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COMBINATORIAL METHOD FOR SOLVING NONLINEAR EQUATIONS IN FINITE-DIMENSIONAL AND INFINITE-DIMENSIONAL SPACES

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A large number of problems in linear and nonlinear analysis, convex and nonconvex optimization, control theory, mathematical economics, and some other fields of applied mathematics, can be reduced to solving an equation of form

$$0 \in F(x), \qquad x \in \Omega, \tag{*}$$

where Ω is a given subset of a linear space X and F is a multi-valued mapping from Ω into a linear space Y. Obviously, if $Y = X$ and $F(x) = f(x) - x$, then solving the equation (*) amounts to finding a fixed point of the mapping f.

Since the appearance of Scarf's pioneering work [9], combinatorial methods for finding fixed points have attracted much attention (see [4], where an extensive bibliography on this subject can be found). In some recent papers [10] [11] [12], the author has developed a new combinatorial algorithm for finding fixed points, which allows to overcome a major difficulty of Scarf's original algorithm, by using a procedure different from the methods known in the literature under the names of «homotopy» and «sandwich» methods. The present paper is a continuation of this work. Specifically, starting from the above mentioned algorithm, we shall attempt to construct methods for solving equations (*) under very general hypotheses on Ω and F. In this way, apart from some new results on the solvability condition of equations (*) (for example, the inward* boundary condition to be introduced in section 1, the general approximation scheme for reducing to finite dimensions in section 2, some results on quasi-variational inequalities in section 3, we shall obtain an *algorithmic proof*, based on a unified combinatorial approach, for many known fixed point propositions. These include some recent and general results in this field, such as fixed point theorems for inward and outward mappings [2] [7] or fixed point theorems on non compact domains [6].

As a matter of notations, we shall denote, for a set A in a linear topological space X: int $A =$ the interior of A; ri $A =$ the relative interior of A;

$\bar{A}$ = the closure of A; $co\ A$ = the convex hull of A; $aff\ A$ = the affine hull of A; $\partial A = \bar{A} \setminus int\ A$ (the topological boundary of A); $\partial_a A = \{x \in A : (\exists u \in X)(\forall \varepsilon > 0)(\exists \lambda, |\lambda| \leqslant \varepsilon)\ x + \lambda u \notin A\}$ (the algebraic boundary of A);

$$I_A(x) = \overline{\bigcup_{\lambda \geqslant 0} \lambda(A - x)} \text{ (the tangent cone of } A \text{ at } x)$$

$N_A(x) = \{y \in X^* : (\forall u \in A)\ \langle y, u - x\rangle \geqslant 0\}$ (the inward normal cone of A at x), where, as usual, X^* is the dual space of X, $\langle y, x\rangle$ is the value of the functional $y \in X^*$ at point $x \in X$. Also we shall write $x_k \underset{w}{\rightarrow} x$ to mean the convergence in the weak topology of X.

§ 1. EQUATIONS IN FINITE-DIMENSIONAL SPACES.

1. Equations in a simplex.

Let there be given in a finite-dimensional euclidean space X a n-simplex $S = [a_0, a_1, \dots, a_n]$, i.e. the convex hull of $n+1$ affinely independent points $a_0, a_1, \dots, a_n$ (called the vertices of S). For every $i = 0, 1, \dots, n$, the convex hull of n points a_j ($j \neq i$) is a $(n-1)$-simplex which will be called the i-th face of S and denoted by D_i.

We start from the following result.

Theorem 1. Assume that a mapping $L : S \rightarrow R^n$ satisfies the condition

$$\forall x \in ri\ D_i\ L(x) = b_i\ (i = 0, 1, \dots, n) \tag{1.1}$$

where $b_0, b_1, \dots, b_n$ are such that

$$0 \in int\ co\{b_0, b_1, \dots, b_n\}. \tag{1.2}$$

Then there exists a point $\bar{x} \in S$ with $n+1$ sequences $\{x_i^{(k)}\} \subset S$ such that $x_i^{(k)} \rightarrow \bar{x}\ (k \rightarrow \infty)$ for every $i = 0, 1, \dots, n$ and

$$\forall k \quad 0 \in co\{L(x_0^{(k)}), L(x_1^{(k)}), \dots, L(x_n^{(k)})\} \tag{1.3}$$

Proof. Using the algorithm provided in [10], [11], [12], we can find for every given $\varepsilon > 0$ a set U in S such that

$$|U| = n + 1,\ diam\ U < \varepsilon,\ 0 \in co\ L(U). \tag{1.4}$$

By taking $\varepsilon = \varepsilon_k \downarrow 0$ we obtain for each k a set $U_k = \{x_0^{(k)}, x_1^{(k)}, \dots x_n^{(k)}\} \subset S$ such that $diam\ U_k < \varepsilon_k$ and $0 \in co\ \{L(x_0^{(k)}),\ L(x_1^{(k)}), \dots, L(x_n^{(k)})\}$. Since S is compact and $\varepsilon_k \downarrow 0$, we can assume, by taking subsequences if necessary, $x_i^{(k)} \rightarrow \bar{x} \in S\ (k \rightarrow \infty)$ for every $i = 0, 1, \dots, n$.

Note that if U satisfies (1.4), then any point $\dot{x} \in U$ can be used as an approximation of $\bar{x}$, the level of accuracy being measured by *diam U*. Thus the mentioned algorithm allows to compute $\bar{x}$ with any required accuracy.

It turned out that this result can be applied to solve equation (*) under very general hypotheses on Ω and F. Let us first consider the simplest and fundamental case.

Definition 1. A multivalued mapping F from a simplex $S = [a_0, a_1, \dots a_n]$ into R^n is said to be a *S-mapping* if

$$\forall x \in ri\ D_i \qquad b_i \in F(x) \tag{1.5}$$

where $b_i \neq 0$ is a normal of D_i directed inward S, i.e. $\langle b_i, x' - x \rangle \geqslant 0$ for every $x \in D_i$ and every $x' \in S$.

Theorem 2. Let $S = [a_0, a_1, \dots, a_n]$ be a simplex containing 0 in its interior and let F be a S-mapping, which is upper semi-continuous and such that $F(x)$ is nonempty, convex, compact for every $x \in S$. Then the equation $0 \in F(x)$ has a solution in S.

We first show:

Lemma 1. If $0 \in int\ S$ and $b_i \neq 0$ is an inward normal of D_i, then (1.2) holds.

Indeed, S can be described as the set of all x satisfying

$$\langle b_i, x \rangle \leqslant \alpha_i \qquad (i = 0, 1, \dots, n),$$

where α_i is some positive number. Now, the system

$$\langle b_i, x \rangle < 0, \qquad (i = 0, 1, \dots, n)$$

is inconsistent, since if z is any solution of it then for every $\lambda > 0$ we have $\langle b_i, \lambda z \rangle < 0 < \alpha_i$ $(i = 0, 1, \dots n)$, i.e. $\lambda z \in S$, conflicting with the boundedness of S. Therefore, by the Farkas—Minkowski Lemma, there exist numbers $\theta_i \geqslant 0$ $(i = 0, 1, \dots, n)$ satisfying $\sum_{i=0}^{n} \theta_i = 1$ and $(\forall z \in X) \sum_{i=0}^{n} \langle b_i, z \rangle \geqslant 0$. The latter relation implies $\Sigma \theta_i b_i = 0$. But if $\theta_{i_0} = 0$, then

$$0 = \langle \sum_{i=0}^{n} \theta_i b_i, -a_{i_0} \rangle = - \sum_{i \neq i_0} \theta_i \langle b_i, a_{i_0} \rangle = - \sum_{i \neq i_0} \theta_i \alpha_i < 0,$$

a contradiction. Hence, $\theta_i > 0$ for every i, i.e. (1.2) holds. □

Proof of Theorem 2. If we take $L: S \to R^n$ to be a section of the mapping F, such that $L(x) = b_i$ for every $x \in ri\ D_i$ $(i = 0, 1, \dots, n)$, then by the previous Lemma, L satisfies the conditions of Theorem 1. Hence there exists $\bar{x} \in S$ with $\bar{x} = \lim_{k \to \infty} x_i^{(k)}$ $(i = 0, 1, \dots, n)$, where $x_i^{(k)}$ satisfy (1.3). But the set $F(S)$ is compact because of the upper semi-continuity of F (see for ex. [4]). Therefore one can assume $L(x_i^{(k)}) \to u_i$ $(i = 0, 1, \dots, n)$ as $k \to \infty$. Since $L(x_i^{(k)}) \in F(x_i^{(k)})$, it follows from the closedness of F that $u_i \in F(\bar{x})$ $(i = 0, 1, \dots, n)$, and hence $0 \in F(\bar{x})$, by using (1,3) and the convexity of $F(\bar{x})$. □

2. Equations in a convex compact set.

Condition (1.5) is a kind of boundary condition for a mapping F defined on a simplex S. We now generalize this condition to the case where F is defined on a convex compact set C.

Let X denote a Hausdorff locally convex space, X^* its dual. Let C be a closed convex set in X, F a multivalued mapping from X into a Hausdorff locally convex space Y (in the sequel usually $Y = X$ or $Y = X^*$).

Definition 2. We say that a multivalued mapping F from X into X satisfies the inward boundary condition on a convex set $C \subset X$ if for every $x \in \partial_a C$

$$(\forall v \in N_C(x))\ (\exists u \in F(x)) \qquad \langle v, u \rangle \geqslant 0. \tag{1.6}$$

We say that a multivalued mapping F from X into X^* satisfies the inward* boundary condition on a convex set $C \subset X$ if for every $x \in \partial_a C$:

$$(\forall u \in I_C(x))\ (\exists v \in F(x)) \qquad \langle v, u \rangle \geqslant 0. \tag{1.6}^*$$

Lemma 2. If $F(x)$ is convex and weakly compact, then

$$(1.6) \Leftrightarrow F(x) \cap I_C(x) \neq \phi. \tag{1.7}$$

If $F(x)$ is convex and weakly* compact, then

$$(1.6)^* \Leftrightarrow F(x) \cap N_C(x) \neq \phi. \tag{1.7}^*$$

In any case (1.7) $\Rightarrow$ (1.6); (1.7)* $\Rightarrow$ (1.6)*.

