t_{\tau(k+1)}) = v(S_k, t_{\tau(k+1)}). \tag{18}$$

On the other hand, it follows from (14) and our assumption that

$$v(P_k, t_{\tau(k)}) = v(S_k, t_{\tau(k+1)}). \tag{19}$$

Using (18) and (19), we obtain then

$$\left.\frac{d\rho^{P_k}}{dt}\right|_{t=t_{\tau(k)}} > \left.\frac{d\rho^{S_k}}{dt}\right|_{t=t_{\tau(k)}}.$$

But this is impossible, since the degree of P_k is no greater than k and the function $S_k(z)$ has no fewer than k zeros in $\{v(z) \geqslant t_{\tau(k)}\}$. This contradiction proves (15) in case ii)

iii) $\tau(k) < \tau(k+1)$ and $N_u(t_{\tau(k)}) < \left.\frac{d\rho^f}{dt}\right|_{t=t_{\tau(k)}}$.

From the construction of $P_k(z)$ and the properties of the Nowton polygon it follows that in this case we have:

$$\left.\frac{d\rho^{P_k}}{dt}\right|_{t=t_{\tau(k)}} \leqslant \left.\frac{d\rho^{S_k}}{dt}\right|_{t=t_{\tau(k)}} \leqslant N_u(t_{\tau(k)}) + 1 \leqslant \left.\frac{d\rho^f}{dt}\right|_{t=t_{\tau(k)}}. \tag{20}$$

From (14) and (20) it follows that

$$v(P_k, t_{\tau(k+1)}) \geqslant v(f, t_{\tau(k+1)}). \tag{21}$$

Inequalities (17) and (21) give

$$v(S_k, t_{\tau(k+1)}) \geqslant v(f, t_{\tau(k+1)}). \tag{22}$$

Note that $\tau(k) < \tau(k+1)$ if and only if $N_u(t_{\tau(k)}) = k+1$.

On the other hand, $S_k(z)$ has $k+1$ zeros of ordinal no less than $t_{\tau(k)}$, hence (19) holds in case iii) not only for $t > t_{\tau(k)}$, but for $t \geqslant t_{\tau(k)}$ as well.

Now let N be an arbitrary positive integer, and let k be large enough so that $t_{\tau(k)} < t_N$. By (18), we have the inequality:

$$|(v(S_k, t_{\tau(k)}) - v(\Phi_u, t_{\tau(k)})) - (v(S_k, t_N) - v(\Phi_u, t_N))| \leqslant t_1$$

in case i) and ii), and the inequality:

$$|(v(S_k, t_{\tau(k-1)}) - v(\Phi_u, t_{\tau(k+1)})) - (v(S_k, t_N) - v(\Phi_u, t_N))| \leqslant t_1 \tag{23}$$

in case iii).

Thus, in cases i) and ii) we have:

$$v(S_k, t_N) \geqslant v(\Phi_u, t_N) - t_0 + v(S_k, t_{\tau(k)}) - v(\Phi_u, t_{\tau(k)}).$$

It follows from 15) that:

$$v(S_k, t_N) \geqslant v(\Phi_u, t_N) - t_0 + v(f, t_{\tau(k)}) - v(\Phi_u, t_{\tau(k)}).$$

Similarly, using inequalities (22) and (23), in case iii) we obtain:

$$v(S_k, t_N) \geqslant v(\Phi_u, t_N) - t_0 + v(f, t_{\tau(k+1)}) - v(\Phi_u, t_{\tau(k+1)}).$$

Since $f \in o(\Phi_u)$ and $\lim_{k\to\infty} \tau(k) = \infty$ in all cases we have:

$$\lim_{k\to\infty} v(S_k, t_N) = \infty$$

This means that the sequence $\{P_k\}$ converges to some p-adic analytic function $P(z)$.

