ACTA MATHEMATICA VIETNAMICA

TOM 4, N° 2 (1979)

COOPERATIVE GAMES WITH MULTIPAYOFFS

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I. INTRODUCTION.

There are two widely used models for the study of n-person games, the strategic games (in the normal or extensive form) and the cooperative games (with or without side payments).

In this paper we are concerned with cooperative games with multipayoffs.

We shall suppose that each player i has a finite set $M(i)$ of criteria and that each coalition S has a set of criteria $M(S) = \bigcup_{i \in S} M(i)$.

We denote by $N = \{1, 2, ..., n\}$ the set of players and we shall work in a Euclidean space whose dimension is equal to the total number m of criteria of all players (m = carninality of $M(N)$). Let E^m be the Euclidean space of dimension m and let $E^{M(S)}$ be the subspace of E^m made up of all points $x \in E^m$ with coordinates $x_i = 0$ for all $i \notin M(S)$.

It is not excluded that $M(i) \cap M(j) \neq \phi$ in which case side payments are allowed between players i, j and some parts of players ultilities (represented by vectors of $E^{M(i)}$ and $E^{M(j)}$) are transferable.

The cooperative games to be defined in this paper will generalize Neumann-Morgenstern' cooperative games with side payments. (if $M(i) \cap M(j) \neq \phi$ for certain i, j) and also Aumann' cooperative games without side payments (if $M(i) \cap M(j) = \phi$ for every i, j). Note that if we extend these games in Aumann's way to cooperative games with multipayoffs without side payments, then we shall obtain again a subclass of the games under consideration.

Thus, these games with side payments will constitute an essential generalization of the cooperative games usually considered in game theory.

The basic definition will be given in section 2. In section 3 we shall establish for games with multipayoffs some assertions which in cooperative game theory are well known. Some properties of balanced games will be studied in the three last sections. Here we shall show that the cover of a game $\langle N, v \rangle$ is the smallest totally balanced game containing v and that the class of market games with ultility functions coincides with the class of totally balanced games in which $v(S)$ is convex for each $S \subset N$. Finally, we shall establish the nonemptiness of the core of a balanced game.

2. DEFINITIONS.

Denote by $M = M(N)$ and $E_+^{M(S)}$ the nonnegative orthant of $E^{M(S)}$.

Points x^i of $E^{M(i)}$ are called payoff vectors of player i, and a vector $x = (x^1, x^2, \dots, x^n)$ where $x^i \in E^{M(i)}$ is called a payoff vector. The characteristic mapping of the game is the mapping that associates to each $S \subset N$ a subset $v(S)$ of $E^{M(S)}$.

We assume:

$$v(S) = C_S - E_+^{M(S)}$$

where C_S is a compact subset of $E^{M(S)}$.

A pair $\langle N, v \rangle$ is then called a cooperative game with multipayoffs.

Let $M' \subset M'' \subset M$ and $A \subset E^{M'}$ For the sake of convenience denote $A_{M'} = A \times 0_{M'' \setminus M'}$, where $0_{M'' \setminus M'}$ is the origin of $E^{M'' - M'}$. Let $Po(A)$ denote the set of effective points of A, i, e., the set of points $z \in A$ for which there is no $z \in A$ such that $z' > z$ *).

A payoff vector $x = (x^1, x^2, \dots, x^n)$ is said to be individually rational if $x^i \in Po\ (v(i)) + E_+^{M(i)}$ for all $i \in N$, and group rational if there is no payoff vector y such that $\bar{y} \in v(N)$ and $\bar{y} > \bar{x}$, where:

$$\bar{x} = \sum_{i \in N} x^i_M .$$

The set of imputation $I(v)$ is the set of individually rational payoff vectors x such that $\bar{x} \in v(N)$.

We shall distinguish two important classes of cooperative games with multipayoffs.

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*) If $x, y \in E^{M'}$, then $x < y$ means $x_\nu < y_\nu$ for every $\nu \in M'$.

The first class corresponds to the case $M(S) \equiv M$ for all $S \subset N$ and contains the wellknown Neumann-Morgenstern cooperative games as a subclass. Actually, in this case we obtain the Von Neumann-Morgenstern multipayoff cooperative games in which side payments are allowed and some parts of ultilities are transferable.

A proper subclass of the above class consists of games $\langle N, v \rangle$ with:

$$v(S) = a_S - E^M_+, \; a_S \in E^M .$$

The second class corresponds to the case $M(i) \cap M(j) = \phi$ for every pair $i, j \in N$, $i \neq j$. We then obtain the cooperative games without side payments and with multipayoffs which constitute a direct generalization of Aumann cooperative games without side payments.

Let now x, y be payoff vectors. For $S \subset N$ we write $x \underset{S}{\succ} y$ iff $x^i > y^i$ for all $i \in S$ and $\bar{x}^S = \sum_{i \in S} (x^i)_{M(S)} \in v(S)$, $x \succ y$ iff $x \underset{S}{\succ} y$ for some $S \subset N$.

We say that y is dominated by x if $x \succ y$. Let K be a set of payoff vectors. Just as in Neumann-Morgenstern theory a solution of K is by definition a subset D of K such that no two members of D dominate each other, and every member of K not in D is dominated by some number of D. The core of K (denoted by $C(K)$) is the set of members of K not dominated by any other member of K.

A solution of the game is a solution of $I(v)$ and a *core of the game* is a core of $I(v)$.

3. SOME ELEMENTARY PROPERTIES.

Let $I_g(v)$ denote the set of group rational payoff vectors x such that $\bar{x} \in v(N)$.

