ACTA MATHEMATICA VIETNAMICA
TOM 5, N° 1 (1980)

MULTIFREQUENCY EXCITATION OF PARAMETRIC OSCILLATIONS OF DYNAMICAL SYSTEMS

NGUYỄN VĂN ĐẠO
Institute of Mechanics, Hanoi

Let us consider the parametric oscillations of nonlinear systems described by differential equation of form

$$\ddot{x} + \omega^2 x + \varepsilon \sum_{k=1}^{r} P_k(x) \sin\theta_k + \varepsilon Q(x, \dot{x}) = 0 \tag{0.1}$$

here $\theta_k = \nu_k t + \Gamma_k$, $k = 1, 2, \ldots, r$, and $\nu_1, \nu_2, \ldots, \nu_r$ are positive incommensurable numbers, $P_k(x)$ are polynomials of x, $Q(x, \dot{x})$ — nonlinear function of x, $\dot{x}$.

A lot of physical and technical problems are led to investigation of equation (0. 1). For instance, oscillation of a pendulum whose suspension point accomplishes the complicated oscillations and the transversal oscillation of beam under the action of complicated longitudinal force are described by equation (0. 1).

It is supposed that there exists a resonant relation of type

$$n^* \Omega + c_1^* \nu_1 + \ldots + c_r^* \nu_r = 0 \tag{0.2}$$

where $\Omega > 0$, and $n^*, c_1^*, \ldots, c_r^*$ are integers. The number n^* is different from zero, but some of c_i^* may be disappeared. A rather simple problem when $Q(x, \dot{x}) \equiv 0$, $P_k(x) = \lambda_k x$ has been considered by Ischuch V.V. and Iaschuch V. T. [1]

This paper consists of three sections.

In section 1 a general theory of construction of approximate solutions of equation (0.1) by asymptotic method of nonlinear oscillctions is given.

Section 2 is devoted to investigation of monofrequency resonant case when the relation

$$n^* \Omega + c_s^* \nu_s = 0$$

takes place. It has been found that this is the most important case of multifrequency excitation and that in the first approximation only the component of external excitation with resonant frequency (ν_s) has influence on the parametric oscillation. The influence of other nonresonant components is found only in the second approximation.

In section 3 the multifrequency resonant case when

$$n^* \Omega + c_1^* \nu_1 + ... + c_r^* \nu_r = 0$$

is true with more than one of $c_1, c_2, ..., c_r$ is considered. In this case the influence of external excitation on parametric oscillation is found only in the second approximation.

§ 1. CONSTRUCTION OF APPROXIMATE SOLUTIONS

Following to asymptotic method of nonlinear mechanics the solution of equation (0.1) is found under the asymptotic expansion [2]:

$$x = a\cos(\xi + \psi) + u_1(a, \xi + \psi, \theta) + \varepsilon^2 u_2(a, \xi + \psi, \theta) + \varepsilon^3 ... \quad (1.1)$$

where

$$\theta = (\theta_1, \theta_2, ..., \theta_r), \quad (1.2)$$

and the amplitude a and phase ψ are determined from the equations

$$\frac{da}{dt} = \varepsilon A_1(a, \psi) + \varepsilon^2 A_2(a, \psi) + \varepsilon^3 + ...,$$

$$\frac{d\psi}{dt} = \omega - \Omega + \varepsilon B_1(a, \psi) + \varepsilon^2 B_2(a, \psi) + \varepsilon^3 + ... \quad (1.3)$$

Function ξ satisfies the condition

$$\frac{d\xi}{dt} = \Omega, \quad n^* \xi + c_1^* \theta_1 + ... + c_r^* \theta_s \equiv 0 \quad (1.4)$$

Substituting the expression (1.1) into (0.1) and comparing the coefficients of $\varepsilon, \varepsilon^2 ...$ we obtain the following equations

$$\omega^2 \frac{\partial^2 u_1}{\partial(\xi+\psi)^2} + 2\omega \sum_{k=1}^{r} \nu_k \frac{\partial^2 u_1}{\partial\theta_k \partial(\xi+\psi)} + \sum_{k,j=1}^{r} \nu_j \nu_k \frac{\partial^2 u_1}{\partial\theta_j \partial\theta_k} + \omega^2 u_1 =$$

$$= \left[(\omega + \Omega) a \frac{\partial B_1}{\partial\psi} + 2\omega A_1\right] \sin(\xi + \psi) - \left[(\omega - \Omega) \frac{\partial A_1}{\partial\psi} - 2\omega a B_1\right] \cos(\xi + \psi) -$$

$$- \sum_{k=1}^{r} P_k [a\cos(\xi + \psi)] \sin\theta_k - Q[a\cos(\xi + \psi), -\omega a \sin(\xi + \psi)], \quad (1.5)$$

and

$$\omega^2 \frac{\partial^2 u_2}{\partial(\xi+\psi)^2} + 2\omega \sum_{k=1}^{r} \nu_k \frac{\partial^2 u_2}{\partial\theta_k \partial(\xi+\psi)} + \sum_{k,j=1}^{r} \nu_j \nu_k \frac{\partial^2 u_2}{\partial\theta_j \partial\theta_k} + \omega^2 u_2 =$$

$$= \left[2A_1 B_1 + + aA_1 \frac{\partial B_1'}{\partial a} + aB_1 \frac{\partial B_1}{\partial\psi} + 2\omega A_2 + (\omega - \Omega) a \frac{\partial B_2}{\partial\psi}\right] \sin(\xi + \psi) -$$

$$- \left[A_1 \frac{\partial A_1}{\partial a} + B_1 \frac{\partial A_1}{\partial\psi} - aB_1^2 - 2\omega a B_2 + (\omega - \Omega) \frac{\partial A_2}{\partial\psi}\right] \cos(\xi + \psi) + R_2^{\circ},$$