It remains to prove that $P(z) \equiv f(z)$. Since u is an interpolation sequence for $P(z)$, $v(P, t) = \lim v(P_n, t) \geqslant v(f, t)$ we have $P(z) \in o(\Phi_u)$. Set $g(z) = P(z) - f(z)$. It follows from the assumption and from what was just said that $g(z) \in o(\Phi_u)$. On the other hand, $g(u_i) = 0$, $i = 0, 1, 2, \dots$ we hence obtain

$$\frac{d\rho^g}{dt} \geqslant N_u(t) = \frac{d\rho^{\Phi_u}}{dt} \text{ for } t > 0. \tag{24}$$

But then $g(z) \equiv 0$, since otherwise (24) would contradict the fact that $g(z) \in o(\Phi_u)$. Sufficiency is proved.

As an obvious corollary, we obtain the following theorem.

Theorem 2.2. Let $f(z)$ be any function in $\mathcal{H}$, and let $u = \{u_i\}_{i=0}^{\infty}$ be a sequence of points in T satisfying the conditions: $N_u(t) < \infty$ for every $t > 0$, and $v(u_i) \geqslant v(u_{i+1})$. Then u is an interpolation sequence for all functions in $o(f)$ the function

$$N(t) = N_u(t) - \frac{d\rho^f}{dt}$$

is bounded from below for $t > 0$.

In fact, under the conditions in the theorem, it is easy to see that the class $o(f)$ is contained in the class $o(\Phi_u^f)$.

Corollary 2.1. (an important special case). The sequence $\{\Upsilon - 1, \Upsilon \in \mathcal{M}_{p^n}, n \geqslant 1\}$ is an interpolation sequence for all functions in $o(\log)$.

In fact, take for $f(z)$ the function $\log(1+z)$, and let u be the sequence in the corollary. Then $N(t) \equiv 0$, by the description of the Newton polygon of the log function. It is known that the p-adic Mellin-Mazur transform of a slowly increasing measure ([3, 6]) is a function of class $o(\log)$. Hence, such a transform is completely determined by its values on the set of Dirichlet characters.

An analogous result holds for the p-adic Mellin-Mazur transform of an h-admissible measure ([3]). It is known that all measure corresponding to parabolic modular forms are h-admissible for suitable h. It then follows that the Mellin-Mazur transform is a function of class $o(\log^h)$, and so is completely determined by its values on the set of characters of the form χz^k, $k = 0, 1, 2, \ldots, h-1$, where χ is a Dirichlet character (see § 1). Note that is this case, letting $f(z)$ denote the Mellin-Mazur transform, we have:

$$N(t) = N_u(t) - d\rho^f/dt \equiv 0$$

where u is the sequence of points $\{g^{k\Upsilon} - 1\}$.

§ 3. p - ADIC CONTINUATION OF ANALYTIC FUNCTIONS

As mentioned above, results of Amice and Mahler give conditions under which a function given on a set of integers (in the case of Mahler), on a «very well distributed» sequence or on certain other sequences of points, can be extended to a continuous, analytic or locally analytic function on Z_Δ^*. In this section we use results of the preceding sections to study extensions of a function given on an arbitrary sequence of points to an analytic function on T. We shall consider sequences satisfying the basic conditions of the preceding section and written in the same form as in § 2.

Theorem 3. 1. Let $u = \{u_i\}_{i=0}^{\infty}$ be a sequence of points in T, and let $\alpha = \{\alpha_i\}_{i=0}^{\infty}$ be a sequence of values in C_p. Further, let $\{P_n(z)\}$ be the sequence of polynomials satisfying the conditions :

$$\deg P_n(z) \leqslant n, \; P_n(u_i) = \alpha_i, \; i=0,\dots,n.$$

Then :

1) If the condition

$$\| P_n \| = \sup_{z \in T} | P_n(z) | = o(\| \Phi_u \|_{t_{\tau(n)}})$$

s fulfilled as $n \to \infty$, where the notation is as in § 2, then there exists an analytic unction $f(z) \in \mathcal{H}$ such that

$$f(u_i) = \alpha_i, \; i = 0,1,2,\dots$$

nd $f(z) = \lim P_n(z)$. In other words, under these conditions $\{u_i\}$ is an interpolation sequence for the function $f(z)$.