Proposition 1. *A solution of $I(v)$ is a solution of $I_g(v)$ and conversely.*

Proof. Let B be a solution of $I(v)$, then $I(v) = B + domB$. Let $x \in B$. If $x \notin I_g(v)$ then there exists a payoff vector y such that $\bar{y} > \bar{x}$ and $\bar{y} \in v(N)$. Take the payoff vector $z = (z^1, \dots, z^n)$ where for each $i \in N$, $v \in M(i)$, z^i_v is defined by:

$$z^i_v = x^i_v + \frac{(\bar{y})_v - (\bar{x})_v}{|N(v)|} \tag{1}$$

with $N(v) = \{i : v \in M(i)\}$.

Then $(\bar{z})_v = (\bar{y})_v$ for each $v \in M$, i. e, $\bar{z} = \bar{y} \in v(N)$ and $z^i > x^i$, i.e., $z \underset{N}{\succ} x$. Thus z is an imputation and $z \underset{N}{\succ} x$. But $x \in B$ hence $z \notin B$, i. e., there exists an imputation $z' \in B$ and a subset S' of N such that $z' \underset{S'}{\succ} z$. Hence $z' \underset{S'}{\succ} x$, i. e., $z' \succ x$ and $x \in dom\ B$.

This contradiction shows that $x \in I_g(v)$. Hence $B \subset I_g(v)$. Moreover, every point of $I_g(v)$ not in B, is also a point of $I(v)$ not in B, therefore it is dominated by some member of B and no two members of B dominate each other, i. e. B is a solution of $I_g(v)$.

Conversely, let D be a solution of $I_g(v)$. Then $I_g(v) = D + dom_{I_g(v)} D$. Obviously, $D \subset I(v)$ and no two members of D dominate each other.

Let $x \in I(v) \setminus I_g(v)$. Then there is a payoff vector y such that $\bar{y} \in v(N)$ and $\bar{y} \succ \bar{x}$. Consider now a payoff vector z defined by (1). Then z is an imputation and $z \underset{N}{\succ} x$. If $z \in I_g(v)$ (indeed, $z' = z$ if $z \in D$ and $z' \neq z$ if $z \in dom_{I_g(v)} D$).

If $z \in I(v) \setminus I_g(v)$, then in the same way as before starting from z we obtain imputation z_1 such that $z_1 \underset{N}{\succ} z$. Continuing this procedure, we shall have two possibilities: either after a finite number of steps we obtain an imputation $z^* \in D$ and $z^* \succ x$, or we get an infinite sequence of imputations $(x, z, z_1, \ldots)$. Since $v(N) = C_N - E^m_+$ with C_N compact, there exists $z^{**} \in I(v)$ and $z_k \to z^{**}$, $z^{**} \underset{N}{\succ} z_k$ for all k. Obviously, $z^{**} \in I_g(v)$. Then we can choose an imputation $z^0 \in D (z^0 = z^{**}$ if $z^{**} \in D$ and $z^0 \neq z^{**}$ if $z^{**} \in dom_{I_g(v)} D)$ such that $z^0 \succ x$. This implies $x \in dom_{I(v)} D$ and so $I(v) = D + dom D$.

Proposition 2. If $N = \{1,2\}$ and $v(1)_M + v(2)_M \subset v(1,2)$ then the game has a unique solution, which is just the core of the game.

Proof. We have:

$$I(v) = \left\{ x = (x^1, x^2) : \begin{array}{ll} x^i \in Po(v(i)) + E^{M(i)}_+ & i = 1,2 \\ x^1_M + x^2_M \in v(1, 2) & \end{array} \right.$$

First, we shall show that:

$$C(I(v)) = I(v) \cap \{ x = (x^1, x^2) : \bar{x} \in \partial v(1,2) \}$$

where ∂A is the boundary of A. Indeed, the preference $\underset{1}{\succ}$ and $\underset{2}{\succ}$ being not defined on $I(v)$, we have only to consider the preference $\underset{(1,2)}{\succ}$ on the set of imputations.

Let x be an imputation with $\bar{x} \in int\ v(1, 2)$. Then there exists $y = (y^1, y^2)$ such that $y^1 > x^1$, $y^2 > x^2$ and $\bar{y} \in v(1, 2)$. Thus $y \in v(1, 2)$ and $y \succ x$, i.e., $x \notin C(I(v))$, hence the assertion.

Since C_{12} is a compact set, $\partial v(1, 2)$ is compact and $Po(\partial v(1,2)) \neq \phi$. Furthermore, since $Po(v(1)) + Po(v(2)) \subset v(1, 2)$ there are vectors $z \in Po\ (\partial v(1, 2))$, $x'^1 \in Po(v(1))$, $x'^2 \in Po(v(2))$ such that:

$$x'^1_M + x'^2_M \leqslant z.$$

If $x'^1_M + x'^2_M = z$, then $x' = (x'^1, x'^2) \in C(I(v))$. On the other hand, if $x'^1_M + x'^2_M \neq z$ there is an index v such that

$(x'^1)_v + (x'^2)_v < z_v$. Now take:

$$u^1_v = \begin{cases} z_v & \text{if } v \in M(1) \setminus M(2) \\ x'^1_v + \dfrac{z_v - (x'^1_v + x'^2_v)}{2} & \text{if } v \in M(1) \cap M(2) \end{cases}$$

$$u^2_v = \begin{cases} z_v & \text{if } v \in M(2) \setminus M(1) \\ x'^2_v + \dfrac{z_v - (x'^1_v + x'^2_v)}{2} & \text{if } v \in M(1) \cap M(2) \end{cases}$$

Then $u^1 \in x^1 + R^{M_1}_+$, $u^2 \in x^2 + R^{M_2}_+$, $u^1 + u^2 = z$. This implies $u \in C(I(v))$, i.e., $C(I(v)) \neq \phi$.