Proof. The last assertion being obvious, it is enough to prove the first part of the Lemma. Suppose $F: X \to 2^X$ and $F(x)$ is convex and weakly compact. Then by the minimax theorem (see e.g. [5])

$$\inf_{v \in N_C(x)} \ \sup_{u \in F(x)} \langle v, u \rangle = \sup_{u \in F(x)} \ \inf_{v \in N_C(x)} \langle v, u \rangle$$

so that (1.6) is equivalent to

$$(\exists u \in F(x))\ (\forall v \in N_C(x)) \quad \langle v, u \rangle \geqslant 0.$$

But $I_C(x) = \{u : \langle v, u \rangle \geqslant 0\ \forall v \in N_C(x)\}$, hence (1.6) $\Leftrightarrow$ (1.7).

Similarly, if $F : X \to 2^{X^*}$ and $F(x)$ is convex and weakly* compact, then the minimax theorem shows that (1.6)* is equivalent to

$$(\exists v \in F(x))\ (\forall u \in I_C(x)) \quad \langle v, u \rangle \geqslant 0,$$

which is nothing else than (1.7)*.

The inward boundary condition has been previously considered by some authors (see e.g. [7]). But it seems that the inward* boundary condition is introduced here for the first time. In the simplest cases $F(x) \neq \phi$ and $F(x) \subset I_C(x)$ (then (1.6) holds), or $F(x) \subset N_C(x)$ (then (1.6)* holds).

Theorem 3. Let F be a multivalued mapping from a convex compact set C in R^n into R^n, such that $F(x)$ is nonempty, convex and compact for every $x \in C$. If F is upper semi-continuous and if: a) either F satisfies the inward boundary condition on C; b) or *int* $C \neq \phi$ and F satisfies the inward* boundary condition on C, then the equation $0 \in F(x)$ has a solution in C.

In order to prove this theorem, our method is to extend F to an S-mapping over a simplex $S \supset C$, containing 0 in its interior, such that the equation $0 \in F(x)$ is not solvable in $S \setminus C$: then, by theorem 2, it has a solution in C.

The extension of F requires some properties of convex sets in R^n, which we shall need also for Banach spaces. Therefore we shall establish these properties in Banach spaces.

Let us recall that a norm $\|.\|$ in a linear normed space X is said to be smooth if the unit ball in X has at each boundary point just one supporting hyperplane; it is said to be rotund (or strictly convex) if no open line segment in the unit ball meets the unit sphere. It is known (see [3]) that every reflexive Banach space has an equivalent norm which is both smooth and rotund. We say that a linear normed space X has property (H) if: $x_k \underset{W}{\rightarrow} x$ and $\|x_k\| \rightarrow \|x\|$ imply $\|x_k - x\| \rightarrow 0$ (for example, the spaces L^p with $p > 1$, have this property; see [7]).

Lemma 3. Let C be a convex closed set in a reflexive Banach space X with a rotund norm. For every $x \in X$ there is just one $y = \pi(x) \in C$ satisfying

$$\| y - x \| = \min_{u \in C} \| u - x \| . \tag{1.8}$$

The mapping $\pi: X \rightarrow C$ is weakly continuous in the sense that: $x_k \rightarrow x$ implies $\pi(x_k) \underset{w}{\rightarrow} \pi(x)$. If C is compact or X has property (H), then π is continuous.

Proof. Consider a sequence $y_k \in C$ such that $\| y_k - x \| \rightarrow \mu = \inf_{u \in C} \| u - x \|$.

Since μ is obviously finite and $\geqslant 0$, the sequence $\{y_k\}$ is bounded, and so, by using subsequences if necessary, one can assume $y_k \underset{w}{\rightarrow} y$. But the convex set C being closed is also weakly closed, so that $y \in C$. Furthermore, the convex function $\varphi(x) = \| x \|$ being continuous is weakly lower semi-continuous, so that

$$\underline{\lim} \| y_k - x \| \geqslant \| y - x \| ,$$

and hence $\| y - x \| = \mu$.

If $\| y' - x \| = \mu$ for some other $y' \in C \; (y' \neq y)$, then the open line segment joining y, y' must lie in C (because C is convex), and so for every u of this line segment one has $\| u - x \| \geqslant \mu$; since on the other hand $\| u - x \| < \mu$ by the rotundity of the norm, we get a contradiction. Therefore $y = \pi(x)$ is uniquely defined.

Suppose $x_k \rightarrow x$, and let $y_k = \pi(x_k)$, $y = \pi(x)$. Since $\|y - x\| \leqslant \| y_k - x \| \leqslant \| y_k - x_k \| + \| x_k - x \| \leqslant \| y - x_k \| + \| x_k - x \|$, we have $\| y_k - x \| \rightarrow \| y - x \|$. Therefore if X has property (H), $y_k \rightarrow y$. If C is compact there is a subsequence along which $y_k \rightarrow \bar{y}$ and since $\| \bar{y} - x \| = \| y - x \|$, it follows that $\bar{y} = y$ (by the uniqueness property), i.e. $y_k \rightarrow y$. Thus in these two cases, π is continuous. In the general case, the sequence $\{y_k\}$ is bounded (because $\| y_k - x \| \rightarrow \| y - x \|$), and so there is a subsequence along which $y_k \underset{w}{\rightarrow} \bar{y}$; then, as previously, $\bar{y} = y$, i.e. $y_k \underset{w}{\rightarrow} y$. Hence π is weakly continuous. □

Observe that $\pi(x) = x$ for $x \in C$ and $\pi(x) \in \partial_a C$ for $x \notin C$. We call $\pi()$ the projection of x on C. If X is a Hilbert space, it is easily seen that $\pi(x)$ is characterized by the condition $\pi(x) - x \in N_C(\pi(x))$, i.e.

$$(\forall u \in C) \quad \langle x - \pi(x),\, u - \pi(x)\rangle \leqslant 0.$$

In the case where X is a reflexive Banach space, let $j: X \to X^*$ be the mapping such that $\langle j(x), x\rangle = \|x\|^2$, $\|j(x)\| = \|x\|$. For every $u \in X$ such that $\|u\| \leqslant \|x\|$ one has $\langle j(x), u-x\rangle \leqslant \|j(x)\| \cdot \|u\| - \langle j(x), x\rangle = \|x\| (\|u\| - \|x\|) \leqslant 0$ i.e. $j(x)$ is the «outward normal» of the ball of radius $\|x\|$, at point x: from the smoothness of the norm, $j(x)$ is uniquely determined for each given x.

Lemma 4. Let C be a convex closed set in a reflexive Banach space X, (with a smooth norm). For every closed ball B in X (centered at 0) the set $C' = C + B$ is convex closed, and for every $x \in \partial C'$:

$$N_C(\pi(x)) \supset N_{C'}(x) = \{\lambda j(\pi(x) - x) : \lambda \leqslant 0\},$$

where $\pi(x)$ denotes the projection of x on C.

Proof. The convex set C being closed is weakly closed, and since B is weakly compact, the set $C' = C + B$ is weakly closed, and hence closed. Let r denote the radius of B. Since $x \in \partial C' \subset C'$, we have $x \in z + B$ with $z \in C$, consequently $\|\pi(x) - x\| \leqslant \|z - x\| \leqslant r$. But the inequality $\|\pi(x) - x\| < r$ would imply $x \in int\,(\pi(x) + B) \subset int\, C'$, contrary to the fact $x \in \partial C'$. Therefore, $\|\pi(x) - x\| = r$, and $x \in \partial B'$ where $B' = \pi(x) + B$. Obviously, $N_{B'}(x) = \{\lambda j(\pi(x) - x)) : \lambda \leqslant 0\}$; On the other hand, $N_{C'}(x) \subset N_{B'}(x)$ and $N_{C'}(x) \neq \{0\}$ because C' has a nonempty interior. Hence $N_{C'}(x) = N_{B'}(x)$. For every $z \in C$ we have $z + x - \pi(x) \in C'$ (because $x - \pi(x) \in B$), therefore $\langle j(\pi(x) - x),\, z - \pi(x)\rangle = \langle j(\pi(x) - x),\, (z + x - \pi(x)) - x\rangle \leqslant 0$, which means $j(\pi(x) - x) \in N_C(\pi(x))$. □

Finally, before proceeding to the proof of Theorem 3, it is convenient to recall the following fact which we shall frequently use:

If C is a convex set (in a Hausdorff locally convex space) with an interior point ω, then for every $x \notin \operatorname{int} C$, the set

$$N_C^{\omega}(x) = \{y \in N_C(x) : \langle y, \omega - x\rangle = 1\} \tag{1.9}$$

is nonempty, convex and weakly* compact.

Proof of Theorem 3

I — Case where a) holds (i.e. F satisfies the inward boundary condition). By translating if necessary we can assume that 0 is a relative interior point of C. Let $S = [a_0, a_1, \ldots, a_n]$ denote a simplex such that $C \subset int\, S$. We shall extend F to a S-mapping F' in the following way.

For every $x \in C$, set $F'(x) = F(x)$. For every $x \in S \setminus C$ let $\pi(x)$ denote the projection of x on C, let $\sigma(x)$ denote the point where the line through x and $\pi(x)$ meets the boundary of S, and let $\lambda(x) = \dfrac{\|x - \pi(x)\|}{\|\sigma(x) - \pi(x)\|}$. From the continuity of π (Lemma 3) it easily follows that $\sigma(x)$ and $\lambda(x)$ are also continuous. Set

$$F'(x) = \lambda(x) . N^{\circ}_S(\sigma(x)) + (1 - \lambda(x)) . f(x) \quad (1.10)$$

where

$$N^{\circ}_S(\sigma(x)) = \{y \in N_S(\sigma(x)) : \langle y, -\sigma(x)\rangle = 1\} \text{ (see (1.9))}$$

$$f(x) = \{y \in F(\pi(x)) : \langle \pi(x) - x, y\rangle \geqslant 0\} \quad (1.11)$$

Since $\pi(x) - x \in N_C(\pi(x))$, it follows from the inward boundary condition that $f(x)$ is nonempty; moreover, $f(x)$ is obviously convex and compact.

We thus obtain a multivalued mapping F' from S into R^n such that $F'(x)$ is nonempty, convex, compact for every $x \in S$. If $x \in riD_i$ (the i-th face of S) then $\sigma(x) = x$, $\lambda(x) = 1$, while $N^{\circ}_S(\sigma(x))$ reduces to a normal of D_i directed inwards S, so that F is a S-mapping. To verify that F' is upper semicontinuous, we observe first that the mapping $z \mapsto N^{\circ}_S(z)$ being obviously upper semi-continuous, the set $N^{\circ}_S(\partial S)$ is compact. Now, if $x_k \in S \setminus C$, $x_k \to x$, $y_k \to y$, $y_k \in F'(x_k)$. then $y_k = \lambda(x_k)u_k + (1 - \lambda(x_k))v_k$ with $u_k \in N^{\circ}_S(\sigma(x_k))$ $v_k \in f(x_k)$, and by compactness there is a subsequence along which $u_k \to u$, $v_k \to v$; from the continuity of $\pi(x)$, $\lambda(x)$, $\sigma(x)$ and the upper semi-continuity of the mappings $N^{\circ}_S(.)$, $F(.)$, we then have $u \in N^{\circ}_S(\sigma(x))$, $v \in f(x)$, $y = \lambda(x)u + (1 - \lambda(x))v$, which implies $y \in F'(x)$ (since for $x \in \partial C : \lambda(x) = 0$ and $f(x) \subset F(x) = F'(x)$). On the other hand, if $x_k \in C$, $x_k \to x$, $y_k \to y$, $y_k \in F'(x_k)$, then it is obvious that $y \in F(x) = F'(x)$. Thus, F' is closed and hence, upper semi-continuous.