2) Conversely, if there exists an analytic function $g(z)$ in the class $o(\Phi_u)$ which satisfies the conditions

$$g(u_i) = \alpha_i, \; i=0,1,2,\dots,$$

hen the sequence of polynomials $P_n(z)$ satisfies the following condition :

$$\| P_n \| = o\,(p^{n t_{\tau(n)}} \| \Phi_u \|_{t_{\tau(n)}}).$$

Proof. Let the polynomial $P_n(z)$ have the expansion

$$P_n(z) = \sum_{k=0}^{n} a_k(n)\, z^k.$$

By assumption, we have :

$$v(a_k^{(n)}) \geqslant v(\Phi_u, t_{\tau(n)}) + d(n), \; k = 0,1,2,\dots$$

where $d(n) \to \infty$ as $n \to \infty$. It hence follows that :

$$v(P_n, t_{\tau(n)}) \geqslant \min_k \{v(\Phi_u, t_{\tau(n)}) + d(n) + k\, t_{\tau(n)}\}$$

$$= v(\Phi_u, t_{\tau(n)}) + d(n).$$

Arguments similar to those used to prove (23) above give :

$$\text{either } v(S_n, t_{\tau(n)}) = v(\Phi_u, t_{\tau(n)}) + d(n),$$

$$\text{or else } v(S_n, t_{\tau(n+1)}) = v(\Phi_u, t_{\tau(n+1)}) + d(n+1).$$

Hence, for any fixed N we have :

$$v(S_n, t_N) - v(\Phi_u, t_N) \geqslant v(S_n, t_{\tau(n)}) - v(\Phi_u, t_{\tau(n)}) = d(n).$$

Consequently,

$$\lim_{n \to \infty} v(S_n, t_N) = \infty$$

Thus, the sequence of polynomials $\{P_k(z)\}$ converges in the topology of $\mathcal{H}$ to a p-adic analytic function $f(z)$, where: $f(u_i) = \alpha_i$, $i = 0, 1, 2,\ldots$ The first part of theorem is proved.

Now let $g(z) \in o(\Phi_u)$ be an analytic function on T for which

$$g(u_i) = \alpha_i, \qquad i = 0, 1, 2, \ldots$$

Using Lazard's lemma, we see that the polynomial $P_n(z)$ satisfies:

$$v(P_n, t_{\tau(n)}) \geqslant v(g, t_{\tau(n)}).$$

Consequently, for all k with $0 \leqslant k \leqslant n-1$

$$v(a_k^{(n)}) + kt_{\tau(n)} \geqslant v(g, t_{\tau(n)}) = v(\Phi_u, t_{\tau(n)}) + d(n),$$

where $d(n) \to \infty$ as $n \to \infty$;

$$v(a_k^{(n)}) \geqslant v(\Phi_u, t_{\tau(n)}) + d(n) - kt_{\tau(n)} \geqslant v(\Phi_u, t_{\tau(n)}) + d(n) - nt_{\tau(n)}.$$

Thus,

$$\|P_n\| \leqslant \|\Phi_u\|_{t_{\tau(n)}} \cdot p^{nt_{\tau(n)} - d(n)};$$

that is,

$$\|P_n\| = o(p^{nt_{\tau(n)}} \|\Phi_u\|_{t_{\tau(n)}}).$$

The theorem is proved.

Corollary 3.1. If the sequence $u = \{u_i\}_{i=0}^{\infty}$ satisfies the condition

$$nt_{\tau(n)} < \infty,$$

then there exists an anlytic function $f(z)$ taking the values α_i at the points u_i if and only if the sequence of polynomials $\{P_n\}$ satisfies the condition

$$\|P_n\| = o(\|\Phi_u\|_{t_{\tau(n)}}).$$

Remark When $u = \{g^k\gamma - 1\}$, we have:

$$nt_{\tau(n)} \leqslant p^{\tau(n)}/\varphi(p^{\tau(n)}) - p/(p-1); \quad \|\Phi_u\|_{t_{\tau(n)}} = \|\log^h\|_{t_{\tau(n)}}$$

and $\tau(n) = k$ for $p^{k-1} < n \leqslant p^k$, $t_k = 1/\varphi(p^k)$.
Consequently, a function on the corresponding sequence of interpolation polynomials $\{P_n\}$ satisfies: $\|P_n\| = o(p^{hn})$. This is a result of Amice ([2]).