Moreover if A is a solution of the game, then $A = I(v) \setminus dom\ A$, hence $C(I(v)) \subset A$ and $A \neq \phi$.

Now we claim that $A = C(I(v))$. Indeed, assume $x \in A \setminus C(I(v))$, i.e., there exists a payoff vector $y \in I(v)$ such that $y \underset{1,2}{\succ} x$. Then there is $z \in A$ such that $z \underset{(1,2)}{\succ} y$ since $y \notin A$. Consequently, $z \underset{1,2}{\succ} x$, which is impossible.

Thus the game has a unique solution, which is just the core of the game.

Now we shall give a necessary condition for the nonemptiness of the core analogous to a well-known assertion in [3,6].

For each $S \subset N$, define the characteristic vector of S, $e_S \in E^n$ to be:

$$e_S(i) = \begin{cases} 1 & \text{if } i \in T \\ 0 & \text{if } i \notin T \end{cases}$$

A collection $\mathcal{S}$ of subsets S of N is said to be *balanced* if one can assign to each S in $\mathcal{S}$ a non-negative number $\Upsilon(S)$ such that:

$$\sum_{S \in \mathcal{S}} \Upsilon(S)\, e_S = e_N \tag{2}$$

The set of these $\Upsilon(S)$ is denoted by $B(\mathcal{S})$.

Let J be a balanced collection of N.

Proposition 3. If $C(I(v)) \neq \phi$, then for each $S \in J$, there exists a vector $u^S \in v(S)$ such that $u^S \notin v(S) - int\,(E_+^{M(S)})$ and for any $\Upsilon \in B(J)$ the following inclusion holds

$$\sum_{S \in J} \Upsilon(S)\,(u^S)_M \in v(N)$$

Proof. Let $x \in C(I(v))$. Then $x^i \in Po(v(i)) + E_+^{M(i)}$ for all $i \in N$ and $x^S \notin v(S) - int\,(E_+^{M(S)})$ for each $S \subset N$. Hence there are vectors $u^S \in \partial v(S)$ such that $u^S \leqslant \overline{x^S}$. It is easy to see that $u^S \notin v(S) - int\,(E_+^{M(S)})$ and

$$\sum_{S \in J} \Upsilon(S)(u^S)_M \leqslant \sum_{S \in J} \Upsilon(S)(\overline{x}^S)_M = \sum_{S \in J} \Upsilon(S)\Big(\sum_{i \in S} x^i_M\Big) = \sum_{i \in N}\Big(\sum_{\substack{S \in J \\ S \ni i}} \Upsilon(S)\Big) x^i_M =$$

$$= \overline{x} \in v(N).$$

Remark 1. In the N.M. game $\langle N, v\rangle$ with $v(S) = a^S - E_+^M$, $a^S \in E^M$ we have vector $u^S \in a^S - (E_+^M)$ and $u^S \notin a^S - int(E_+^M)$, i.e., $u^S \in a^S - \partial(E_+^M)$.

When $|M| = 1$. $E^M = E^1$ the game $\langle N, v\rangle$ reduces to the classical N.M. cooperative game with $v(S) = a^S - E_+^1$. Then we shall have numbers $u^S = a^S$ such that for each $\Upsilon \in B(J)$

$$\sum_{S \in J} \Upsilon(S) a^S \leqslant v(N).$$

This is the well-known balanced condition given in [3, 6].

4. THE COVER OF A GAME.

The notion of cover of a side payment game has been introduced by Shaley and Shubik [8]. The cover of a game without side payments has been considered by Billera [2]. Here we shall extend this notion to games with multipayoffs.

Let $S \subset N$. Denote by $B(S)$ the set of all balancing weights for balanced collections on S, i.e.,

$$B(S) = \Big\{\Upsilon : 2^S \to E_+^1 : \sum_{\substack{T \subset S \\ T \ni i}} \Upsilon(T) = 1 \text{ all } i \in S\Big\}$$

where $2^S = \{T \subset S,\ T \neq \phi\}$.

Let there be given a game with multipayoffs $\langle N, v\rangle$. For each $\Upsilon \in B(S)$ we define

$$v_\Upsilon = \sum_{T \subset S} \Upsilon(T)\,(v(T)_{M(S)}$$

The cover of $\langle N, v\rangle$ is the game with multipayoffs $\langle N, \overline{v}\rangle$ where the mapping $\overline{v}$ is defined for each $S \subset N$ by:

$$\overline{v}(S) = \bigcup\{v_\Upsilon : \Upsilon \in B(S)\}.$$

Let $B^m(S)$ be the set of weights for minimal balanced collections on S, i. e. $B^m(S) = \{\Upsilon : B(S) : \{e_T \mid \Upsilon(T) > 0\}$ is linear independent$\}$.

In [6] it has been proved that $B(S) = conv\ B^m(S)$ when $B^m(S)$ considered as a subset of $E^{2|S|-1}$, where conv A is the convex hull of A.