From Theorem 2, there exists $\bar{x} \in S$ satisfying $0 \in F'(\bar{x})$. Assume that $\bar{x} \in S \setminus C$. Then $0 = \lambda(\bar{x})u + (1 - \lambda(\bar{x})) . v$, for some $u \in N^{\circ}_S((\sigma(\bar{x}))$, $v \in f(\bar{x})$. $0 < \lambda(\bar{x}) < 1$ ($0 < \lambda(\bar{x})$ because $\bar{x} \notin C$; $\lambda(\bar{x}) < 1$, for otherwise $\lambda(\bar{x}) = 1$ and we would have $0 = u$, conflicting with the relation $\langle u, -\sigma(\bar{x})\rangle = 1$ which follows from the definition of $N^{\circ}_S(\sigma(\bar{x}))$). But, by (1.11), $\langle \pi(\bar{x}) - \bar{x}, v\rangle \geqslant 0$, hence

$$\langle v, \pi(\bar{x}) - \sigma(\bar{x})\rangle \geqslant 0. \quad (1.12)$$

On the other hand $\langle u, \pi(\bar{x}) - \sigma(\bar{x})\rangle \geqslant 0$, since $u \in N^{\circ}_S(\bar{x})$). This, together with, (1.12) and the relation $0 = \lambda(\bar{x})u + (1 - \lambda(\bar{x}))v$, $0 < \lambda(\bar{x}) < 1$ implies $\langle u, \pi(\bar{x}) - \sigma(\bar{x})\rangle = 0$. Thus $\pi(\bar{x})$ must lie in the supporting hyperplane of S (at $\sigma(\bar{x})$) that is normal to u. But this is impossible, because $\pi(\bar{x}) \in int\ S$. Therefore, $\bar{x} \in C$, and hence $0 \in F(\bar{x})$.

II. – Case where b) holds (i. e. $int\ C \neq \phi$ and F satisfies the inward* boundary condition). By translating we can assume $0 \in int\ C$. Let $S = [a_0, a_1, .., a_n]$ denote as before a simplex such that $C \subset int\ S$.

For every $x \in S \setminus C$, denote by $p(x)$ and $s(x)$ the points where the ray issued from 0 and passing through x cuts ∂C and ∂S, respectively, and let $\lambda(x) = \frac{\| x - p(x) \|}{\| s(x) - p(x) \|}$. It can easily be shown that $p(x)$, $s(x)$ and hence $\lambda(x)$, are continuous functions of x. Set for every such $x \in S \setminus C$;

$$F'(x) = \lambda(x) . N_S^o (s(x)) + (1 - \lambda(x)) . f(x),$$

where N_S^o (.) has the same meaning as previously, and

$$f(x) = \{ y \in F(p(x)) : \langle y, p(x) \rangle \leqslant 0 \} \tag{1.13}$$

Since $-p(x) \in I_C(p(x))$, the inward* boundary condition ensures that $f(x)$ is nonempty; moreover $f(x)$ is obviously convex and compact.

Thus, by setting $F'(x) = F(x)$ for $x \in C$, we obtain a multivalued mapping F' from S into R^n. As previously, we can verify that F' satisfies all conditions of theorem 2 (in particular, is a S-mapping). By virtue of this Theorem, there exists $\bar{x} \in S$ such that $0 \in F'(\bar{x})$ and it is not difficult to see that $\bar{x} \in C$. Indeed, if $\bar{x} \in S \setminus C$, then $0 = \lambda(\bar{x}) . u + (1 - \lambda(\bar{x})) . v$, with $u \in N_S^o (s(\bar{x}))$, $v \in f(\bar{x})$, $0 < \lambda(\bar{x}) < 1$. By (1.13), $\langle v, p(\bar{x}) \rangle \leqslant 0$, hence $\langle v, s(\bar{x}) \rangle \leqslant 0$; but from the definition of $N_S^o (s(\bar{x}))$: $\langle u, s(\bar{x}) \rangle \leqslant 0$. Therefore, $\langle u, s(\bar{x}) \rangle = 0$, i.e. 0 lies in a supporting hyperplane of S at $s(\bar{x})$, which obviously conflicts with the fact $0 \in int\ S$. Thus, $\bar{x} \in C$, and hence $0 \in F(\bar{x})$. ◻

3. Equations in an arbitrary compact set.

Consider now the equation $0 \in F(x)$, $x \in \Omega$, where Ω is a compact (but non necessarily convex) set in R^n.

Theorem 4. Let Ω be a compact set in R^n, let F be a upper semi-continuous multivalued mapping from Ω into R^n, such that $F(x)$ is nonempty, convex and compact for every $x \in \Omega$. Assume there are a convex compact set $C \supset \Omega$ and an upper semi-continuous multivalued mapping G from C into R^n, such that $G(x)$ is convex, compact for every $x \in C$, while $0 \notin G(x) \neq \phi$ for every $x \in C \setminus \Omega$, and such that, moreover: a) either G satisfies the inward boundary condition on C; b) or int $C \neq \phi$ and G satisfies the inward* boundary condition on C. Then there exists a point $\bar{x} \in \Omega$ such that:

1) either $\bar{x} \in int\ \Omega$ and $0 \in F(\bar{x})$;

2) Or $\bar{x} \in \partial\Omega$ and $0 \in \theta F(\bar{x}) + (1 - \theta) . G(\bar{x})$ for $0 \leqslant \theta \leqslant 1$.

Proof. By replacing if necessary $G(x)$ by ϕ for every $x \in int\ \Omega$, we may assume $G(x) = \phi$ for every $x \in int\ \Omega$. Furthermore, by setting $F(x) = \phi$ for every $x \in C \setminus \Omega$, we can consider F as defined on all of C. Let

$$H(x) = co \{ F(x), G(x) \}.$$

Then $H(x)$ is nonempty, convex, compact for every $x \in C$ and the mapping H is upper semi-continuous and satisfies condition a) or b) in Theorem 3. Therefore, by this Theorem, one can find an $\overline{x} \in C$ such that $0 \in H(\overline{x})$. The conclusion follows immediately. □

Corollary. Let Ω be a closed set in R^n, containing 0 in its interior. Let f be a upper semi-continuous multivalued mapping from Ω into a compact set in R^n, such that $f(x)$ is nonempty, convex, compact for every $x \in \Omega$. If either of the following conditions holds: a) for every $x \in \partial\Omega$ and for every $\lambda > 1$: $\lambda x \notin f(x)$; b) for every $x \in \partial\Omega$ there is $\lambda = \lambda(x) \leqslant 1$ such that $\lambda x \in f(x)$, then f has a fixed point.

Proof. Denote by C a ball around 0, containing $f(\Omega)$ in its interior. By applying Theorem 4 with $\Omega \cap C$ playing the role of Ω in this Theorem, and $F(x)=f(x)-x$, $G(x)=-x$, one can find $\overline{x}$ such that: 1) either $\overline{x} \in int(\Omega \cap C)$ and $0 \in F(\overline{x})$, i. e. $\overline{x} \in f(\overline{x})$; 2) or $\overline{x} \in \partial(\Omega \cap C)$ and $0 \in \theta F(\overline{x}) - (1-\theta)\overline{x}$ for $0 < \theta \leqslant 1$ ($\theta > 0$ because $x \neq 0$ for every $x \in \partial(\Omega \cap C)$). In the latter case, we can assume $\theta < 1$ (otherwise $\overline{x}$ is a fixed point), so $\overline{x} \in \theta f(\overline{x})$ with $0 < \theta < 1$, which implies $\overline{x} \notin \partial C$, since $f(\Omega) \subset int\, C$. Therefore, if case 2) holds, then $\overline{x} \in \partial\Omega$; since this conflicts with condition a), there is, by condition b), a number $\lambda \leqslant 1$ such that $\lambda\overline{x} \in f(\overline{x})$, and the convexity of the set $f(\overline{x})$ then implies $\overline{x} \in f(\overline{x})$. □

§2. EQUATIONS IN INFINITE-DIMENSIONAL SPACES

1. General approximation scheme.

In this section we shall study the solvability of the equation (*), in the general case where Ω is a compact set in a Hausdorff locally convex space X, and F is a multivalued mapping from Ω into a Hausdorff locally convex space Y. We shall always assume that $F(x)$ is nonempty, convex and compact for every $x \in \Omega$ and that F is upper semi-continuous.