§ 4. DETERMINATION OF FUNCTION BY ITS ZEROS

The question of the extent to which a p-adic analytic function is determined by its zeros has often been discussed in the literature. In some sense, results of Lazard's ([5]) and Van der Put ([11]) can be considered answers to this quesion. Here we consider the problem from another point of view: what points must be «added» to the set of zeros of a function in order for the resulting set to completely determined the p-adic analytic function?

Theorem 4.1. (uniqueness of p-adic analytic function) Two p-adic analytic functions coincide in $\{|z|<1\}$ if and only if they have the same zoros (counting multiplicity) and coincide on some other infinite set of points u satisfying the condition assumed at the beginning of § 2.

Proof. Let $f(z)$ and $g(z)$ be p-adic analytic functions satisfying the assumptions of the theorem. Then $\varphi(z) = f(z)/g(z)$ is a p-adic analytic function which is nonzero everywhere in T, and so is bounded in T. Hence, $\varphi(z)$ belongs to $o(\Phi_u)$, where u is the sequence where $f(z)$ and $g(z)$ coincide. By Theorem 2.1. $\{u_i\}$ is an interpolation sequence for $\varphi(z)$. On the other hand, by the definition of $\varphi(z)$, all of the interpolation polynomials for $\varphi(z)$ are identically equal to 1, hence $\varphi(z)\equiv 1$. The theorem is proved.

§ 5. THE p-ADIC L-FUNCTION ASSOCIATED TO AN ELLIPTIC CURVE

Let E be a Weil elliptic curve of conductor N, let p be a prime, $(p, N) = 1$ and suppose that E has good reduction at p. In [9] Mazur and Swinnerton — Dyer associated to each such curve E a p-adic L-function $L_p(E/A, s)$, where A is a finite abelian extension of Q. They stated the following conjecture: for all E, A, the p-adic L-function $L_p(E/A, s)$ is not identically zero. Here we give a partial affirmation of this conjecture.

We first recall the definition of Mazur and Swinnerton—Dyer ([9]). Let E be a Weil curve of conductor N, $(p, N) = 1$, and let Δ_0 be a fixed integer prime to p. Set $\Delta_n = \Delta_0 p^n$ and $Z_\Delta = \varprojlim Z/\Delta_n Z$. We set $H = H_1(E, Z)$ and let

$$\varphi : Q/Z \to H$$

be the modular symbol associated to E ([9]). Since φ is an $H \otimes Z_p$-valued eigenfunction for the operator T_p, we can construct the measure $\mu = \mu^{\Delta,\varphi}$. Now if A is a finite abelian extension of Q of conductor m, we write m in the form $m = \Delta_0 p^n$ and construct the measure $\mu^{\Delta,\varphi}$ for this Δ_0. The p-adic Mellin—Mazur transform corresponding to the measure $\mu^{\Delta,\varphi}$ is called the p-adic L-function associated to E:

$$L_p(E/A, s) = \prod_{\chi} L_p(E, \chi, s) \in Z_p[[s]]$$

where the product is taken over all characters belonging to the extension A/Q, and

$$L_p(E, \chi, s) = \int_U u^{s-1} \chi(u)\, d\mu^{\Delta,\varphi}.$$

Theorem 5.1. Let $N = l^n$, where l is a prime and n is any positive integer, and let the prime p be a primitive root modulo N. Then for every finite abelian extension A of Q of conductor $m = p^r$ and every elliptic curve E of conductor N,

$$L_p(E/A, s) \not\equiv 0.$$

Note that here $\Delta_0 = 1$. By the definition of $L_p(E/A, s)$ Theorem 5.1 follows from the following theorem.

Theorem 5.2. Let $\Phi(z)$ be a cusp form of weight 2 for $\Gamma_0(N)$ and let $\Phi(z)$ be an eigen-function for all of the Hecke operators T_m, $(m, N) = 1$. If $\Delta_0 = 1$ and N, p satisfy the conditions in Theorem 5.1, then the p-adic Mellin — Mazur transform $L_p(\Phi, \chi, s) \not\equiv 0$ for every character of the group Z^*_Δ.

The proof is based on several lemmas.

Lemma 5.1. If $L_p(\Phi, \overline{\chi}, s) \equiv 0$ for some character $\overline{\chi}$ of Z^*_Δ, then $\mu_\Phi \equiv 0$, where μ_Φ is the measure corresponding to $\Phi(z)$ ([3, 6]).