Since $B^m(S)$ is a finite set, we may denote $|B^m(S)| = k$ and

$$P^k = \{\alpha : \alpha_j \geqslant 0,\ \sum_{j=1}^{k} \alpha_j = 1\}.$$

Proposition 4. *For each $S \subset N$ we have*

$$\overline{v}(S) = \bigcup_{\alpha \in P^k} \left\{ \sum_{\Upsilon \in B^m(S)} \alpha_\Upsilon\ v_\Upsilon \right\}$$

Proof. Indeed

$$\bigcup_{\alpha \in P^k} \left\{ \sum_{\Upsilon \in B^m(S)} \alpha_\Upsilon v_\Upsilon \right\} = \bigcup_{\alpha \in P^k} \left\{ \sum_{\Upsilon \in B^m(S)} \alpha_\Upsilon \left\{ \sum_{T \subset S} \Upsilon(T)\,(v(T))_{M(S)} \right\}\right\}$$

$$= \bigcup_{\alpha \in P^k} \left\{ \sum_{T \subset S} \left(\sum_{\Upsilon \in B^m(S)} \alpha_\Upsilon \Upsilon(T)\ v(T)_{M(S)} \right\} = \right.$$

$$\bigcup_{\theta \in B(S)} \left\{ \sum_{T \subset S} \theta(T)\,(v(T))_{M(S)} \right\} = \bigcup \{ v_\theta .\ \theta \in B(S)\} = \overline{v}(S).$$

Denote

$$\overline{C_S} = \bigcup_{\alpha \in P^k} \left\{ \sum_{\Upsilon \in B^m(S)} \left\{ \sum_{T \subset S} \Upsilon(T)\,(C_T)_{M(S)} \right\}\right\}$$

It is easy to see that $\overline{C}_S$ is compact (since $v(T) = C_T - E_+^{M(T)}$, with C_T compact) and we have:

$$\overline{v}(S) = C_S - E_+^{M(S)}.$$

Thus $\langle N, \overline{v} \rangle$ is also a game.

Remark 2. If for each $S \subset N$, $v_S = C_S - E_+^{M(S)}$ with C_S convex and compact, then:

$$\overline{v}(S) = \text{conv}\left\{ \bigcup v_\Upsilon : \Upsilon \in B^m(S) \right\}$$

and $\overline{v}(S) = \overline{C}_S - E_+^{M(S)}$ with $\overline{C}_S$ being also convex.

In the case $M(i) = i$ for all $i \in N$, $M(i) \cap M(j) = \phi$ for $i \neq j$, we get proposition 3.1 in [2].

A game $\langle N, v \rangle$ is said to be balanced if:

$$v(N) = \bigcup \{v_\Upsilon : \Upsilon \in B(N)\}.$$

It is totally balanced if for each $S \subset N$

$$v(S) = \bigcup \{ v_\Upsilon : \Upsilon \in B(S)\}.$$

Theorem 1. $\langle N, \overline{v} \rangle$ is the smallest totally balanced game containing $\langle N, v \rangle$.

Proof. Since the function Υ° with $\Upsilon^\circ(T) = 1$ if $T = S$ and $\Upsilon^\circ(T) = 0$ for each $T \subset S$, $T \neq S$ is in $B(S)$ we have for each $S \in N$

$$v(S) = v_{\Upsilon_0} \subset \bigcup \{v_\Upsilon : \Upsilon \in B(S)\} = \overline{v}(S)$$

Now we shall first show that $\langle N, \overline{v} \rangle$ is totally balanced. It suffices to verify the condition:

$$\overline{v}(S) = \bigcup \{\overline{v}_\Upsilon : \Upsilon \in B(S)\}$$

where $\overline{v}_\Upsilon = \sum_{T \subset S} \Upsilon(T)(\overline{v}(T))_{M(S)}$.

Then $\overline{v}(S) = \overline{v}_{\Upsilon_0}$ since $\Upsilon^\circ \in B(S)$.

Conversely, for each $x \in B(S)$, since $\bar{v}(T) = \bigcup \{v_\delta : \delta \in B(T)\}$, it follows that for each $y \in \bar{v}_\Upsilon$ there are $\delta_T \in B(T)$ satisfying:

$$y = \sum_{T \subset S} \Upsilon(T)(v_{\delta_T})_{M(S)} = \sum_{T \subset S} \Upsilon(T)(\sum_{U \subset T} \delta_T(U) v(U))_{M(S)})$$

$$= \sum_{U \subset S} (\sum_{T \subset U} \Upsilon(T)\, \delta_T(U))(v(U))_{M(S)}$$

$$= \sum_{U \subset S} \lambda_s(U)\, v(U)_{M(S)}$$

where

$$\lambda_S(v) = \sum_{\substack{T \subset S \\ T \supset U}} \Upsilon(T) . \delta_T(U).$$

We claim that $\lambda_S \in B(S)$. Indeed for each $i \in S$.

$$\sum_{\substack{U \subset S \\ U \ni i}} \lambda_S(U) = \sum_{\substack{U \subset S \\ U \ni i}} (\sum_{\substack{T \supset U \\ T \subset S}} \Upsilon(T)\, \delta_T(U)) =$$

$$= \sum_{\substack{T \subset S \\ T \supset U \ni i}} \Upsilon(T)(\sum_{\substack{U \subset T \\ U \ni i}} \delta_T(U)) = \sum_{\substack{T \subset S \\ i \in T}} \Upsilon(T) = 1$$

since $\delta_T \in B(T)$, $\Upsilon \in B(S)$.

Thus $y \in \sum_{U \subset S} \lambda_S(U)\, v(U) = v_\lambda$, i. e., $y \in \overline{v}(S)$. This implies $\overline{v}_\Upsilon \subset \overline{v}(S)$ for all $\Upsilon \in B(S)$.