. . . . (1.6)

where

$$R_2^0 = u_1 \frac{\partial F_1^0}{\partial x_0} + \left[A_1\cos(\xi+\psi) - aB_1\sin(\xi+\psi) + \omega\frac{\partial u_1}{\partial(\xi+\psi)} + \sum_{k=1}^{r} \nu_k \frac{\partial u_1}{\partial\theta_k}\right]\frac{\partial F_1^0}{\partial x_0} -$$

$$-\left[2A_1\sum_{k=1}^{r}\nu_k\frac{\partial^2 u_1}{\partial\theta_k\partial a} + 2\omega A_1\frac{\partial^2 u_1}{\partial(\xi+\psi)\partial a} + 2B_1\sum_{k=1}^{r}\nu_k\frac{\partial^2 u_1}{\partial\theta_k\partial(\xi+\psi)} + 2\omega B_1\frac{\partial^2 u_1}{\partial(\xi+\psi)^2} +\right.$$

$$\left. + (\omega-\Omega)\frac{\partial B_1}{\partial\psi}\frac{\partial u_1}{\partial(\xi+\psi)} + (\omega-\Omega)\frac{\partial A_1}{\partial\psi}\frac{\partial u_1}{\partial a}\right], F_1^0 = -\sum_{k=1}^{r} P_k(x_0)\sin\theta_k - Q(x_0, \dot{x}_0),$$

$$x_0 = a\cos(\xi+\psi),\ \dot{x}_0 = -\omega a\sin(\xi+\psi). \tag{1.7}$$

Let us expand $P_k(x_0)$, $Q(x_0, \dot{x}_0)$ in the Fourier series. We find

$$P_k(x_0) = P_k[a\cos(\xi+\psi)] = \sum_{m\geqslant 0} D_{km}(a)\cos m(\xi+\psi),\ Q(x_0, \dot{x}_0) =$$

$$= Q[a\cos(\xi+\psi), -\omega a\sin(\xi+\psi)] = \sum_{m\geqslant 0}[f_m(a)\cos m(\xi+\psi) + g_m(a)\sin m(\xi+\psi)], \tag{1.8}$$

here,

$$D_{k0}(a) = \frac{1}{2\pi}\int_0^{2\pi} P_k(a\cos\Phi)d\Phi,$$

$$f_0(a) = \frac{1}{2\pi}\int_0^{2\pi} Q(a\cos\Phi, -\omega a\sin\Phi)d\Phi, \tag{1.9}$$

$$D_{km}(a) = \frac{1}{\pi}\int_0^{2\pi} P_k(a\cos\Phi)\cos m\Phi d\Phi,$$

$$f_m(a) = \frac{1}{\pi}\int_0^{2\pi} Q(a\cos\Phi, -\omega a\sin\Phi)\cos m\Phi d\Phi,$$

$$g_m(a) = \frac{1}{\pi}\int_0^{2\pi} Q(a\cos\Phi, -\omega a\sin\Phi)\sin m\Phi d\Phi.$$

We shall find the function u_1 in expanded form with undefined coefficients

$$u_1 = \sum_{n,c_1\ldots,c_r}\{u_{1nc_1\ldots c_r}\sin[n(\xi+\psi) + c_1\theta_1 + \ldots + c_r\theta_r] +$$

$$v_{1nc_1\ldots c_r}\cos[n(\xi+\psi) + c_1\theta_1 + \ldots + c_r\theta_r]\} \tag{1.10}$$

To determine the unknown coefficients $u_{1nc_1\ldots c_r}$, $v_{1nc_1\ldots c_r}$ and functions A_1, B_1 we substitute expressions (1.8)-(1.10) into (1.5) and we compare the coefficients of $\sin[n(\xi+\psi) + c_1\theta_1 + \ldots + c_r\theta_r]$, $\cos[n(\xi+\psi) + c_1\theta_1 + c_2\theta_2 + \ldots + c_r\theta_r]$. By substituting (1.8) and (1.10) into (1.5) we get

$$\sum_{n,c_1\ldots c_r} [\omega^2 - (n\omega + \sum_{i=1}^{r} c_i \nu_i)^2] \{ u_{1nc_1\ldots c_r} \sin [n(\xi + \psi) + \ldots + c_r \theta_r] + v_{1nc_1\ldots c_r}$$

$$\cos [n(\xi + \psi) + c_1\theta_1 + \ldots + c_r\theta_r] \} = \left[(\omega - \Omega)\, a \frac{\partial B_1}{\partial\psi} + 2\,\omega A_1\right] \sin (\xi + \psi) -$$

$$- \left[(\omega - \Omega) \frac{\partial A_1}{\partial\psi} - 2\omega a B_1\right] \cos (\xi + \psi) - \sum_{m\geqslant 0} [f_m(a)\cos m(\xi+\psi) + g_m(a)\sin m(\xi+\psi)] -$$

$$- \frac{1}{2} \sum_{i=1}^{r} \sum_{m\geqslant 0} D_{im}(a) \{ \sin [m(\xi + \psi) + \theta_i] - \sin[m(\xi + \psi) - \theta_i] \} \tag{1.11}$$