Proof. By definition, we have:

$$L_p(\Phi, \chi, s) = \int_U u^{s-1} \overline{\chi}(u)\, d\mu_\Phi(u). \tag{25}$$

For $u \in U$ we have the expansion

$$u^{s-1} = \exp((s-1)\log u) = \sum_{n=0}^{\infty} \frac{(\log u)^n}{n!} (s-1)^n, \tag{26}$$

where the series converges for s such that $\operatorname{ord}_p(s-1) \geqslant 1/(p-1) - \operatorname{ord}(\log u)$. We hence obtain:

$$\int_U u^{s-1}\overline{\chi}(u)\, d\mu_\Phi = \sum_{n=0}^{\infty} \left[\int_U \frac{(\log u)^n}{n!} \overline{\chi}(u)\, d\mu_\Phi\right](s-1)^n. \tag{27}$$

Thus, $L_p(\Phi, \overline{\chi}, s) \equiv 0$ if and only if for all $n \geqslant 0$ we have:

$$\int_U (\log u)^n\, \overline{\chi}(u)\, d\mu_\Phi = 0. \tag{28}$$

For every wild character $\chi \in X(U)$ there exists $x \in T$ such that

$$\chi(u) = (x+1)^{\log u/\log\gamma}.$$

Consequently, we have:

$$\chi(u) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{\log(1-x)}{\log\gamma}\right)^n (\log u)^n. \tag{29}$$

From (28) and (29) we thus obtain

$$\int_U \chi(u)\, \overline{\chi}(u)\, d\mu = 0. \tag{30}$$

Equation (30) holds for all characters $\chi \in X(U)$. Since $\overline{\chi}$ is fixed, we have

$$\int_U \chi(u)\, d\mu_\Phi(u) = 0 \quad \text{for all } \chi \in X(U). \tag{31}$$

Thus, $\mu_\Phi \equiv 0$ on Z_Δ by the Amice isomorphism (see § 1). The lemma is proved.

Lemma 5. 2. If $\Delta_0 = 1$ and $\mu_\Phi \equiv 0$, then $\int_0^{b/p^m} \Phi(z)\,dz = 0$ for all $m \geqslant 0$ and all $b \bmod p^m$.

Proof. We have the following formula for μ_Φ ([3,6]) : $\mu_\Phi(a + (\Delta p^m)) =$

$= M_m \int_{a/\Delta p^m}^{i\infty} \Phi(z)\,dz + M_{m-1} \int_{a/\Delta p^{m-1}}^{i\infty} \Phi(z)dz$, where M_m, M_{m-1} are nonzero constants. Hence, the lemma follows easily by induction if we prove the following equalities :

$$\int_0^{i\infty} \Phi(z)\,dz = 0 \text{ and } \int_0^{b/p} \Phi(z)dz = 0 \ (b = 1, \ldots, p-1) \tag{32}$$

Thus, it remains to prove (32). Since $\mu_\Phi \equiv 0$, we have $L_p(E/A, s) \equiv 0$. Then, by a result of Mazur ([9], Corollary 1, § 9. 6), we have $L(E, 1) = 0$ and so $\int_0^{i\infty} \Phi(z)\,dz = 0$.

Now let χ be a Dirichlet character modulo p. If $\Phi(z)$ has the form $\Phi(z) = \sum_{n=1}^{\infty} \lambda_n e^{2\pi inz}$, then we set

$$\Phi\chi(z) = \sum_{n=1}^{\infty} \lambda_n \chi(n) e^{2\pi inz}.$$

Further, let $G(\chi)$ be the Gauss sum

$$G(\chi) = \sum_{k=1}^{p-1} \chi(k) e^{2\pi ik/p}.$$

We then have the following equation [10] :

$$\Phi\chi(z) = \frac{G(\chi)}{p} \sum_{b=1}^{p-1} \chi^*(b)\, \Phi\left(z + \frac{b}{p}\right),$$

where $\chi^*(b) = \chi^{-1}(-b)$. From the functional equation for the Mellin — Mazur transform [3, 6], we have :

$$\int_0^{i\infty} \Phi\chi(z)\,dz = 0.$$

Thus.
$$\sum_{b=1}^{p-1} \chi^*(b) \int_0^{i\infty} \left(z + \frac{b}{p}\right) dz = 0. \tag{33}$$

Equation (33) holds for all primitive character $\chi \bmod p$, $\chi \not\equiv 1$, so that we obtain a system of $p-2$ equations in the $p-1$ unknowns $\int_0^{i\infty} \Phi\left(z + \frac{b}{p}\right) dz$, $(b = 1, \ldots, p-1)$.