To complete the proof, it remains to show that if $\langle N, \tilde{v} \rangle$ is a totally balanced game with $v \subset \tilde{v} \subset \overline{v}$, then $\tilde{v} = \overline{v}$.

Indeed, if the restriction of v to S is balanced, i. e., $v(S) = \cup \{v_\delta : \delta \in B(S)\}$, then $v(S) = \overline{v}(S)$. Therefore $\langle N, v \rangle$ is totally balanced if and only if $v = \overline{v}$. Moreover, for each $S \subset N$, $v(S) \subset \tilde{v}(S)$, which implies $v_\Upsilon \subset \tilde{v}_\Upsilon = \sum_{T \subset S} \Upsilon(T)$ $(\tilde{v}(T)_{M(S)}$ for each $\Upsilon \in B(S)$ and hence

$$\overline{v}(S) = \cup \{v_\Upsilon : \Upsilon \in B(S)\} \subset \cup \{\tilde{v}_\nu : \Upsilon \in B(S)\} = v(\tilde{S}) = \tilde{v}(S). \text{ i.e. } \tilde{v} = \overline{v}.$$

5. THE MARKET WITH VECTOR-FUNCTIONS.

Let Z be a Hausdorff topological vector space and for each $i \in N$, let X^i and Y^i be non-empty subsets of Z which will be referred to as the consumption and production sets for i. For each agent i, there are given an initial endowment $w^i \in X^i - Y^i$ and vector-ultility function $u^i : X^i \to E^{Q(i)}$, where:

$$u^i = \{ u^i_\nu : X^i \to E, \ \nu \in Q(i) \}$$

The collection $\{(X^i, Y^i, u^i, w^i), i \in N\}$ is called a market with vector-functions.

The class of markets examined by Shapley [7] and Billera [2] can be considered as a subclass of markets with vector-functions when $Q(i) \equiv 1$ for all $i \in N$.

As usual we can associate to a market with vectorfunctions a cooperative game with multipayoffs as follows.

For each $S \subset N$ denote $Q(S) = \bigcup_{i \in S} Q(i)$. $x^S = (x^i, i \in S)$ and

$$u^i_{Q(S)}(x^i) = \begin{cases} u^i_\nu(x^i) & \text{for } \Upsilon \in Q(i) \\ 0 & \text{otherwise} \end{cases}$$

We define:

$$V(S) = \left\{ v \in E^{Q(S)} : \begin{array}{ll} \exists x^S & v \leqslant \sum_{i \in S} u^i_{Q(S)}(x^i) \\ x^i \in X^i, & \sum_{i \in S} (x^i - w^i) \in \sum_{i \in S} Y^i \end{array} \right\}$$

We shall prove that the mapping V so defined is the characteristic mapping of a cooperative game with multipayoffs, which will be called the market game.

Proposition 5. For each $i \in N$, suppose that:

i) X^i, Y^i are compact, convex subsets of Z.

ii) $u^i_\nu : X^i \to E^1$ is a concave, upper semicontinuous function for each $Q(i)$.

Then the corresponding market game $\langle N, v\rangle$ is a cooperative game with multipayoffs, the characteristic mapping of which is:

$$V(S) = C_S - E_+^{Q(S)}$$

where $C_S \subset E^{Q(S)}$ is compact and convex.

Proof. It is easy to verify that $V(S)$ is the projection on $E^{Q(S)}$ of the set $A_S \times E^{Q(S)} \cap B_S$, where A_S is the projection on X^i of the set

$$\{(x^i, y^i, i \in S) : \sum_{i \in S} (x^i - \omega^i - y^i) = 0,\ x^i \in X^i,\ y^i \in Y^i\}$$

and

$$B_S = \{(x^S, v) : v \leqslant \sum_{i \in S} u^i_{Q(S)}(x^i)\}.$$

The assertion follows immediately.

Theorem 2. A market game $\langle N, V\rangle$ is totally balanced.

Proof. We shall show that for each $S \subset N$

$$V(S) = \cup \{V_\gamma : \gamma \in B(S)\}.$$

It is enough to verify that $V_\gamma \subset V(S)$ for each $\gamma \in B(S)$. Indeed, if $v^T \in V(T)$ then there exists $x^T = \{x^i : i \in T,\ x^i \in X^i\}$ such that

$$\sum_{i \in T} (x^i - w^i) \in \sum_{i \in T} Y^i$$

and

$$v^T_{Q(S)} \leqslant \sum_{i \in T} u^i_{Q(S)}(x^i)$$

Hence for $\gamma \in B(S)$

$$\sum_{T \in S} \gamma(T)\ v^T_{Q(S)} \leqslant \sum_{T \subset S} \gamma(T) \left(\sum_{i \in T} u^i_{Q(S)}(x^i)\right) =$$

$$= \sum_{i \in S} \left(\sum_{\substack{T \subset S \\ T \ni i}} \gamma(T)\ u^i_{Q(S)}(x^i)\right) \leqslant \sum_{i \in S} u^i_{Q(S)}(\tilde{x}^i)$$

where $\tilde{x}^i = \sum_{\substack{T \subset S \\ T \ni i}} \gamma(T)\ x^i \in X^i$.

On the other hand we have $y^i \in Y^i$ such that:

$$\sum_{i \in S} (x^i - w^i) = \sum_{i \in T} y^i \text{ hence:}$$

$$\sum_{T \subset S} \gamma(T) \sum_{i \in T} (x^i - w^i)) = \sum_{T \subset S} \gamma(T) \left(\sum_{i \in T} y^i\right)$$

$$\sum_{i \in S} \tilde{x}^i - w^i = \sum_{i \in S} \tilde{y}^i \in \sum_{i \in S} Y^i.$$

where $\tilde{y}^i = \sum_{\substack{T\subset S \\ T\ni i}} \gamma(T) y^i$.