§ 2. MONOFREQUENCY RESONANT CASE

In this section we consider the resonant case when there exists a resonant relation

$$n^*\Omega + c_s^* \nu_s = 0 \tag{2. 1}$$

where n^*, c_s^*, are integers, $1 \leqslant s \leqslant r$, $n^* . c_s^* < 0$. Comparing with (0. 2) we have

$$c_1^* = c_2^* = \ldots = c_{s-1}^* = c_{s+1}^* = \ldots = 0, \qquad c_s^* \neq 0.$$

The last terms in (1. 11) will contain $\sin (\xi + \psi)$, $\cos (\xi + \psi)$ if

$$(m \pm 1)(\xi + \psi) \pm \theta_i = \chi. \tag{2. 2}$$

On the other hand from (1. 3) we have

$$n^*(\xi + \psi) + c_s^* \theta_s = n^*\psi. \tag{2.3}$$

The comparison of formulae (2. 2) and (2. 3) gives

$$m = \mp 1 + \sigma n^*, \qquad i = s, \qquad \pm 1 = \sigma c_s^*, \qquad \chi = \sigma n^*\psi$$

or

$$m = \mp 1 \pm \frac{n^*}{c_s^*}, \qquad \chi = \pm \frac{n^*}{c_s^*} \psi. \tag{2. 4}$$

Since $D_{sm} = 0$ if $m < 0$ and $n^* . c_s^* < 0$, $D_{s, \frac{n^*}{c_s^*} - 1} = 0$, then we have the following terms containing $\sin(\xi + \psi)$, $\cos(\xi + \psi)$ in (1.11) :

$$\left[(\omega - \Omega)\, a \frac{\partial B_1}{\partial\psi} + 2\,\omega A_1\right] \sin(\xi + \psi) - \left[(\omega - \Omega) \frac{\partial A_1}{\partial\psi} - 2\,\omega a B_1\right] \cos(\xi + \psi) =$$

$$= f_1(a)\cos(\xi + \psi) + g_1(a)\sin(\xi + \psi) + \frac{1}{2} D_{s, \frac{n^*}{c_s^*} + 1} \sin \left[(\xi+\psi) + \frac{n^*}{c_s^*} \psi\right] -$$

$$- \frac{1}{2} D_{s, -\frac{n^*}{c_s^*} + 1} \sin \left[(\xi+\psi) - \frac{n^*}{c_s^*} \psi\right] + \frac{1}{2} D_{s, -\frac{n^*}{c_s^*} - 1} \sin \left[(\xi+\psi) + \frac{n^*}{c_s^*} \psi\right].$$

From here we obtain

$$2\omega A_1 + (\omega - \Omega)\, a \frac{\partial B_1}{\partial \psi} =$$

$$= g_1(a) + \frac{1}{2}\left(D_{s,\frac{n^*}{c_s^*}+1} + D_{s,-\frac{n^*}{c_s^*}-1} - D_{s,-\frac{n^*}{c_s^*}+1}\right)\cos\frac{n^*}{c_s^*}\psi,$$

$$2\omega a B_1 - (\omega - \Omega)\frac{\partial A_1}{\partial \psi} =$$

$$= f_1(a) + \frac{1}{2}\left(D_{s,\frac{n^*}{c_s^*}+1} + D_{s,-\frac{n^*}{c_s^*}-1} + D'_{s,-\frac{n^*}{c_s^*}+1}\right)\sin\frac{n^*}{c_s^*}\psi \qquad (2.5)$$

If $\omega - \Omega = O(\varepsilon)$ then we have

$$A_1 = \frac{1}{2\omega} g_1(a) + \frac{1}{4\omega}\left(D_{s,\frac{n^*}{c_s^*}+1} + D_{s,-\frac{n^*}{c_s^*}-1} - D_{s,-\frac{n^*}{c_s^*}+1}\right)\cos\frac{n^*}{c_s^*}\psi,$$

$$B_1 = \frac{1}{2a\omega} f_1(a) + \frac{1}{4a\omega}\left(D_{s,\frac{n^*}{c_s^*}+1} + D_{s,-\frac{n^*}{c_s^*}-1} + D_{s,-\frac{n^*}{c_s^*}+1}\right)\sin\frac{n^*}{c_s^*}\psi$$

(2.6)