Note that for all characters $\chi \bmod p$, $\chi \not\equiv 1$, we have : $\sum_{b=1}^{p-1} \chi^*(b) = 0$. This equation, together with the independence of characters, gives a system of solutions to the equations (33) of the form

$$\int_0^{i\infty} \left(z + \frac{b}{p}\right) dz = c, \tag{34}$$

where c is an arbitrary constant, $b = 1, \ldots, p-1$.

On the other hand, from the formula for the operator T_p we obtain :

$$0 = \lambda_p \int_0^{i\infty} \Phi(z)\, dz = \int_0^{i\infty} (\Phi \mid T_p)(z)\, dz =$$

$$= \int_0^{i\infty} \Big\} \sum_{\substack{d/p \\ b=0, \ldots, d-1}} \int_0^{i\infty} \Phi\left(\frac{d^{-1}pz+b}{d}\right) d\left(\frac{d^{-1}pz+b}{d}\right) \Big\} = \sum_{b=1}^{p-1} \int_0^{i\infty} \Phi\left(z + \frac{b}{p}\right) dz.$$

It follows from this and (34) that

$$\int_0^{i\infty} \Phi\left(z + \frac{b}{p}\right) dz = 0, \; b = 1, \ldots, p-1.$$

The lemma is proved.

Before proceding to the next lemma, we recall the Manin homomorphism ([8]).

Let H denote the upper half-plane, and let $X_N(C)$ denote the Riemann surface which is the standard compactification of $\Gamma_0(N)\backslash H$. For every pair $\alpha, \beta \in H \cup Q \cup \{i\infty\}$, let $\{\alpha, \beta\} \in H_1(X_N, R)$ denote the homology class on $X_N(C)$ of the image of the geodesic from α to β in H. Consider the mapping:

$$\xi : \Gamma_0(N) \to H_1(X_N, Z), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \left\{ 0, \frac{b}{d} \right\} \tag{35}$$

Manin proved that ξ is a surjective group homomorphism.

Now let φ be the differential form on $X_N(C)$ induced by the form $\Phi(z)\,dz$. Then, by Lemma 5.2, we have :

$$\int_{\{0,\, b/p^m\}} \varphi = 0 \tag{36}$$

for all $m \geqslant 0$, $b \bmod p^m$. For each homology class $\{\alpha, \beta\} \in H_1(X_N, R)$ we set

$$\{\alpha, \beta\}\varphi = \int_{\{\alpha, \beta\}} \varphi.$$

We now prove that, under the assumptions of Theorem 5.2, we have $\left\{0, \frac{b}{d}\right\}\varphi = 0$ or all $\left\{0, \frac{b}{d}\right\} \in H_1(X_N, Z)$. We note that for all $b \bmod p^m$ there exists a matrix

$$\begin{pmatrix} a & b \\ cN & p^m \end{pmatrix} \in \Gamma_0(N),$$

nce $(p, N) = 1$. Consequently, $\left\{0, \frac{b}{p^m}\right\} \in H_1(X_N, Z)$.

Lemma 5.3. Under the conditions of Theorem 5.2, if $\left\{0, \frac{b}{p^m}\right\}\varphi = 0$ for all $, \frac{b}{p^m}\right\} \in H_1(X_N, Z)$ and $m \geqslant 0$, then $\left\{0, \frac{1}{d}\right\}\varphi = 0$ for all $d \in Z$, $(d, N) = 1$.