This shows that $\sum_{T\in S} \gamma(T) v^T_{Q(S)} \in V(S)$, i.e., $V_\gamma \subset V(S)$. Thus we have proved that a market game is totally balanced.

Let $\langle N, v\rangle$ be a cooperative game with multipayoffs. We associate to it a market with vector functions in the following way. We choose a positive number b_0 such that:

$C_S = [-b_0, b_0]^{m(S)} \subset A$ where $A = [-b_0, b_0]^m$ since C_S is compact for each $S \subset N$

For each $i \in N$, let

$$X^i = A \times \{0_{E^n}\} \subset E^{m+n}$$

$$Y^i = conv\left(\bigcup_{S\subset N} (C_S)_M \times \{-e_S\}\right) U\,(0_{E^m}, 0_{E^n})) \subset E^{m+n}$$

$$w^i = (0, e_i) = (0, 0) - (0, e_i) \subset X^i - Y^i$$

$u^i : E^{m+n} \to E^{M(i)}$ defined for each $z = (z^i, \bar{z})$, $z^i \in E^{M(i)}$ $\bar{z} \in E^{M \smallsetminus M(i)}$ by $u^i(z) = z^i$.

We thus obtain a market with vector-functions generated by $\langle N, v\rangle$.

Theorem 3. Let $\langle N, v\rangle$ be a cooperative game with multipayoffs, where C_S is convex for each $S \subset N$. If $\langle N, V^*\rangle$ is a market game cooresponding to the market with vector-functions generated by $\langle N, v\rangle$, then $V^* = \bar{v}$.

Proof.

1) For each $S \subset N$ we have:

$$V^*(S) = \left\{ z \in E^{M(S)} : \begin{array}{c} \exists x^i \in X^i,\ z \leqslant \sum_{i\in S} (u^i(x^i))_{M(S)} \\ \sum_{i\in S} (x^i - \omega^i) \in \sum_{i\in S} Y^i \end{array} \right\}$$

Thus there are for each $i \in S$, $y^i + Y^i$, $t^i_R \in C_R$ $\alpha^i_R \geqslant 0$, $\sum_{R\subset N} \alpha^i_R = 1$

such that

$$y^i = \sum_{R\subset N} \alpha^i_R((t^i_R)_M,\ -e^1_R)$$

But $\quad x^i = (\bar{x}^i, 0_{E^n}),\ \bar{x}^i \in A,\ -w^i = (0_{E^m}, -e_i)$.

Hence

$$\sum_{i\in S} (\bar{x}^i, 0_{E^n}) + (0_{E^m}, -e^i) = \sum_{i\in S} \left(\sum_{R\subset N} \alpha^i_R (t^i_R)_M,\ -e_k\right)$$

and

$$-e_S = \sum_{i\in S} - e_i = \sum_{i\in S} \left(\sum_{R\subset N} \alpha^i_R(-e_R)\right) = \sum_{R\subset N} \left(\sum_{i\in S} \alpha^i_R\right) - e_R$$

Thus $\alpha^i_R > 0$ implies $R \subset S$ and if we define $\gamma(R) = \sum_{i\in S} \alpha^i_R$ then we have

$$\sum_{\substack{R\subset S\\ R\ni i}} \gamma(R) = \sum_{\substack{R\subset S\\ R\ni i}} \left(\sum_{i\in S} \alpha^i_R\right) = \sum_{R\subset N} \alpha^i_R = 1, \text{ i.e., } \gamma \in B(S)$$

and

$$x = \sum_{i\in S} \overline{x}^i = \sum_{i\in S} \left(\sum_{R\subset N} \alpha^i_R (t^i_R)_M\right) = \sum_{\substack{R\subset S\\ \gamma(R)>0}} \gamma(R)\cdot \frac{1}{\gamma(R)} \left(\sum_{i\in S} \alpha^i_R (t^i_R)_M\right)$$

Since C_S is convex for each $S \subset N$, we have:

$$\sum_{i\in S} \frac{1}{\gamma(R)} \alpha^i_R \left(t^i_R\right)_M \in (c_R)_M$$

$$\sum_{i\in S} (u^i(x^i))_{M(S)} = \sum_{i\in S} (\overline{x}^i)_{M(S)} = \overline{x}$$

with

$$\sum_{i\in S} (\overline{x}^i)_{M(S)} \in \sum_{R\in S} \gamma(R)(c_R)_{M(S)} \subset \overline{v}(S)$$

Thus $z \in \overline{v}(S)$, i.e., $V^*(S) \subset \overline{v}(S)$.

2) Finally, we shall show that $v(S) \subset V^*(S)$ for each $S \subset N$. Indeed, let $z \in v(S)$. Then there exists $z' \in C_S$ such that $z \leqslant z'$.