By comparison of the higher harmonics in (1. 11) we get

$$\omega^2(1-n^2)\, u_{1nc_1=0,\dots,c_r=0} = -g_n(a), \qquad n \neq 1,$$

$$\omega^2(1-n^2)\, v_{1nc_1=0,\dots,c_r=0} = -f_n(a) \qquad n \neq 1$$

$$[\omega^2 - (n\omega + \nu_i)^2]\, u_{1nc_1=0,\dots,c_{i-1}=0,\, c_i=1,\, c_{i+1}=0,\dots,c_r=0} = \frac{-1}{2} D_{in}(a) \quad (i \neq s)$$

$$[\omega^2 - (n\omega - \nu_i)^2]\, u_{1nc_1=0,\dots,c_{i-1}=0,\, c_i=-1,\, c=0,\dots,c_r=0} = \frac{1}{2} D_{in}(a) \quad (i \neq s)$$

$$[\omega^2 - n\omega + \nu_s)^2]\, u_{1nc_1=0,\dots,c_{s-1}=0,\, c_s=1,\, c_{s+1}=0,\dots,c_r=0} = -\frac{1}{2} D_{sn}(a)\left(n \neq 1 + \frac{n^*}{c_s^*}\right)$$

$$[\omega^2 - (n\omega - \nu_s)^2]\, u_{1nc_1=0,\dots,c_{s-1}=0,\, c_s=-1,\, c_{s+1}=0,\dots,c_r=0} = \frac{1}{2} D_{sn}(a)\left(n \neq -\frac{n^*}{c_s^*} \pm 1\right)$$

(2. 7)

And therefore we have

$$u_1 = -\frac{f_0}{\omega^2}(a) + \sum_{n\geqslant 2}\frac{1}{(n^2-1)\omega^2}\left[g_n(a)\sin n(\xi+\psi) + f_n(a)\cos n(\xi+\psi)\right]$$

$$+\frac{1}{2}\sum_{\substack{i=1\\ i\neq s}}^{r}\sum_{n\geqslant 0}\frac{D_{in}(a)\sin[n(\xi+\psi)+\theta_i]}{(n\omega+\nu_i)^2-\omega^2} +$$

$$+\frac{1}{2}\sum_{\substack{i=1\\ i\neq s}}^{r}\sum_{n\geqslant 0}\frac{D_{in}(a)\sin[n(\xi+\psi)-\theta_i]}{\omega^2-(n\omega-\nu_i)^2}+$$

$$+\frac{1}{2}\sum_{\substack{n\neq 1+\frac{n^*}{c_s^*}\\ n\geqslant 0}}\frac{D_{sn}(a)\sin[n(\xi+\psi)+\theta_s]}{(n\omega+\nu_s)^2-\omega^2}+$$

$$+\frac{1}{2}\sum_{\substack{n\geqslant 0\\ n\neq -\frac{n^*}{c_s^*}\mp 1}}\frac{D_{sn}(a)\sin[n(\xi+\psi)-\theta_s]}{\omega^2-(n\omega-\nu_s)^2} \tag{2.8}$$

To determine the functions A_2, B_2 we compare the coefficients of $\sin(\xi+\psi)$ $\cos(\xi+\psi)$ in (1.6). First we write $\overset{o}{R_2}$ in the expanded form

$$R_2^o=\sum_{nc_1\ldots c_r}\{R^o_{21nc_1\ldots c_r}\cos[n(\xi+\psi)+c_1\theta_1+\ldots+c_r\theta_r]+$$

$$R^o_{22nc_1\ldots c_r}\sin[n(\xi+\psi)+c_1\theta_1+\ldots+c_r\theta_r]\} \tag{2.9}$$

where

$$R^o_{210\ldots 0}=\frac{1}{2\pi}\int_0^{2\pi}R_2^o\,d(\xi+\psi)\,d\theta_1\ldots d\theta_r,$$

$$R^o_{21nc_1\ldots c_r}=\frac{1}{\pi}\int_0^{2\pi}R_2^o\cos[n(\xi+\psi)+c_1\theta_1+\ldots c_r\theta_r]\,d(\xi+\psi)\,d\theta_1\ldots d\theta_r$$

$$R^o_{22nc_1\ldots c_r}=\frac{1}{\pi}\int_0^{2\pi}R_2^o\sin[n(\xi+\psi)+c_1\theta+\ldots c_r\theta]\,d(\xi+\psi)d\theta_1..d\theta_r$$

In (2. 9) the terms containing $\cos(\xi+\psi)$, $\sin(\xi+\psi)$ correspond to that n and $c_1,\ldots c_r$ for which

$$n(\xi+\psi)+c_1\theta_1+\ldots+c_r\theta_r=\pm(\xi+\psi)+\alpha$$

On the other hand have

$$n^*(\xi+\psi)+c_s^*\theta_s=n^*\psi.$$

From here it follows that

$$n\mp 1=\mu\, n^*,\ c_1=c_2=\ldots=c_{s-1}=c_{s+1}=\ldots=c_r=0,$$

$$\mu=\frac{c_k}{c_s^*},\ \alpha=\mu\, n^*\psi.$$

Thus we can write R_2^o (2.9) in the form

$$R_2^o=C(a,\psi)\cos(\xi+\psi)+S(a,\psi)\sin(\xi+\psi)+\ldots \tag{2.10}$$

where dots denote the terms which do not contain $\sin(\xi+\psi)$, $\cos(\xi+\psi)$ and