Proof. Since $(d, N) = 1$, it is clear that $\left\{0, \frac{1}{d}\right\} \in H_1(X_N, Z)$. Choose an element $\begin{pmatrix} a & 1 \\ cN & d \end{pmatrix} \in \Gamma_0(N)$ in the preimage of $\left\{0, \frac{1}{d}\right\}$ under the mapping ξ. Since p is imitive root modulo N, there exists $m \geqslant 0$ and $y \in Z$ such that $p^m = d + yN$. hence have:

$$\begin{pmatrix} a & 1 \\ cN & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -yN & 1 \end{pmatrix} \begin{pmatrix} a & 1 \\ (c-ay)N & p^m \end{pmatrix}$$

n the homomorphism (35) we obtain

$$\left\{0, \frac{1}{d}\right\}\varphi = \{0, 0\}\varphi + \{0, 1/p^m\}\varphi = 0.$$

emma is proved.

Since $\{0, -b/d\} = \{0, b/(-d)\}$, we may assume that $b \geqslant 0$. We shall prove $0, \frac{b}{d}\right\}\varphi = 0$ for all $\left\{0, \frac{b}{d}\right\} \in H_1(X_N, Z)$ by induction on b.

i) and lemma 5.3, it now suffices to prove the following lemma:

Lemma 5.4. *If* $\left\{0, \frac{b}{d}\right\}\varphi = 0$ *for all* $\left\{0, \frac{b}{d}\right\} \in H_1(X_N, Z)$ *and* $b < b_0$, *then* $d\}\varphi = 0$ *for all* $\{0, b_0/d\} \in H_1(X_N, Z)$.

Proof. Let $\{0, b_0/d\}$ be an element of $H_1(X_N, Z)$. Choose a matrix $\begin{matrix} b_0 \\ d \end{matrix}\Big) \in \Gamma_0(N)$ in the preimage of $\{0, b_0/d\}$ under ξ. Represent a in the form: $+ xb_0$, where $0 < a_0 < b_0$. By assumption, $N = l^n$, where l is a prime and sitive integer. We consider the two possible cases: $(x, N) = 1$ and $(x+1, N) = 1$.

i) $(x, N) = 1$. Then there exists $\alpha, \gamma \in Z$, such that $\alpha x + \gamma N = 1$. We have the following equation of matrices in $\Gamma_0(N)$:

$$\begin{pmatrix} a & b_0 \\ cN & d \end{pmatrix} = \begin{pmatrix} -ax - b_0\gamma N & a_0 \\ -(c\alpha + d\gamma)N & cN - dx \end{pmatrix} \begin{pmatrix} -x & -1 \\ \gamma N & -\alpha \end{pmatrix}.$$

Then from the homomorphism (35) we have:

$$\{0, b_0/d\}\varphi = \{0, a_0/(cN - xd)\}\varphi + \{0, 1/\alpha\}\varphi.$$

Hence, $\{0, b/d\}\varphi = 0$ by our assumption, since $0 < 1$, $a_0 < b_0$

ii) $(x+1, N) = 1$. There exist α, γ such that $\alpha(x+1) + \gamma N = 1$. It is easy to verify the following equation of matrices in $\Gamma_0(N)$:

$$\begin{pmatrix} a & b_0 \\ cN & d \end{pmatrix} = \begin{pmatrix} a\alpha + b_0\gamma N & b_0 - a_0 \\ (c\alpha + d\gamma)N & -cN + dx + d \end{pmatrix} \begin{pmatrix} x+1 & 1 \\ -\gamma N & \alpha \end{pmatrix}.$$

From the group homomorphism (35) we obtain:

$$\{0, b_0/d\}\varphi = \{0, (b_0 - a_0)/(-cN + dx + d)\}\varphi + \{0, 1/\alpha\}\varphi.$$

Since $0 < b_0 - a_0 < b_0$, by our assumption we have: $\{0, b_0/d\}\varphi = 0$. The lemma is proved.

Thus, it follows from Lemmas 5.1 — 5.4. that, under the conditions of Theorem 5.2, if $L_p(\Phi, \chi, s) \equiv 0$ for any character χ, then $\left\{0, \frac{b}{d}\right\} = 0$ for all $\left\{0, \frac{b}{d}\right\} \in H_1(X_N, Z)$. But this means that φ has zero period. Consequently, $\varphi(z) \equiv 0$. This completes the proof of Theorems 5.1 and 5.2.

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