In order to solve the equation (*) we shall proceed according to the following general scheme:

(i) Select two families of linear (usually finitedimensional) spaces X_ν, Y_ν ($\nu \in N$), and two families of mappings $p_\nu : dom p_\nu \subset X \to X_\nu$, $q_\nu : dom q_\nu \subset Y \to Y_\nu$ such that : every p_ν, q_ν is linear and for every set $N' \subset N$ cofinal with N in the ordering « $\nu_1 \prec \nu_2$ iff $X_{\nu_1} \subset X_{\nu_2}$ », we have:

$$x_\nu \in X,\ x_\nu \underset{N'}{\to} \overline{x},\ (\forall \nu \in N')\ p_\nu(x_\nu) = 0 \Rightarrow \overline{x} = 0 \tag{2.1}$$

$$y_\nu \in Y,\ y_\nu \underset{N'}{\to} \overline{y},\ (\forall \nu \in N')\ q_\nu(y_\nu) = 0 \Rightarrow \overline{y} = 0 \tag{2.2}$$

(ii) For every $\nu \in N$, solve the equation

$$0 \in F_\nu(u),\ u \in \Omega_\nu \tag{2.3}$$

where $\Omega_\nu = p_\nu(\Omega)$, F_ν is the mapping from Ω_ν into Y_ν defined by

$$F_\nu(u) = co\,[q_\nu(\cup\{F(x) : x \in \Omega, p_\nu(x) = u\})]. \tag{2.4}$$

Theorem 5. Under the stated hypotheses, if for every $\nu \in N$ the equation (2.3) has a solution $u_\nu = p_\nu(x_\nu)$, $x_\nu \in \Omega$, then every cluster point $\bar{x}$ of the net $\{x_\nu\}$ is a solution of the equation $0 \in F(x)$ in Ω. This proposition follows from

Lemma 5. Assume that $N' \subset N$ is cofinal with N and for every $\nu \in N'$ we have

$$v_\nu \in co\,[q_\nu(\cup\{F(x) : x \in \Omega.\ p_\nu(x) = u_\nu\})], \tag{2.5}$$

$$x_\nu \in \Omega,\ p_\nu(x_\nu) = u_\nu,\ x_\nu \underset{N'}{\to} \bar{x} \in \Omega. \tag{2.6}$$

Then there exist a subnet $\{v_{\nu_\mu}, \mu \in M\}$ of the net $\{v_\nu, \nu \in N'\}$ and a net $\{y_{\nu_\mu}\}$ such that

$$v_{\nu_\mu} = q_{\nu_\mu}(y_{\nu_\mu}),\quad y_{\nu_\mu} \underset{M}{\to} \bar{y} \in F(\bar{x}). \tag{2.7}$$

Proof. Let $\{V_\alpha, \alpha \in A\}$ denote a local base of convex closed neighbourhoods of $0 \in Y$. In view of the upper semi-continuity of F there is for each α a neighbourhood U_α of $0 \in X$ such that

$$(\forall x \in \bar{x} + U_\alpha) \qquad F(x) \subset F(\bar{x}) + V_\alpha. \tag{2.8}$$

We have

$$v_\nu = \sum_{i \in I_\nu} \lambda_{\nu i} q_\nu(w_{\nu i}) \text{ with } w_{\nu i} \in F(x_{\nu i}),$$

$$x_{\nu i} \in \Omega, \quad p_\nu(x_{\nu i}) = u_\nu, \quad \lambda_{\nu i} > 0, \quad \sum_{i \in I_\nu} \lambda_{\nu i} = 1, \ |I_\nu| < \infty.$$

From this it is not hard to see that for some $\nu_0 = \nu_0(\alpha)$:

$$(\forall \nu \succ \nu_0)(\forall i \in I_\nu) \qquad x_{\nu i} \in \bar{x} + U_\alpha. \tag{2.9}$$

Indeed, otherwise there exists a set $N'' \subset N'$ cofinal with N' such that

$$(\forall \nu \in N'')(\exists i \in I_\nu) \qquad x_{\nu i} \notin x + U_\alpha. \tag{2.10}$$

Since Ω is compact, it can be assumed that $x_{\nu i} \xrightarrow[N'']{} \hat{x} \in \Omega$. We have

$$(\forall \nu \in N'') p_\nu(x_{\nu i} - x_\nu) = p_\nu(x_{\nu_i}) - p_\nu(x_\nu) = u_\nu - u_\nu = 0,$$

and $(x_{\nu i} - x_\nu) \xrightarrow[N'']{} (\hat{x} - \bar{x})$, therefore by (2.1) $\hat{x} = \bar{x}$, i.e.

$$x_{\nu i} \xrightarrow[N'']{} \bar{x},$$

contrary to (2.10).

Thus, there is $\nu_0 = \nu_0(\alpha)$ satisfying (2.9). Then, by (2.8)

$$(\forall \nu \succ \nu_0)(\forall i \in I_\nu) \qquad F(x_{\nu i}) \subset F(\bar{x}) + V_\alpha,$$

and hence

$$w_{\nu i} \in F(\bar{x}) + V_\alpha, \quad \sum_{i \in I_\nu} \lambda_{\nu i} w_{\nu i} \in F(\bar{x}) + V_\alpha,$$

since $F(\bar{x}) + V_\alpha$ is a convex set. Setting $y_\nu = \sum_{i \in I_\nu} \lambda_{\nu i} w_{\nu i}$, we obtain

$q_\nu(y_\nu) = v_\nu$ and

$$(\forall \nu \succ \nu_0(\alpha)) \qquad y_\nu \in F(\bar{x}) + V_\alpha \tag{2.11}$$

Denote by M the set of all pairs $\mu = (\nu, \alpha) \in N \times A$ such that $\nu \succ \nu_0(\alpha)$ and set $\nu_\mu = \nu$ for every $\mu = (\nu, \alpha)$. Then it is easy to see that $\{v_{\nu_\mu}, \mu \in M\}$ is a subnet of the net $\{v_\nu, \nu \in N\}$. Indeed, M is a directed set with the ordering

$$\mu = (\nu, \alpha) \prec \mu' = (\nu', \alpha') \Leftrightarrow (\nu \prec \nu') \ \& \ (\alpha \prec \alpha'),$$

since for any given $\mu_1 = (\nu_1, \alpha_1) \in M$, $\mu_2 = (\nu_2, \alpha_2) \in M$, if we take $\alpha \succ \{\alpha_1, \alpha_2\}$ and $\nu \succ \{\nu_0(\alpha), \nu_1, \nu_2\}$, then $\mu = (\nu, \alpha) \in M$ and $\mu \succ \{\mu_1, \mu_2\}$. Moreover, given an arbitrary ν_1, if we take $\alpha \in A$ and $\nu \succ \{\nu_0(\alpha), \nu_1\}$, then we shall have $\mu = (\nu, \alpha) \in M$ with $\nu_\mu = \nu \succ \nu_1$.

From (2.11) we can write, for every $\mu = (\nu, \alpha) \in M$:

$$y_{\nu_\mu} = \bar{y}_{\nu_\mu} + \varepsilon_{\nu_\mu} \text{ with } \bar{y}_{\nu_\mu} \in F(\bar{x}),\ \varepsilon_{\nu_\mu} \in V_\alpha.$$

Since $F(\bar{x})$ is compact, it can be assumed that $\bar{y}_{\nu_\mu} \xrightarrow[M]{} \bar{y} \in F(\bar{x})$. But $\varepsilon_{\nu_\mu} \xrightarrow[M]{} 0$, therefore $y_{\nu_\mu} \xrightarrow[M]{} \bar{y}$. □

Proof of Theorem 5. For every $\nu \in N$ there is by hypothesis an $x_\nu \in \Omega$ such that $p_\nu(x_\nu) = u_\nu$, $0 \in F_\nu(u_\nu)$. By the compactness of Ω there exists $N' \subset N$ cofinal with N such that $x_\nu \xrightarrow[N']{} \bar{x}$. It suffices now to apply the previous Lemma, with $v_\nu = 0$ $(\forall \nu)$ and to observe that, since $0 = q_{\nu_\mu}(y_{\nu_\mu})$ for every μ we must have $\bar{y} = 0$ by (2.2). □

2. Case $Y = X$.

In order to apply Theorem 5 we construct the net $\{X_\nu, Y_\nu, p_\nu, q_\nu\}$ in the following way.

Let Z_ν be finite-dimensional subspaces of X^*, whose union $\bigcup_{\nu \in N} Z_\nu$ is dense in X^* in the weak* topology and is such that for every element y of this union there is $\nu_0 = \nu_0(y)$ satisfying $(\forall \nu \succ \nu_0)$ $y \in Z_\nu$. Such Z_ν always exist: for example we can take $\{Z_\nu, \nu \in N\}$ to be a family of finite-dimensional subspaces which is exhaustive for X^* in the sense that for every finite-dimensional subspace Z of X^* there exists $Z_\nu \supset Z$ for some $\nu \in N$.

Then let $X_\nu = Y_\nu = Z_\nu^*$ and let $p_\nu = q_\nu : X \to X_\nu$ be the linear mapping that sends every $x \in X \subset X^{**}$ to the functional x restricted to $Z_\nu \subset X^*$.

Lemma 6. The above constructed net $\{X_\nu, Y_\nu, p_\nu, q_\nu\}$ satisfies conditions (2.1), (2.2); moreover, p_ν, q_ν are continuous and surjective.

Proof. Suppose that $N' \subset N$ is cofinal with N and

$$x_\nu \in X,\ x_\nu \underset{N'}{\rightarrow} \overline{x}\ ,\ (\forall \nu \in N')\ p_\nu(x_\nu) = 0$$

(i.e. $\forall y \in Z_\nu \cdot \langle y, x_\nu \rangle = 0$). Consider an arbitrary $y \in X^*$. Since the union of all Z_ν is weakly* dense in X^* there is, for an arbitrarily given $\varepsilon > 0$, an element y_ε of this union satisfying $| \langle y_\varepsilon, \overline{x} \rangle - \langle y, \overline{x} \rangle | < \frac{\varepsilon}{2}$. On the other hand, since $x_\nu \underset{N'}{\rightarrow} \overline{x}$ there is ν_0 such that $(\forall \nu \succ \nu_0)\ | \langle y_\varepsilon, x_\nu \rangle - \langle y_\varepsilon, \overline{x} \rangle | < \frac{\varepsilon}{2}$. Furthermore, it can be assumed that $y_\varepsilon \in Z_\nu$ $(\forall \nu \succ \nu_0)$, i.e. $\langle y_\varepsilon, x_\nu \rangle = \langle y_\varepsilon, p_\nu(x_\nu) \rangle = 0$. Hence $| \langle y, \overline{x} \rangle | < \varepsilon$ and so, ε being arbitrary, $\langle y, \overline{x} \rangle = 0\ (\forall y \in X^*)$ i.e. $\overline{x} \doteq 0$. Thus (2.1) holds. The continuity of p_ν is obvious. Finally, if $u \in X_\nu$, then u can be extended to a linear continuous functional x over X^*: then $x \in X$ and $p_\nu(x) = u$, proving the surjectivity of p_ν. □

Lemma 7. Let C be a convex set in X and let G be a multivalued mapping from X into X, satisfying the inward boundary condition on C. If $p_\nu : X \rightarrow X_\nu$ is a continuous surjective linear mapping, and $C_\nu = p_\nu(C)$, then the mapping $\Gamma_\nu : u \mapsto \Gamma_\nu(u) = p_\nu(\cup \{ G(x) : x \in X,\ p_\nu(x) = u \})$ satisfies the inward boundary condition on C_ν.

Proof. Suppose $u \in \partial_a C_\nu$ and $y \in N_{C_\nu}(u)$. There is then $v \in X_\nu$ such that $(\forall \lambda > 0)\ u + \lambda v \notin C_\nu$. Let $w \in X$ be the element that satisfies $p_\nu(w) = v$. For every $x \in C$ such that $p_\nu(x) = u$ we have $(\forall \lambda > 0)\ p_\nu(x + \lambda w) = u + \lambda v \notin C_\nu$, so $x + \lambda w \notin C$, and hence $x \in \partial_a C$, Since $y \in N_{C_\nu}(u)$, $(\forall u' \in C_\nu)\ \langle y, u' - u \rangle \geqslant 0$.