For each $v \in M(S)$ denote $M(v, S) = \{j \in S : v \in M(j)\}$ We have $|M(v, S)| \geqslant 1$. For each $i \in S$, $v \in M$ define the vector x^i by

$$x^i_v = \begin{cases} 0 & v \notin M(i) \\ \dfrac{z_{v'}}{|M(v,S)|} & v \in M(i) \end{cases}$$

Since $z' \in C_S$ and $(0, 0) \in Y^i$ hence $\left(x^i, -\frac{e_S}{|S|}\right) \in Y^i$. Thus $(x^i, 0_{E^n}) \in X^i$, $\left(x^i - \frac{es}{S}\right) \in Y^i$ and

$$\sum_{i\in S} (x^i, o_{E^n}) - w^i = \sum_{i\in S} (x^i, o_{E^n}) - (o, e_i) = \sum_{i\in S} \left(x^i, -\frac{e_S}{|S|}\right) \in \sum_{i\in S} Y^i$$

$$\sum_{i\in S} (u^i(x^i))_{M(S)} = \sum_{i\in S} (x^i)_{M(S)} = z' \geqslant z.$$

Thus $z \in V^*(S)$, i.e., $\langle N, V^* \rangle$ is a totally balanced game containing $\langle N, v \rangle$, i.e., $v \subset V^*$. This implies $V^* = \overline{v}$.

Corollary. Every totally balanced game with multipayffos, where C_S is convex for each $S \subset N$, is a market game with vector-functions satisfying conditions i) and ii) of Proposition 5 and vice versa.

6. THE CORE OF A BALANCED GAME.

Let $\langle N, v \rangle$ be a balanced game, i.e., $v(N) = \cup \{ v_\gamma : \gamma \in B(N) \}$ and let $\langle N, \overline{v} \rangle$ be its cover. From section 4 we know that $\langle N, \overline{v} \rangle$ is totally balanced and $v(S) \subset \overline{v}(S)$ for each $S \subset N$.

Proposition 6. The core of $\langle N, v \rangle$ contains the core of $\langle N, \overline{v} \rangle$.

Indeed, if for each $x \in I(v)$ we denote

$$D_v(x) = \{ y \in I(v) : \exists S \subset N \sum_{S \in i} (y^i)_{M(S)} \in v(S) \; y \underset{S}{\succ} x \}$$

then

$$C(I(v)) = \{ x \in I(v) : D(x) = \phi \}$$

But $v(i) = \overline{v}(i)$ and $v(N) = \overline{v}(N)$ hence $I(v) = I(\overline{v})$ and $D_v(x) = D_{\overline{v}}(x)$. Hence

$$C(I(v)) = \{ x \in (v) : D_v(x) = \phi \} \supset$$
$$\{ x \in I(v) : D_{\overline{v}}(x) = \phi \} = c(I(\overline{v}))$$

Now we consider a game $\langle N, v \rangle$ with $v(S)$ convex for each $S \in N$. Denote by P the standard simplex in E^m, i.e.,

$$P = \{ p \in E^m_+ , \; \sum_v p_v = 1 \}$$

We shall prove the nonemptiness of the core of a balanced game by an argument analogous to that used by Aubin in [4].

Let $S \subset N$, $p \in P$. We define:

$$g(S, p) = \max_{u \in v(S)} \langle u_M , p \rangle = \max_{n \in C_S} \langle u_M , p \rangle$$

Proposition 7. [4]. The function $g(S,.)$ is convex and continuous and

$$v(S) = \{ x \in E^{M(S)} : \; g(s, p) - \langle p, x_M \rangle \geqslant 0 \; \forall p \in P \}$$

Let $x^i \in E^{M(i)}$, $y^i = \langle p, x^i_M \rangle \in E^1$.

Proposition 8. Let the game $\langle N, v \rangle$ be balanced Then for each $p \in P$ the linear programming problem:

$$(\mathcal{P}_p) - \begin{cases} \min \sum_{i \in N} y^i \\ \sum_{i \in S} y^i \geqslant g(S, p) \qquad \forall S \subset N \end{cases}$$

has a solution $y^* = \{ y^{i*}, i \in N \}$ such that $\sum_{i \in N} y^{i*} = g(N, p)$

Proof. The dual problem $\mathcal{P}_p^*$ of $\mathcal{P}_p$ is

$$\mathcal{P}_p^*) \begin{cases} \max \sum_{S\subset N} \gamma_S \quad g(S, p) & \\ \sum_{S\ni i} \gamma_S = 1 & \forall i \in N \\ \gamma_S \geqslant 0 & \forall s \subset N \end{cases}$$

Because $B(N) \neq \phi$, $\mathcal{P}_p^*$ has a solution $\{\gamma_S^*: S \subset N\}$ and $\mathcal{P}_p$ has a solution $\{y^{i*}, i \in N\}$.

Let $B_0 = \{S \subset N: \gamma_S^* > 0\}$. Obviously B_0 is a balanced collection of N and

$$\forall S \in B_0 \sum_{i\in S} y^{i*} = g(S, p)$$

We have:

$$\sum_{i\in N} y^{i*} = \sum_{i\in S} (1) \cdot y^{i*} = \sum_{i\in N} (\sum_{S\ni i} \gamma_S^*) y^{i*} = \sum_{S\in B_0} \gamma_S^* (\sum_{i\in S} y^{i*}) = \sum_{S\in B_0} \gamma_S^* g(S, p) =$$
$$= \sum_{S\in B_0} \gamma_S^* \max_{u\in v(S)} \langle p, u_M \rangle = \max \{ \langle p, u \rangle : u \in \sum_S \gamma_S^* (v(S))_M \}$$
$$\leqslant \max \{ \langle p, u \rangle : u \in v(N) \} = g(N, p)$$

On the other hand the inequality $g(N, p) \leqslant \sum_{i\in N} y^i$ is a constraint of the problem $\mathcal{P}_p$. Since C_S is compact, we can choose for each $S \subset N$ two vectors a_S, $b_S \in E^{M(S)}$ such that:

$$C_S \subset (a_S + E_+^{M(S)}) \cap (b_S + E_+^{M(S)})$$

(hence $v(S) \subset b_S - E_+^{M(S)}$) and $\forall p \in P \ \langle p, (b_S)_M \rangle \geqslant g(S, p)$.