$$C(a,\psi) = \sum_{\mu} [(R^{o\mu+}_{21} + R^{o\mu-}_{21}) \cos \mu n^* \psi + (R^{o\mu+}_{22} + R^{o\mu-}_{22}) \sin \mu n^* \psi],$$

$$S(a,\psi) = \sum_{\mu} [(R^{o\mu-}_{21} - R^{o\mu+}_{21}) \sin \mu n^* \psi + (R^{o\mu+}_{22} - R^{o\mu-}_{22}) \cos \psi n^* \psi];$$

$$R^{o\mu+}_{21} = R^{o}_{21,\,1+\mu n^*,\,c_1=o,\dots,c_{s-1}=o,c_s=\mu c^*_s,\,c_{s+1}=o,\dots,c_r=o},$$

$$R^{o\mu-}_{21} = R^{o}_{21,\,-1+\mu n^*,\,c_1=o,\dots,c_{s-1}=o,c_s=\mu^*_s,\,c_{s+1}=o,\dots,c_r=o},$$

$$R^{o\mu+}_{22} = R^{o}_{22,\,1+\mu n^*,\,c_1=o,\dots,c_{s-1}=o,c_s=\mu c^*_s,\,c_{s+1}=o,\dots,c_r=o},$$

$$R^{o\mu-}_{22} = R^{o}_{22,\,-1+\mu n^*,\,c_1=o,\dots,c_{s-1}=o,c_s=c^*_s,\,c_{s+1}=o,\dots,c_r=o}. \qquad (2.11)$$

Substituting (2.10) into (1.6) and comparing coefficents of $\cos(\xi+\psi)$, $\sin(\xi+\psi)$ we obtain

$$-2\omega A_2 + (\Omega-\omega) a \frac{\partial B_2}{\partial\psi} = 2A_1B_1 + aA_1 \frac{\partial B_1}{\partial a} + a B_1 \frac{\partial B_1}{\partial \psi} + S(a,\psi),$$

$$2a\omega B_2 + (\Omega-\omega) \frac{\partial A_2}{\partial\psi} = A_1 \frac{\partial A_1}{\partial a} + B_1 \frac{\partial A_1}{\partial\psi} - aB_1^2 - C(a,\psi).$$

If $\omega - \Omega = O(\varepsilon)$ then we have

$$A_2 = -\frac{1}{2\omega}\left[2A_1B_1 + aA_1 \frac{\partial B_1}{\partial\psi} + aB_1 \frac{\partial B_1}{\partial\psi} + S(a,\psi)\right], \qquad (2.12)$$

$$B_2 = \frac{1}{2a\omega}\left[A_1 \frac{\partial A_1}{\partial a} + B_1 \frac{\partial A_1}{\partial\psi} - aB_1^2 - C(a,\psi)\right].$$

Thus in the first approximation we have

$$x = a\cos\left(-\frac{c_s}{n^*}\theta_s + \psi\right) \qquad (2.13)$$

here a and ψ are determined by equations:

$$\frac{da}{dt} = \frac{\varepsilon}{2\omega} g_1(a) + \frac{\varepsilon}{4\omega}\left(D_{s,\frac{n^*}{c^*_s}+1} + D_{s,-\frac{n^*}{c^*_s}1} - D_{s,-\frac{n^*}{c^*_s}+1}\right)\cos\frac{n^*}{c^*_s}\psi,$$

$$\frac{d\psi}{dt} = \omega - \Omega + \frac{\varepsilon}{2a\omega} f_1(a) + \frac{\varepsilon}{4\omega a}\left(D_{s,\frac{n^*}{c^*_s}+1} + D_{s,-\frac{n^*}{c^*_s}-1} + D_{s,-\frac{n^*}{c^*_s}+1}\right)$$

$$\sin\frac{n^*}{c^*_s}\psi. \qquad (2.14)$$

The refinement of the first approximation is

$$x = a\cos\left(-\frac{c_s^*}{n^*}\theta_s + \psi\right) + \varepsilon\left\{\frac{-f_0(a)+}{\omega^2}\sum_{n\geqslant 2}\frac{g_n(a)\sin n(\xi+\psi)+f_n(a)\cos n(\xi+\psi)}{(n^2-1)\omega^2}\right.$$

$$+\frac{1}{2}\sum_{\substack{i=1\\ i\neq s}}^{r}\sum_{n\geqslant 0}\frac{D_{in}(a)\sin[n(\xi+\psi)+\theta_i]}{(n\omega+\nu_i)^2-\omega^2}+\frac{1}{2}\sum_{\substack{i=1\\ i\neq s}}^{r}\sum_{n\geqslant 0}\frac{D_{in}(a)\sin[n(\xi+\psi)-\theta_i]}{\omega^2-(n\omega-\nu_i)^2}+$$

$$\left.+\frac{1}{2}\sum_{\substack{n\neq 1+\frac{n^*}{c_s^*}\\ n\geqslant 0}}\frac{D_{sn}(a)\sin[n(\xi+\psi)+\theta]}{(n\omega+\nu_s)^2-\omega^2}+\frac{1}{2}\sum_{\substack{n\neq -\frac{n^*}{c_s^*}\pm 1\\ n\geqslant 0}}\frac{D_{sn}(a)\sin[n(\xi+\psi)-\theta_s]}{\omega^2-(n\omega-\nu_s)^2}\right\}, \quad (2.15)$$

$$\theta_s = -\frac{n^*}{c_s^*}(\xi+\psi) - \frac{n^*}{c_s^*}\psi$$

in which a and ψ satisfy the equations (2.14).