But $y \circ p_\nu$ is a linear continuous functional on X and $(\forall x' \in C)\ \langle y \circ p_\nu, x' - x \rangle = \langle y, p_\nu(x') - p_\nu(x) \rangle = \langle y, p_\nu(x') - u \rangle \geqslant 0$ (because $p_\nu(x') \in C_\nu$). which means that $y \circ p_\nu \in N_C(x)$ for every $x \in G$ satisfying $p_\nu(x) = u$. Since G satisfies the inward boundary condition on C and $x \in \partial_a C$ there is $z \in G(x)$ such that $\langle y \circ p_\nu, z \rangle \geqslant 0$, or else $\langle y, p_\nu(z) \rangle \geqslant 0$. So for every $y \in N_{C_\nu}(u)$ there is $s = p_\nu(z)$ satisfying $\langle y, s \rangle \geqslant 0$ with $z \in G(x)$ and $x \in \partial_a C$, $p_\nu(x) = u$. □

Theorem 6. Let Ω be a compact set in a Hausdorff locally convex space X, let F be a upper semi-continuous multivalued mapping from Ω into X, such that $F(x)$ is nonempty, convex and compact for every $x \in \Omega$. Assume there are a convex compact set $C \supset \Omega$ and an upper semi-continuous multivalued mapping G from C into X, such that $G(x)$ is convex, compact for every $x \in C$ while $0 \notin G(x) \neq \phi$ for every $x \in C \setminus \Omega$, and such that, moreover, G satisfies the inward boundary condition on C. Then there exists a point $\overline{x} \in \Omega$ such that:

1) either $\overline{x} \in int\ \Omega$ and $0 \in F(\overline{x})$;

2) or $\overline{x} \in \partial\Omega$ and $0 \in \theta F(\overline{x}) + (1 - \theta)\ G(\overline{x})$ for $0 \leqslant \theta \leqslant 1$.

Proof. As in the proof of Theorem 4 we can assume $G(x) = \phi$ for every $x \in int\ \Omega$, and $F(x) = \phi$ for every $x \in C \setminus \Omega$. Then the mapping H defined on C by

$$H(x) = co\ \{ F(x),\ G(x) \}$$

is upper semi-continuous, satisfies the inward boundary condition on C, and $H(x)$ is nonempty, convex, compact for every $x \in C$.

Let X_ν, Y_ν, p_ν, q_ν be chosen as above and let $\Omega_\nu = p_\nu(\Omega)$, $C_\nu = p_\nu(C)$. For every $u \in C_\nu$ we set

$$\Delta_\nu(u) = \{x \in C : p_\nu(x) = u\},$$

$$B_\nu(u) = \cup\{H(x) : x \in \Delta_\nu(u)\}, \ H_\nu(u) = co\ p_\nu(B_\nu(u)).$$

Since p_ν is continuous, C_ν is a convex compact set in X_ν; furthermore, $\Delta_\nu(u)$ being a closed subset of the compact set C is itself compact and, since H is upper semi-continuous, the set $B_\nu(u)$ is also compact (see[1]). The mapping B_ν is closed, since if $u_\sigma \to u_0$, $y_\sigma \to y_0$ and $y_\sigma \in B_\nu(u_\sigma)$, then $y_\sigma \in H(x_\sigma)$ with $p_\nu(x_\sigma) = u_\sigma$, $x_\sigma \in C$ and, C being compact, we can assume $x_\sigma \to x_0 \in \Omega$, hence $y_0 \in H(x_0)$ (by the closedness of H) and $p_\nu(x_0) = u_0$ (by the continuity of p_ν), i.e. $y_0 \in B_\nu(u_0)$. Then the range of B_ν being contained in a compact set, it follows that B_ν, and hence, $p_\nu \circ B_\nu$, is upper semicontinuous. Since, moreover, for every $u \in C_\nu$ the set $p_\nu(B_\nu(u))$ is compact, the mapping H_ν also is upper semi-continuous (see [1]). By Lemma 7, H_ν satisfies the inward boundary condition on C_ν. Hence, by Theorem 3, one can compute a point u_ν satisfying $0 \in H_\nu(u_\nu)$, and $u_\nu \in C_\nu$, i.e. $u_\nu = p_\nu(x_\nu)$ with $x_\nu \in C$. Since C is compact, there is for the net $\{x_\nu, \nu \in N\}$ a cluster point $\bar{x} \in C$ and by Theorem 5,

$$0 \in H(\bar{x}) = co\{F(\bar{x}), G(\bar{x})\}$$

Obviously, $\bar{x} \in \Omega$ and is the required point. □

Corollary. Let C be a convex compact set in a Hausdorff locally convex space X, and let F be a multivalued mapping from C into X such that $F(x)$ is nonempty, convex and compact for every $x \in C$. If F is upper semi-continuous and satisfies the inward boundary condition on C, then the equation $0 \in F(x)$ has a solution in C.

Proof. Apply the previous Theorem, with $\Omega = C$, $G = F$. If one is interested only in the existence problem, and not in the algorithmic question, then this Corollary is a known recent result in fixed point theory [7] and generalizes a classical and long known theorem of Kakutani.

3. Case Y = X*.

As a counter part of Theorem 6, we have the following result for the case $Y = X^*$ (in what follows, X^* is supposed to be endowed with the weak* topology, or any finer locally convex topology).

Theorem 6* Let Ω be a compact set in a Hausdorff locally convex space X, let F be a upper semi-continuous multivalued mapping from Ω into X^*, such that $F(x)$ is nonempty, convex and compact for every $x \in \Omega$. Assume there are a convex compact set $C \supset \Omega$ such that aff $C = X$ and a upper semi-continuous multivalued mapping G from C into X, such that $G(x)$ is convex, compact for every $x \in C$ while $0 \notin G(x) \neq \phi$ for every $x \in C \setminus \Omega$, and such that, moveover, G satisfies the inward* boundary condition on C. Then there exists a point $\bar{x} \in \Omega$ such that:

1) either $\bar{x} \in int\ \Omega$ and $0 \in F(\bar{x})$;

2) or $\bar{x} \in \partial\Omega$ and $0 \in \theta F(\bar{x}) + (1-\theta)\, G(\bar{x})$ for $0 \leqslant \theta \leqslant 1$.

Proof. As previously, we can assume $G(x) = \phi$ for every $x \in int\ \Omega$, $F(x) = \phi$ for every $x \in C \setminus \Omega$ and consider the mapping H defined on C by $H(x) = co\{F(x), G(x)\}$. This is a upper semi-continuous mapping satisfying the inward* boundary condition on C; also $H(x)$ is nonempty, convex and compact for every $x \in C$.

Now take a family of finite-dimensional subspaces X_ν, $\nu \in N$, of X, such that their union is weakly dence in X and for every element x of this union there is $\nu_0 = \nu_0(x)$ satisfying $(\forall \nu \succ \nu_0)\ x \in X_\nu$; X_ν being equal to the linear hull of $C \cap X_\nu$ (by translating, we can assume that $0 \in \Omega$, so that X is the linear hull of C). Such a family is always available: for example, we can take N to be the collection of all finite subsets of C, and X_ν to be the linear hull of $\nu \in N$.

Let $Y_\nu = X_\nu^*$ and let p_ν be the identity mapping of X_ν (i.e. p_ν is defined on X_ν and $p_\nu(x) = x$ for every $x \in X_\nu$), let q_ν be the restriction of functionals to X_ν(i.e. for every $y \in X^*$, $q_\nu(y) \in X_\nu^*$ is the functional y restricted to X_ν).

We have $\Omega_\nu = p_\nu(\Omega) = \Omega \cap X_\nu$, $C_\nu = p_\nu(C) = C \cap X_\nu$.

Let us verify that:

I, p_ν, q_ν are linear continuous surjective mappings satisfying (2.1), (2.2),

II. The mapping $H_\nu(u) = q_\nu(H(u))$ satisfies the inward* boundary condition on C_ν.

Indeed, that p_ν, q_ν are linear continuous and surjective is obvious (the surjectivity of q_ν being a consequence of the Hahn-Banach Theorem). Also condition (2.1) is evident. To prove (2.2), suppose $y_\nu \in X^*$, $y_\nu \underset{N'}{\rightarrow} \bar{y}$ and $(\forall \nu \in N')$ $q_\nu(y_\nu) = 0$. If $x \in X$, then it follows from the weak denseness in X of the union of all X_ν that for every given $\varepsilon > 0$ one can find an element x_ε if this union, satisfying $|\langle \bar{y}, x_\varepsilon \rangle - \langle \bar{y}, x \rangle| < \frac{\varepsilon}{2}$. On the other hand, since $y_\nu \underset{N'}{\rightarrow} \bar{y}$ there is ν_0 such that $(\forall \nu \in N', \nu \succ \nu_0) |\langle y_\nu, x_\varepsilon \rangle - \langle \bar{y}, x_\varepsilon \rangle| < \frac{\varepsilon}{2}$. Also it can be assumed that $(\forall \nu \in N', \nu \succ \nu_0)\ x_\varepsilon \in X_\nu$, i.e. $\langle y_\nu, x_\varepsilon \rangle = \langle q_\nu(y_\nu), x_\varepsilon \rangle = 0$. Then $|\langle \bar{y}, x \rangle| < \varepsilon$ and since $\varepsilon > 0$ is arbitrary, $\langle \bar{y}, a \rangle = 0$ for every $x \in X$, hence $\bar{y} = 0$ which shows that (2.2) holds.

To prove property II, it suffices to observe that if $u \in \partial_a C_v$ then $u \in \partial_a C$; therefore for each $w \in I_{C_v}(u) \subset I_C(u)$ there must exist $v \in G(u) \subset H(u)$ such that $\langle v, w \rangle \geqslant 0$, and evidently $q_v(v) \in H_v(u)$, $\langle q_v(v), w \rangle \geqslant 0$.

Thus, properties I, II hold. Clearly Ω_v is compact, C_v is convex, compact, and we may suppose that for every $v \in N$: X_v is the linear hull of $C_v = C \cap X_v$; furthermore, by arguing as in the proof of Theorem 6, we can see that H_v is a upper semi-continuous mapping. Therefore, by Theorem 3, there exists $u_v \in C_v$ satisfying $0 \in H_v(u_v)$. The proof may then be completed just in the same way as with Theorem 6. □

Corollary. Let C be a convex compact set in a Hausdorff locally convex space X, and let E be a multivalued mapping from C into X such that $F(x)$ is nonempty, convex and compact for every $x \in C$. If aff $C = X$ and if F is upper semi-continuous and satisfies the inward* boundary condition on C, then the equation $0 \in F(x)$ has a solution in C.