Now let $P^\varepsilon = \{p \in P: p_v \geqslant \varepsilon \ \forall v\}$ for each $\varepsilon > 0$.

Theorem 4. Every balanced cooperative game with multipayoffs has a nonempty core.

Proof. First, given $\varepsilon > 0$ we define the multivalued mapping:

$$G: \begin{matrix} u = (u^1, \ldots, u^n) \\ u^i \in E^{M(i)} \end{matrix} \mapsto \left\{ \begin{matrix} p : g(N, p) - \langle \overline{u}, p \rangle \leqslant \\ g(N, p) - \langle \overline{u}, p' \rangle \ \forall p' \in P^\varepsilon \end{matrix} \right\}$$

It is easy to sse that G is a closed mapping and $G(u) \neq \phi$, convex, compact for each u. Next define:

$$U: p \in P^\varepsilon \rightarrow \left\{ \begin{matrix} u = (u^1, \ldots, u^n) \ ; \ u^i \geqslant a_i \text{ such that} \\ \{y^i = pu_M^i\} \text{ is a solution of } (\mathcal{P}_p) \end{matrix} \right\}$$

It is not difficult to verify that U is a closed mapping and $U(p) \neq \phi$, convex, compact for each $p \in P$.

Now let $b^\circ = \max_{S \subset N} \{(b_S)_M\}$. Then, setting $y^{io} = p^i b^\circ_i$, we have:

$$\sum_{i \in S} y^{ib} \geqslant \langle p, u_M \rangle \ \forall \ u \in v(S), \text{ hence}$$

$$\sum_{i \in S} y^{ib} \geqslant g(S, p) \qquad \forall \ S \subset N. \text{ Thus}$$

$$\langle p, b^\circ \rangle = \sum_{i \in N} y^{io} \geqslant \sum_{i \in N} y^{i*}$$

por each solution $y^* = \{ y^{i*} \}$ of the problem $(\mathscr{P}_p)$.

But for $p \in P^{\varepsilon}$, we have $p\, u^{i*} = p^{i*}$

$$\varepsilon \sum_{i \in N} u^{i*} \leqslant \sum_{i \in N} p\, u^{i*} = \sum_{i \in N} y^{i*} \leqslant \sum_{i \in N} b^{\circ i}$$

We now put:

$$C = \left\{ u : \ u^i \geqslant a_i, \ \sum_{i \in N} u^i \leqslant \sum_{i \in N} \frac{b^{\circ i}}{\varepsilon} \right\}$$

Clearly C is compact and $U(p) \subset C$ for each $p \in P^{\varepsilon}$. Thus the multivalued mapping Φ defined on $P^{\varepsilon} \times C$ by $\Phi(p, u) = G(u) \times U(p)$ satisfies all conditions of Kakutani' fixed point theoem. Let $(p^{\varepsilon}, \bar{u}^{\varepsilon})$ be a fixed point of Φ. Then we have for $\varepsilon > 0$,

$$\forall p \in p^{\varepsilon}, \ g(N, p) - \langle p, \bar{u}^{\varepsilon} \rangle \geqslant g(N, p^{\varepsilon}) - \langle p^{\varepsilon}, \bar{u}^{\varepsilon} \rangle = 0$$

$$\forall S \subset N, \ \sum_{i \in S} \langle p^{\varepsilon}, u^{i\varepsilon}_M \rangle \geqslant g(S, p^{\varepsilon})$$

On the other hand, when $\varepsilon < \frac{1}{m}$, the vector $p^\circ = \left(\frac{1}{m}, \frac{1}{m}, \dots, \frac{1}{m}\right)$ belongs to P^{ε}, therefore $\langle p^\circ, \bar{u}^{\varepsilon} \rangle \leqslant g(N, p^\circ)$ i. e.,

$$\sum_{v=1}^{m} \frac{1}{m} \left(\sum_{i \in N} u^{i\varepsilon} \right)_v \leqslant g(N, p^\circ)$$

$$(1, 1, \dots, 1) \ (\sum_{i \in N} u^{i\varepsilon}) \leqslant m\, g(N, p^\circ)$$

$$(1, 1, \dots, 1) \sum_{i \in N} (u^{i\varepsilon} - a_i) \leqslant m\, g(N, p^\circ) - (1, 1, \dots, 1) \sum_{i \in N} a_i)$$

$$\sum_{v=1}^{m} (\sum_{i \in N} a^{i\varepsilon} - a_i)_2 \leqslant m\, g(N, p^\circ) - \sum_{v=1}^{m} (\sum_{i \in N} a^{i}_{M})_v = K.$$

The sequence u^{ε} in the compact set has a cluster point $u^* = \{u^{*i}, i \in N\}$ as $\varepsilon \downarrow 0$ and $p^{\varepsilon} \to p^* \in P$.

Finally we obtain:

$$\forall p \in P \qquad g(N, p) - \langle p, \overline{u^*} \rangle \geqslant 0 \quad .$$

i. e., $\overline{u^*} \in v(N)$ and

$$\forall S \subset N \qquad \sum_{i \in S} \langle p^*, u^i_M \rangle \geqslant g(S, p^*)$$

which implies that u^* is a point of the core of the game $\langle N, v \rangle$.

The author is grateful to Prof. N.N. Vorobiov for useful discussions.

Received April 30, 1978

(Revised manuscript received January 2, 1979)

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