In the second approximation function x is of form (2.15) but a and ψ are determined by the following equations

$$\frac{da}{dt} = \frac{\varepsilon}{2\omega}g_1(a) + \frac{\varepsilon}{4\omega}\left(D_{s,\frac{n^*}{c_s^*}+1} + D_{s,-\frac{n^*}{c_s^*}-1} - D_{s,-\frac{n^*}{c_s^*}+1}\right)\cos\frac{n^*}{c_s^*}\psi -$$

$$-\frac{\varepsilon^2}{2\omega}\left[2A_1B_1 + aA_1\frac{\partial B_1}{\partial a} + aB_1\frac{\partial B_1}{\partial\psi} + S(a,\psi)\right],$$

$$\frac{d\psi}{dt} = \omega-\Omega+\frac{\varepsilon}{2a\omega}f_1(a)+\frac{\varepsilon}{4a\omega}\left(D_{s,\frac{n^*}{c_s^*}+1}+D_{s,-\frac{n^*}{c_s^*}-1}+D_{s,-\frac{n^*}{c_s^*}+1}\right)\sin\frac{n^*}{c_s^*}\psi+$$

$$+\frac{\varepsilon^2}{2a\omega}\left[A_1\frac{\partial A_1}{\partial a} + B_1\frac{\partial A_1}{\partial\psi} - aB_1^2 - C(a,\psi)\right]. \quad (2.16)$$

A special case

By making use of the general theory developed in this section we now study the parametric oscillation of nonlinear system of type

$$\ddot{x} + \omega^2 x + \varepsilon x\sum_{i=1}^{r}\lambda_i\sin\theta_i + \varepsilon Q(x,\dot{x}) = 0 \quad (2.17)$$

where λ_i are constants.

Let us consider the subharmonic resonant case, for which

$$2\Omega - \nu_s = 0,$$
$$\omega^2 = \Omega^2 + \varepsilon\Delta$$

here Δ — detuning of frequencies.

By substituting $n^* = 2$, $c_s^* = -1$, $P_s(x) = \lambda_s x$, $P_s(a\cos\Phi) = \lambda_s a\cos\Phi$ in (1.8) we get

$$D_{s,1} = \lambda_s a,\ D_{s,0} = D_{s,n} = 0,\ n \geqslant 2,$$

and therefore the formulae (2.6) become

$$A_1 = \frac{1}{2\omega} g_1(a) + \frac{1}{4\omega} (\lambda_s a) \cos 2\psi,$$

$$B_1 = \frac{1}{2a\omega} f_1(a) - \frac{1}{4\omega} \lambda_s \sin 2\psi.$$

Following formulae (2.13), in the first approximation we have

$$x = a\cos\left[\frac{1}{2}(\nu_s t + \Gamma_s) + \psi\right], \tag{2.18}$$

here a and ψ are determined from equations (2.14):

$$\frac{da}{dt} = \frac{\varepsilon}{2\omega\pi} \int_0^{2\pi} \sin\Phi \cdot Q(a\cos\Phi, -\omega a\sin\Phi)\, d\Phi + \frac{\varepsilon}{4\omega} \lambda_s a\cos 2\psi, \tag{2.19}$$

$$\frac{d\psi}{dt} = \omega - \frac{\nu_s}{2} + \frac{\varepsilon}{2a\omega\pi} \int_0^{2\pi} \cos\Phi \cdot Q(a\cos\Phi, -\omega a\sin\Phi)\, d\Phi - \frac{\varepsilon}{4\omega} \lambda_s \sin 2\psi.$$

The refinement of the first approximation is of form (2.15) with $\xi = \frac{1}{2}\theta_s$,

$$x = a\cos\left(\frac{1}{2}\theta_s + \psi\right) + \varepsilon\left\{-\frac{f_0(a)}{\omega^2} + \sum_{n\geqslant 2} \frac{f_n \cos n(\xi+\psi) + g_n \sin n(\xi+\psi)}{\omega^2(n^2-1)}\right.$$

$$\left. + \frac{a}{2}\sum_{i=1}^{r} \frac{\lambda_i \sin\left(\frac{1}{2}\theta_s + \theta_i + \psi\right)}{\nu_i(\nu_s+\nu_i)} + \frac{a}{2}\sum_{\substack{i=1\\ i\neq s}}^{r} \frac{\lambda_i \sin\left(\frac{1}{2}\theta_s - \theta_i + \psi\right)}{\nu_i(\nu_s-\nu_i)}\right\}, \tag{2.20}$$

in which a and ψ satisfy equations (1.5).