Proof. Apply the previous Theorem, with $\Omega = C$, $G = F$.

Remark. 1) It is obvious that the equation $0 \in F(x)$ is équivalent to $0 \in -F(x)$. Therefore, all the previous results remain valid, if the inequalities in (1.6) and (1.6)* are reversed, i. e. if we replace the inward (inward*) boundary condition by the outward (outward*) boundary condition.

2) Every previous result can of course be restated as a fixed point proposition.

4. Equations in a noncompact set.

The approximation method presented above requires the set Ω to be compact. If this assumption is not satisfied, the problem is much more difficult. However, in the case were the image of Ω under the mapping $x \mapsto F(x)+x$ is relatively compact, essentially the same method can be applied.

As an example, let us establish by this method the following proposition which generalizes a theorem of T.W. Ma and was first proved (in a nonconstructive way) by G.Kayser (see e.g. [6]).

Theorem 7. Let Ω be a closed set in a Hausdorff locally convex space X, containing 0; let f be a upper semi-continuous multivalued mapping from Ω into a compact set in X, such that $f(x)$ is nonempty, convex, compact for every $x \in \Omega$. Assume there exist a closed neighbourhood W of 0 and a closed convex set K in X such that $\Omega = W \cap K$, $f(\Omega) \subset K$, and $\lambda x \notin f(x)$ for every $x \in (\partial W) \cap K$ and every $\lambda > 1$. Then f has a fixed point.

Proof. We can suppose $\lambda x \notin f(x)$ for every $x \in \partial W \cap K$ and every $\lambda \geqslant 1$ (for if $\lambda x \in f(x)$ with $\lambda = 1$, then x is a fixed point). Consider a net $\{V_v, v \in N\}$ of convex, closed, balanced neighbourhoods of 0, coverging to 0, and for every $v \in N$ select a finite subset Q_v of K satisfying $0 \in Q_v$, $f(\Omega) \subset \bigcup_{q \in Q_v} (q + V_v)$. Let X_v

denote the linear hull of Q_ν, let $B_\nu \subset V_\nu$ be a ball in X_ν around 0, and let $K_\nu = K \cap \cap X_\nu$, $W_\nu = W \cap X_\nu$. For each $u \in X_\nu$ denote by $\pi_\nu(u)$ the projection of u on K_ν (in X_ν). Define Ω_ν to be the set of all $u \in W_\nu \cap (K_\nu + B_\nu)$ for which $\pi_\nu(u) \in W_\nu$. Since $\pi_\nu(u) = u$ for $u \in K_\nu$, it follows that Ω_ν contains $W_\nu \cap K_\nu$; moreover, since $V_\nu + u \subset W$ for all u in some neighbourhood of 0 and all $\nu \succ$ some ν_0, we shall have $\pi_\nu(u) \in W_\nu$ for all u in some neighbourhood of 0 in X_ν and all $\nu \succ \nu_0$, which implies that for these ν, Ω_ν is a neighbourhood of 0 in X_ν; finally, Ω_ν is obviously closed, because of the continuity of π_ν and the closedness of W_ν and K_ν. We now show that the mapping $f_\nu(u) = (f(\pi_\nu(u)) + V_\nu) \cap co\ Q_\nu$, defined on Ω_ν, satisfies all conditions in the Corollary of Theorem 4. First observe that for some ν_1 (which can be assumed to be equal to ν_0), we shall have $\lambda x \notin f(x) + V_\nu$ whenever $x \in \partial W \cap K$, $\lambda \geqslant 1$, $\nu \succ \nu_1$. Indeed, otherwise there would exist a set $N' \subset N$ cofinal with N and for each $\nu \in N'$ a point $x_\nu \in \partial W \cap K$, an element $v_\nu \in V_\nu$ and a number $\lambda_\nu \geqslant 1$ such that $\lambda_\nu x_\nu - v_\nu \in f(x_\nu)$. Since $f(\Omega)$ is relatively compact, while $v_\nu \to 0$, we can assume $1/\lambda_\nu \to \mu_0 \in [0,1]$, $\lambda_\nu x_\nu \to y_0$, hence $x_\nu \to x_0 \in \partial W \cap K$, with $x_0 = \mu_0 y_0$ and $\mu_0 > 0$ (because $\mu_0 = 0$ would imply $x_0 = 0$, conflicting with the fact $0 \in int\ W$). Then $\lambda_\nu \to \lambda_0 = 1/\mu_0 \geqslant 1$ and $\lambda_0 x_0 \in f(x_0)$, contrary to the hypothesis.

Consider now any $\nu \succ \nu_0$. For every $u \in \Omega_\nu$, since for each $y \in f(\pi_\nu(u))$ there exists $q \in Q_\nu \subset K_\nu$ satisfying $y \in q + V_\nu$, i.e. $q \in (y + V_\nu) \cap Q_\nu$, it follows that $f_\nu(u)$ is nonempty. Furthermore it is evident that $f_\nu(u)$ is convex, compact and that f_ν is upper semi-continuous. If $u \in \partial\Omega_\nu$ (the boundary in X_ν), then either $u \in \partial W \cap K_\nu$, or $u \notin \text{ri}\ K_\nu$ (for $u \in \text{ri}\ K_\nu$ and $u \in \text{ri}\ W_\nu$ imply $u \in \text{ri}\ (W_\nu \cap K_\nu)$ and hence, $u \in \text{ri}\ \Omega_\nu$): in the first case, we have seen that $\lambda u \notin f(u) + V_\nu$ for $\lambda > 1$ and, consequently, $\lambda u \notin f_\nu(u)$, because $\pi_\nu(u) = u$; in the second case, the fact $0 \in \text{ri}\ \Omega_\nu$ and the convexity of K_ν imply $\lambda u \notin K_\nu$ for $\lambda > 1$, and hence, $\lambda u \notin f_\nu(u)$, since $f_\nu(u) \subset K_\nu$. Thus, Corollary of Theorem 4 applies and yields a point $u_\nu \in f_\nu(u)$, i.e, $u_\nu \in f(\pi_\nu(u_\nu)) + v_\nu$ with $v_\nu \in V_\nu$. Using the relative compactness of $f(\Omega)$ and the fact $v_\nu \to 0$, we can assume $u_\nu \to \bar{x}$, and since $\pi_\nu(u_\nu) - u_\nu \in V_\nu$, it follows that $\pi_\nu(u_\nu) \to \bar{x}$. Finally, the upper semi-continuity of f implies $\bar{x} \in f(\bar{x})$. □

§3. SOME APPLICATIONS

We devote this last section to some illustrative examples of applications to variational and quasivariational inequalities.

1. Variational inequalities.

Theorem 8. Let C be a convex closed set in R^n, let f be a upper semi-continuous multivalued mapping from C into R^n, such that $f(x)$ is nonempty, convex, compact for every $x \in C$, and let φ be a finite lower semi-continuous convex function on C. Assume that either of the following conditions holds:

1) C is compact,

2) There is $a \in C$ such that $\inf_{y \in f(x)} \langle y, x - a \rangle + \varphi(x) \to \infty$ as $\| x \| \to \infty$ (coerciveness assumption). Then one can find $\bar{x} \in C$ and $\bar{y} \in f(\bar{x})$ satisfying

$$(\forall x \in C) \quad \langle \bar{y}, x - \bar{x} \rangle + \varphi(x) - \varphi(\bar{x}) \geqslant 0 \tag{3.1}$$

Proof. First consider the case where hypothesis 1) is fulfilled and assume, additionally, that $int\, C \neq \phi$, namely $0 \in int\, C$.

Select a sequence of convex compact sets $C_\nu \subset int\, C$ such that $0 \in C_\nu$ for every ν, and $C_\nu \subset C_{\nu+1}$, $\bigcup_{\nu=1}^{\infty} C_\nu = C$ (for example we can take C_ν to be the set of all $x \in C$ satisfying $p(x) \leqslant 1 - \varepsilon_\nu$, where p is the gauge of C and $\varepsilon_\nu \downarrow 0$).

Since $C_\nu \subset int\, C$, the function φ has a subdifferential $\partial\varphi(x) \neq \phi$ at each point $x \in C_\nu$. By applying Theorem 4, with C_ν playing the role of C in this Theorem, $\Omega = C_\nu$, $F(x) = f(x) + \partial\varphi(x)$, $G(x) = -\overset{\circ}{N}_{C_\nu}(x) = -\{v \in N_{C_\nu}(x) : \langle v, x \rangle = 1\}$, we obtain a point $x_\nu \in C_\nu$ such that: either $0 \in f(x_\nu) + \partial\varphi(x_\nu)$, or $0 \in \theta(f(x_\nu) + \partial\varphi(x_\nu)) - (1 - \theta)\overset{\circ}{N}_{C_\nu}(x_\nu)$, for $0 < \theta \leqslant 1$ (here $\theta \neq 0$ because $0 \notin \overset{\circ}{N}_{C_\nu}(x_\nu)$ if $x_\nu \in \partial C_\nu$). In both situations we have a point $v_\nu \in f(x_\nu) + \partial\varphi(x_\nu)$ satisfying $(\forall x \in C_\nu)\ \langle v_\nu, x - x_\nu \rangle \geqslant 0$, i.e. we have $y_\nu \in f(x_\nu)$, $z_\nu \in \partial\varphi(x_\nu)$ satisfying

$$(\forall x \in C_\nu) \qquad \langle y_\nu + z_\nu, x - x_\nu \rangle \geqslant 0,$$

and hence

$$(\forall x \in C_\nu) \qquad \langle y_\nu, x - x_\nu \rangle + \varphi(x) - \varphi(x_\nu) \geqslant 0.$$

By the compactness of C, we can assume $x_\nu \to \bar{x} \in C$, $y_\nu \to \bar{y} \in f(\bar{x})$, and since φ is lower semi-continuous: $\underline{lim}\, \varphi(x_\nu) \geqslant \varphi(x)$, so that (3.1) holds.