§ 3. MULTIFREQUENCY RESONANT CASE

In this section it is supposed that more than one of $c_1^* \dots c_r^*$ are different from zero, so that we have a resonant relation

$$n^*\Omega + c_1^* \nu_1 + \dots + c_r^* \nu_r = 0. \tag{3.1}$$

In this case it is easy to show that the last sums in (1.11) do not contain $\sin(\xi+\psi)$, $\cos(\xi+\psi)$. In fact, these terms correspond to the number m satisfying relation

$$m(\xi+\psi) \pm \theta_i = \mp(\xi+\psi) + \chi \tag{3.2}$$

But on the other hand following (1.4) we have

$$n^*(\xi+\psi) + c_1^*\theta_1 + \dots + c_r^*\theta_r = n^*\psi.$$

Therefore, if the relation (3.2) takes place then we must have

$$m \pm 1 = \sigma n^*,\ \chi = \sigma n^* \psi,\ \pm \theta_i = \sigma(c_1^* \theta_1 + \dots + c_r^* \theta_r)$$

or

$$\sigma c_1^* \theta_1 + \dots + (\sigma c_i^* \mp 1)\theta_i + \dots + \sigma c_r^* \theta_r = 0.$$

From here we obtain

$$\sigma c_1^* \nu_1 + \dots + (\sigma c_i^* \mp 1)\nu_i + \dots + c_r^* \nu_r = 0,$$

in which at least two terms are different from zero. This is at variance with assumption about incommesurable numbers $\nu_1, \nu_2, \dots \nu_r$.

By comparison of the terms $\sin(\xi + \psi)$, $\cos(\xi + \psi)$ in (1.5) we get

$$2\omega A_1 + (\omega - \Omega)\, a \frac{\partial B_1}{\partial \psi} = g_1(a),$$
$$2a\omega B_1 - (\omega - \Omega)\frac{\partial A_1}{\partial \psi} = f_1(a). \tag{3.3}$$

If $\omega - \Omega = O(\varepsilon)$ we have

$$A_1 = \frac{1}{2\omega} g_1(a),$$
$$B_1 = \frac{1}{2a\omega} f_1(a). \tag{3.4}$$

Comparing the terms different from $\sin(\xi + \psi)$, $\cos(\xi + \psi)$ in (1.5) we obtain

$$u_1 = -\frac{f_0(a)}{\omega^2} + \sum_{n \geqslant 2} \frac{g_n(a)\sin n(\xi + \psi) + f_n(a)\cos n(\xi + \psi)}{(n^2 - 1)\omega^2} +$$
$$+ \frac{1}{2}\sum_{i=1}^{r}\sum_{n \geqslant 0} \frac{D_{in}(a)\sin[n(\xi + \psi) + \theta_i]}{(n\omega + \nu_i)^2 - \omega^2} + \frac{1}{2}\sum_{i=1}^{r}\sum_{n \geqslant 0} \frac{D_{in}(a)\sin[n(\xi + \psi) - \theta_i]}{\omega^2 - (n\omega - \nu_i)^2} \tag{3.5}$$

To determine the functions A_2, B_2 we use the expansion (2.9) of function R_2^0. The terms of R_2^0 containing $\cos(\xi + \psi)$, $\sin(\xi + \psi)$ correspond to that n and $c_1, \dots, c_n$ for which

$$n(\xi + \psi) + c_1\theta_1 + \dots + c_r\theta_r = \pm(\xi + \psi) + \tau$$

On the other hand following (1.4) we have:

$$n^*(\xi + \psi) + c_1^* \theta_1 + \dots + c_r^* \theta_r = n^*\psi.$$

From here it follows that

$$n \mp 1 = en^*,\ c_i = ec_i^*,\ \tau = en^*\psi$$

where e is a coefficient of proportionality. Thus we have

$$R_2^0 = C^*(a, \psi)\cos(\xi + \psi) + S^*(a, \psi)\sin(\xi + \psi) + \dots\dots \tag{3.6}$$

where

$$C^*(a, \psi) = \sum_{e} \left[(R_{21}^{oe+} + R_{21}^{oe-}) \cos en^*\psi + (R_{22}^{oe+} + R_{22}^{oe-}) \sin en^*\psi \right],$$

$$S^*(a, \psi) = \sum_{e} \left[(R_{21}^{oe-} - R_{21}^{oe+}) \sin en^*\psi + (R_{22}^{oe+} - R_{22}^{oe-}) \cos en^*\psi \right].$$

$$\begin{aligned} R_{21}^{oe+} &= R^{o}_{21,\ 1+en^*,\ c_1 = ec_1^*, \dots, c_r = ec_r^*}, \\ R_{21}^{oe-} &= R^{o}_{21,\ -1+en^*,\ c_1 = ec_1^*, \dots, c_r = ec_r^*}, \\ R_{22}^{oe+} &= R^{o}_{22,\ 1+en^*,\ c_1 = ec_1^*, \dots, c_r = ec_r^*}, \\ R_{22}^{oe-} &= R^{o}_{22,\ -1+en^*,\ c_1 = ec_1^*, \dots, c_r = ce_r^*}. \end{aligned} \tag{3.7}$$

Substituting (3.6) into (1.6) and comparing coefficients of $\cos(\xi+\psi)$, $\sin(\xi+\psi)$ we obtain

$$\begin{aligned} -2\omega A_2 - (\omega-\Omega)\, a\frac{\partial B_2}{\partial\psi} &= 2A_1B_1 + aA_1\frac{\partial B_1}{\partial a} + aB_1\frac{\partial B_1}{\partial\psi} + S^*(a, \psi), \\ 2\omega a B_2 + (\Omega-\omega)\frac{\partial A_2}{\partial\psi} &= A_1\frac{\partial A_1}{\partial a} + B_1\frac{\partial A_1}{\partial\psi} - aB_1^2 - C^*(a, \psi). \end{aligned} \tag{3.8}$$

If $\omega - \Omega = O(\varepsilon)$ then we have

$$\begin{aligned} A_2 &= -\frac{1}{2\omega}\left[2A_1B_1 + aA_1\frac{\partial B_1}{\partial a} + aB_1\frac{\partial B_1}{\partial\psi} + S^*(a, \psi)\right], \\ B_2 &= \frac{1}{2a\omega}\left[A_1\frac{\partial A_1}{\partial a} + B_1\frac{\partial A_1}{\partial\psi} - aB_1^2 - C^*(a, \psi)\right]. \end{aligned} \tag{3.9}$$

Thus in the first approximation we have

$$x = a\cos\left(\frac{c_1^*\theta_1 + \dots + c_r^*\theta_r}{n^*} - \psi\right). \tag{3.10}$$

here a and ψ are determined from equations

$$\begin{aligned} \frac{da}{dt} &= \frac{\varepsilon}{2\omega} g_1(a), \\ \frac{d\psi}{dt} &= \omega - \Omega + \frac{\varepsilon}{2a\omega} f_1(a). \end{aligned} \tag{3.11}$$

The refinement of the first approximation is

$$\begin{aligned} x = {} & a\cos\left[\frac{1}{n^*}(c_1^*\theta_1 + \dots + c_r^*\theta_r) - \psi\right] + \\ & + \varepsilon\Bigg\{-\frac{f(a)}{\omega^2} + \sum_{n\geq 2}\frac{1}{(n^2-1)\,\omega^2}\Bigg[g_n \sin n\left(\frac{1}{n^*}(c_1^*\theta_1 + \dots + c_r^*\theta_r) - \psi\right) + \\ & + f_n(a)\cos n\left(\frac{1}{n^*}(c_1^*\theta_1 + \dots + c_r^*\theta_r) - \psi\right)\Bigg] + \end{aligned}$$

$$+\frac{1}{2}\sum_{i=1}^{r}\sum_{n\geqslant 0}\frac{D_{in}(a)\sin n\left[\frac{1}{n^*}\left(c_1^*\theta_1+\ldots+\left(\frac{n^*}{n}+c_i^*\right)\theta_i+\ldots+c_r^*\theta_r\right)-\psi\right]}{(n\omega+\nu_i)^2-\omega^2}+$$

$$+\frac{1}{2}\sum_{i=1}^{r}\sum_{n\geqslant 0}\frac{D_{in}\sin n\left[\frac{1}{n^*}\left(c_1^*\theta_1+\ldots+\left(c_i^*-\frac{n^*}{n}\right)\theta_i+\ldots+c_r^*\theta_r\right)-\psi\right]}{\omega^2-(n\omega-\nu_i)^2},\quad (3.12)$$

in which a and ψ satisfy the equations (3.11).

In the second approximation we have x in form (3.12) but a and ψ are determined by equations:

$$\frac{da}{dt}=\frac{\varepsilon}{2\omega}g_1(a)-\frac{\varepsilon^2}{2\omega}\left[2A_1B_1+aA_1\frac{\partial B_1}{\partial a}+aB_1\frac{\partial B_1}{\partial\psi}+S^*(a,\psi))\right],$$

$$\frac{d\psi}{dt}=\omega-\Omega+\frac{\varepsilon}{2a\omega}f_1(a)+\frac{\varepsilon^2}{2a\omega}\left[A_1\frac{\partial A_1}{\partial a}+B_1\frac{\partial A_1}{\partial\psi}-aB_1^2-C^*(a,\psi)\right].$$

Conclusion

From the results received it follows that

1. The multifrequency excitation has considerable influence on the parametric oscillations only in monofrequency resonant case (§ 2).

2. For monofrequency resonant case (§ 2) in the first approximation only the component of external excitation with resonant frequency (ν_s) has influence on the parametric oscillation. The influence of other nonresonant components is only found in the second approximation.

3. In multifrequency resonant case (§ 3) the influence of external excitation on parametric oscillation is found only in the second approximation.

Received January 22th, 1979

REFERENCES

1. Ischuch V.V., Iaschuch V.T., Study of equations of Matic type with quasiperiodic coefficients by asymptotic methods. Asymptotic and qualitative method of nonlinear vibrations, Kiev 1971 (in Russian).

2. Bogoliubov N.N., Mitropolski Iu. A., *Asymptotic methods in Theory of nonlinear vibrations*, Moscou 1963 (in Russian).