Thus the theorem has been proved (for the case where assumption 1) holds) under the additional condition $int\ C = \phi$. If this condition does not hold, one can always assume that 0 is a relative interior point of C and consider the linear space X' generated by C. Then every $x \in \mathbf{R}^n$ can be written $x = x' + x''$, with $x' \in X'$, $x'' \in X'^{\perp}$. Let $x' = \pi(x)$ and consider the mapping $g = \pi_\circ f$. By the above, there exist $\bar{x} \in C$ and $\bar{u} \in g(\bar{x})$ satisfying $(\forall x \in C)$ $\langle \bar{u}, x - \bar{x} \rangle + \varphi(x) - \varphi(\bar{x}) \geqslant 0$. If $\bar{u} = \pi(\bar{y})$, $\bar{y} \in f(\bar{x})$, then $\langle \bar{u} - \bar{y}, x - \bar{x} \rangle = 0$, so that (3.1) holds.

Turning to the case where hypothesis 2) is fulfilled we observe that for $r > \| a \|$ and large enough we shall have, for every x satisfying $\| x \| \geqslant r$:

$$\inf_{y \in f(x)} [\langle y, x - a \rangle + \varphi(x) - \varphi(a)] > 0 \tag{3.2}$$

Since the set $C_r = \{x \in C : \| x \| \leqslant r\}$ is compact, there exist, by the preceding argument $\bar{x} \in C_r$ and $\bar{y} \in f(\bar{x})$ such that

$$(\forall x \in C_r) \quad \langle \bar{y}, x - \bar{x} \rangle + \varphi(x) - \varphi(\bar{x}) \geqslant 0 \tag{3.3}$$

Taking $x = a \in C_r$ yields

$$\langle \overline{y}, a - \overline{x} \rangle + \varphi(a) - \varphi(\overline{x}) \geqslant 0. \tag{3.4}$$

From (3.2) it follows that $\| \overline{x} \| < r$, and it remains only to prove the following

Lemma 8. If $\overline{x} \in C_r$ satisfies (3.3) (with $\overline{y} \in f(\overline{x})$ and $\| \overline{x} \| < r$ then (3.1) holds.

Indeed, for every $x \in C \setminus C_r$ there is $x' \in C_r$ such that $x' = \lambda x + (1-\lambda)\, \overline{x}$ with $0 < \lambda < 1$. Then

$$\langle \overline{y}, x - \overline{x} \rangle = \frac{1}{\lambda} \langle \overline{y}, x' - \overline{x} \rangle \geqslant \frac{1}{\lambda} (\varphi(\overline{x}) - \varphi(x')) \geqslant$$

$$\geqslant \frac{1}{\lambda} [\varphi(\overline{x}) - \lambda \varphi(x) - (1-\lambda) \varphi(\overline{x})] = \varphi(\overline{x}) - \varphi(x). \quad \square$$

Theorem 9. Let C be a convex closed set in a reflexive Banach space X let f be a multivalued mapping from C into $Y = X^*$ such that $f(x)$ is nonempty, convex, weakly compact for every $x \in C$, and let φ be a finite lower semi-continuous convex function on C.

Assume that:

1) f is upper semi-continuous, in the weak topology of X^* and the finite topology of X.

2) f is monotonous, i.e. $\langle y_1 - y_2, x_1 - x_2 \rangle \geqslant 0$ for every $x_1, x_2 \in C$ and every $y_1 \in f(x_1)$, $y_2 \in f(x_2)$,

3) There exists $a \in C$ such that

$$\inf_{y \in f(x)} \langle y, x - a \rangle + \varphi(x) \to \infty \text{ as } \| x \| \to \infty. \tag{3.6}$$

Then one can find $\overline{x} \in C$ and $\overline{y} \in f(\overline{x})$ satisfying

$$(\forall x \in C) \ \langle \overline{y}, x - \overline{x} \rangle + \varphi(x) - \varphi(\overline{x}) > 0. \tag{3.7}$$

Proof. Let $\{X_\nu, \nu \in N\}$ denote a net of finite-dimensional subspaces of X, such that for every $x \in X$ there is $\nu_0 = \nu_0(x)$ satisfying $(\forall \nu \succ \nu_0)\ x \in X_\nu$. Let $Y_\nu = X_\nu^*$, $C_\nu = C \cap X_\nu$. Let $r > \| a \|$ be large enough to ensure that for every x satisfying $\| x \| \geqslant r$ we have

$$\inf_{y \in f(x)} [\langle y, x - a \rangle + \varphi(x) - \varphi(a)] > 0$$

As seen in the proof of Theorem 8, for every $\nu \in N$ one can find $x_\nu \in C_\nu$ with $\| x_\nu \| < r$ and $y_\nu \in f(x_\nu)$ such that.

$$(\forall x \in C_\nu) \ \langle y_\nu, x - x_\nu \rangle + \varphi(x) - \varphi(x_\nu) \geqslant 0 \tag{3.8}$$

(we assume that every X_ν contains a). Since x_ν, $\nu \in N$, all belong to the convex closed and bounded set $\{x \in C : \| x \| \leqslant r\}$, it can be assumed that $x_\nu \xrightarrow[w]{} \overline{x} \in C$. Then, using hypotheses 1) and 2) it is not hard to prove that x yields a solution to the variational inequality (3.7) (see e.g. [5], where such a proof has been carried out).

In locally convex spaces we can prove the following proposition, which reduces to a known theorem of F. Browder [2] when $\varphi = 0$, X^* is equipped with the topology of uniform convergence on bounded subsets of X, and $\beta(y, x)$ denotes the value of the functional $y \in X^*$ at point $x \in X$.

Theorem 10. Let C be a convex compact set in a locally convex space X. let f be an upper semi-continuous multivalued mapping from C into X^*, such that $f(x)$ is nonempty, convex, compact for every $x \in C$. Let $\beta : X^* \times X \to R$ be a continuous bilinear functional and let $\varphi : C \to R$ be a lower semi-continuous convex function. Then one can find $\bar{x} \in C$ and $\bar{y} \in f(\bar{x})$ satisfying:

$$(\forall x \in C) \quad \beta(\bar{y}, x - \bar{x}) + \varphi(x) - \varphi(\bar{x}) \geqslant 0. \tag{3.9}$$

Proof. We can assume $0 \in C$. Select a family of finite-dimensional subspaces X_ν, $\nu \in N$, of X, such that X_ν is the linear hull of $C \cap X_\nu$ and for every $x \in C$ there is ν_0 satisfying $x \in C_\nu$ for all $\nu \succ \nu_0$ (e.g. N is the collection of all finite subsets of C and X_ν is the linear hull of $\nu \in N$). Let $C_\nu = C \cap X_\nu$ and define as follows a mapping F from C into X_ν: for every $y \in X^*$, $\beta(y, x)$ being a continuous linear functional in x, there exists a fully determined element $g(y) \in X_\nu$ such that $\langle g(y), x \rangle = \beta(y, x)$ for all $x \in X_\nu$; let $F(x) = g(f(x))$. Clearly, $F(x)$ is nonempty, convex and compact for every $x \in C$. Furthermore, F is upper semi-continuous. Indeed, let $x_\nu \to x_0$, $v_\nu \to v_0$, $v_\nu \in F(x_\nu)$; we have $v_\nu = g(y_\nu)$, $y_\nu \in f(x_\nu)$ and by compactness we can assume $y_\nu \to y_0 \in f(x_0)$; but for every x, $\langle v_0, x \rangle = \lim \langle v_\nu, x \rangle = \lim \beta(y_\nu, x) = \beta(y_0, x) = \langle g(y_0), x \rangle$, hence $v_0 = g(y_0)$. Thus, F is closed and hence, upper semi-continuous, since $F(C)$ is compact. By Theorem 8, one can find $x_\nu \in C_\nu$ and $v_\nu \in F(x_\nu)$ satisfying

$$(\forall x \in C_\nu) \ \langle v_\nu, x - x_\nu \rangle + \varphi(x) - \varphi(x_\nu) \geqslant 0.$$

Remembering that $v_\nu = g(y_\nu)$, $y_\nu \in f(x_\nu)$, we have

$$(\forall x \in C_\nu) \quad \beta(y_\nu, x - x_\nu) + \varphi(x) - \varphi(x_0) \geqslant 0.$$

Using compactness we can assume $x_\nu \to \bar{x} \in C$, $y_\nu \to \bar{y} \in f(\bar{x})$. If $x \in C$, then $x \in C_\nu$ for all $\nu \succ \nu_0 = \nu_0(x)$ and (3.9) follows from the previous relation, by taking into account the continuity of β and the lower semi-continuity of φ.

2. Quasi-variational inequalities.

Theorem 11. Let C be a convex compact set in R^n, let f be a upper semi-continuous multivalued mapping from C into R^n, such that $f(x)$ is nonempty, convex, compact for every $x \in C$, and let E be a continuous multivalued mapping from C into aff C, such that $E(x)$ is nonempty, convex, compact for every $x \in C$ and the mapping $x \mapsto N_{E(x)}(x) \setminus \{0\}$ satisfies the inward or inward* boundary condition on C. Then one can find $\bar{x} \in C \cap E(\bar{x})$ and $\bar{y} \in f(\bar{x})$ satisfying

$$(\forall x \in E(\bar{x})) \quad \langle \bar{y}, x - \bar{x} \rangle \geqslant 0. \tag{3.10}$$

Proof. Without loss of generality, we can suppose $\mathit{int}\ C \neq \phi$ (if necessary we can replace R^n by aff C and each set $f(x)$ by its projection on aff C). Consider first the case where $\mathit{int}\ E(x) \neq \phi$ for every $x \in C$. Let $\Omega = \{x \in C:$

$x \in E(x)\}$. Since the mapping E is continuous, Ω is closed. Let $\Omega' = \{x \in \Omega: x \in int\, E(x)\}$. If $x \in C \setminus \Omega'$ then $x \notin int\, E(x)$, and so the set $N_{E(x)}(x)$ has at least one element $v \neq 0$. Denote by Σ the unit sphere and let $G(x) = co(\Sigma \cap N_{E(x)}(x))$. Clearly $G(x)$ is a convex, compact subset of $N_{E(x)}(x)$ and, by Lemma 9 below, $0 \notin G(x) \neq \phi$ for every $x \in C \setminus \Omega'$.

Suppose $x_\nu \in C \setminus \Omega'$ and $x_\nu \to x_0$. Taking $y_\nu \in \Sigma \cap N_{E(x_\nu)}(x_\nu)$ and a subsequence along which $y_\nu \to y_0$, we shall have $y_0 \in \Sigma$ and $y_0 \in N_{E(x_0)}(x_0)$. Indeed, from the lower semi-continuity of E there exists for every $x \in E(x_0)$ a sequence $x'_\nu \to x$ such that $x'_\nu \in E(x_\nu)$; but $(\forall x' \in E(x_\nu)) \langle y_\nu, x' - x_\nu \rangle \geqslant 0$, hence $\langle y_\nu, x'_\nu - x_\nu \rangle \geqslant 0$, which implies $\langle y_0, x - x_0 \rangle \geqslant 0$, i.e. $y_0 \in N_{E(x_0)}(x_0)$. Since $y_0 \in \Sigma$, we have $y_0 \neq 0$, so that $x_0 \notin int\, E(x_0)$, i.e. $x_0 \in C \setminus \Omega'$. Therefore the set Ω' is open and $C \setminus \Omega \subset C \setminus \Omega'$, $\partial C \in C \setminus \Omega'$.

Furthermore, the above reasoning shows that the mapping $x \to \Sigma \cap N_{E(x)}(x)$ is closed on $C \setminus \Omega'$; hence the mapping G is upper semi-continuous. Applying Theorem 4, with $F(x) = f(x)$ yields a point $\bar{x} \in \Omega$ such that:

a) either $0 \in f(\bar{x})$, so that (3.10) holds with $\bar{y} = 0$;

b) or $0 \in -\theta f(\bar{x}) + (1-\theta) G(\bar{x})$ with $0 < \theta \leqslant 1$ (here $\theta \neq 0$ because $0 \notin G(\bar{x})$): in this case there is $\bar{y} \in f(\bar{x}) \cap N_{E(\bar{x})}(\bar{x})$, i.e. (3.10) is satisfied.

Thus the Theorem has been proved under the condition $int\, E(x) \neq \phi$ for every $x \in C$. If this condition is not satisfied, we can suppose $0 \in int\, C$ and consider the set $C_\varepsilon = \{x : p(x) \leqslant 1 + \varepsilon\}$, where $\varepsilon > 0$ and p is the gauge of C. Denoting by B_ε a ball such that $C + B_\varepsilon \subset C_\varepsilon$, we extend f and E over the whole C_ε in the following way. For $x \in C$ let $\sigma(x) = x$, and for $x \in C_\varepsilon \setminus C$ let $\sigma(x)$ denote the point where the ray through x cuts ∂C; then set $f_\varepsilon(x) = f(\sigma(x))$, $E_\varepsilon(x) = E(\sigma(x)) + B_\varepsilon$. Since C_ε is homothetic to C, it is easily seen that the mapping $x \to N_{E_\varepsilon(x)}(x) \setminus \{0\}$ satisfies the inward (or inward*, resp.) boundary condition on C_ε. Noting that now $int\, E(x) \neq \phi$ for every $x \in C$ and applying the result of the first part of the proof, we can find $x_\varepsilon \in C_\varepsilon \cap (E(\sigma(x_\varepsilon)) + B_\varepsilon)$ and $y_\varepsilon \in f(\sigma(x_\varepsilon))$ such that

$$(\forall x \in E(\sigma(x_\varepsilon)) + B_\varepsilon) \quad \langle y_\varepsilon, x - x_\varepsilon \rangle \geqslant 0 \tag{3.11}$$

Letting $\varepsilon \downarrow 0$ and taking a subsequence along which $x_\varepsilon \to \bar{x}$, $y_\varepsilon \to \bar{y}$, we shall have $\bar{x} \in C \cap E(\bar{x})$ (because $\rho(x_\varepsilon, C) \to 0$ and $\sigma(x_\varepsilon) \to \sigma(\bar{x}) = \bar{x}$), and $\bar{y} \in f(\bar{x})$. For every $x \in E(\bar{x})$, there is, by the lower semi-continuity of E a sequence $x'_\varepsilon \to x$ such that $x'_\varepsilon \in E(\sigma(x_\varepsilon))$ and from (3.11) we get $\langle y_\varepsilon, x'_\varepsilon - x_\varepsilon \rangle \geqslant 0$ hence $\langle \bar{y}, x - \bar{x} \rangle \geqslant 0$.

It remains to prove:

Lemma 9: If a convex set D has an interior point and for some point x we have $0 \notin A \subset N_D(x)$, then $0 \notin co\, A$.

Proof. Assume the contrary, that $0 \in co\ A$. Then there are $v_i \in A$ and $\lambda_i > 0$ satisfying $\Sigma\lambda_i = 1$, $\Sigma\lambda_i v_i = 0$. Since $A \subset N_D(x)$, we have $(\forall u \in D)$ $\langle v_i, u - x\rangle \geqslant 0$ for every i. But $(\forall u \in D)$ $\langle \Sigma\lambda_i v_i, u - x\rangle = 0$, therefore $\langle v_i, u - x\rangle = 0$ for every i. In view of the hypothesis $int\ D = \phi$, this conflicts with the fact $v_i \neq 0$. □

Theorem 12. Let C be a convex closed set in R^n; let f be an upper semi-continuous multivalued mapping from C into R^n such that $f(x)$ is nonempty, convex, compact for every $x \in C$; let E be a continuous multivalued mapping from C into $aff\ C$, such that $E(x)$ is nonempty, convex, compact for every $x \in C$; let E be a continuous multivalued mapping from C into $aff\ C$, such that $E(x)$ is nonempty, convex, compact for every $x \in C$, and $E(x) - x \subset I_C(x)$ for every $x \in \partial_a C$. Assume that the following «coerciveness» condition holds:

There are a point $a \in \bigcap_{x \in C} E(x)$ and a compact set K such that

$$(\forall x \in C \setminus K) \qquad \inf_{y \in f(x)} \langle y, x - a\rangle > 0 \tag{3.12}$$

Then one can find $\bar{x} \in C \cap E(\bar{x})$ and $\bar{y} \in f(\bar{x})$ satisfying

$$(\forall x \in E(\bar{x})) \qquad \langle \bar{y}, x - \bar{x}\rangle \geqslant 0 \tag{3.13}$$

Proof. Select a ball B around a, of radius $r > 0$ so large that $(\forall x \in K)$ $\| x - a \| < r$. Let $C' = C \cap B$, $E'(x) = E(x) \cap B$. It is easy to see that the mapping E' is continuous. Indeed, the upper semi-continuity is obvious. To prove the lower semi-continuity, consider any $y_0 \in E'(x_0)$, $x_0 \in C'$, and any neighbourhood U of y_0 (we can assume U to be a ball centered at y_0). Since the mapping E is continuous, there is a neighbourhood V of x such that, for every $x \in V \cap C'$ the set $E(x)$ meets U at some point y: if $y \notin B$ and $a \notin U$, then the projection z of y_0 on the segment $[a; y]$ belongs to $E(x)$ (since a and y both belong to $E(x)$), and also to U (since $\| z - y_0 \| \leqslant \| y - y_0 \|$) and to B (since $\| z - a \| \leqslant \| y_0 - a \|$) so that in any case we have $E'(x) \cap U \neq \phi$.

On the other hand, $(\forall x \in \partial C')\, E'(x) - x \subset I_{C'}(x)$ (and hence the mapping $x \mapsto N_{E'(x)}(x) \setminus \{0\}$ satisfies the inward* boundary condition on C'). Indeed, if $x \in \partial C'$, then either $x \in \partial C$ or $x \in \partial B \cap int C$: in the first case $E'(x) - x \subset I_C(x) \cap B \subset I_{C'}(x)$, whereas in the second case $E'(x) - x \subset I_B(x) = I_{C'}(x)$. Therefore, by Theorem 11, there are $\bar{x} \in E'(\bar{x})$ and $\bar{y} \in f(\bar{x})$ such that

$$(\forall x \in E'(\bar{x})) \qquad \langle \bar{y}, x - \bar{x}\rangle \geqslant 0.$$

Since $a \in E'(\bar{x})$ we have $\langle \bar{y}, a - \bar{x}\rangle \geqslant 0$ and hence, by (3.12), $\bar{x} \in K$, i.e. $\| \bar{x} - a \| < r$. Therefore, (3.13) follows from (3.14), by Lemma 8. □

Theorem 13. Let C be a convex compact set in a reflexive Banach space X, let f be a multivalued mapping from C into $Y = X^*$, such that $f(x)$ is nonempty, convex, compact for every $x \in C$, and let E be a continuous multivalued mapping from C into $aff\ C$, such that $E(x)$ is nonempty, convex, compact for every $x \in C$ and $E(x) - x \subset I_C(x)$ for every $x \in \partial_a C$. Assume that:

1) f is upper semi-continuous in the weak topology of X^* and the finite topology of X,

2) f is monotonuous and bounded.

Then one can find $\bar{x} \in C \cap E(\bar{x})$ and $\bar{y} \in f(\bar{x})$ satisfying

$$(\forall x \in E(\bar{x})) \qquad \langle \bar{y}, x - \bar{x} \rangle \geqslant 0 \tag{3.15}$$

Proof. For every natural number ν takea ball B_Y (around 0) of radius ε_ν, where $\varepsilon \downarrow 0$. The set $E(C)$ being compact (in view of the continuity of E), there is a finite subset Q_ν of $E(C)$ such that $E(C) \subset \bigcup_{q \in Q_\nu} \left(q + \frac{1}{2} B_Y\right)$. Since $E(C) \subset aff\, C$, each $q^i \in Q_\nu$ has the form $q = \sum_j \lambda_{ij} q_{ij}$, with $q_{ij} \in C$, Let $A_\nu = \bigcup_{i,j} \{q_{ij}\}$.

$$X_\nu = aff\ A_\nu,\ C_\nu = C \cap x_\nu.\ E_\nu(x) = (E(x) + B_Y) \cap \overline{co} E(C) \cap X_\nu$$

Clearly $aff\ C_\nu = X_\nu$ and for every $x \in C$ the set $E_\nu(x)$ is convex, compact (note that $co\,E(C)$ is compact as the closed convex hull of a compact set in a Banach space); moreover, $E_\nu(x)$ is nonempty (indeed, if $v \in E(x)$, then there is $q \in Q_\nu \subset E(C) \cap X_\nu$ such that $v \in q + \frac{1}{2} B_\nu$, hence $q \in (v + B_\nu) \cap E(C) \cap X_\nu \subset E_\nu(x)$). We claim that the mapping E_ν is continuous. Indeed, the upper semi-continuity being obvious, let us prove the lower semi-continuity of E_ν. Consider any $v_0 \in E_\nu(x_0)$ and any neighbourhood U of $0 \in X_\nu$. We have $v_0 = s_0 + w_0$ with $s_0 \in E(x_0)$, $w_0 \in B_\nu$. Let $\alpha \in (0,1)$ be so small that $\alpha(Q_\nu - v_0) \subset U$, and let $U' = \frac{\alpha}{2(1-\alpha)} B_\nu$. Since E is continuous, there exists a neighbourhood V of $0 \in X$ such that for every $x \in (x_0 + V) \cap C$ there is $v \in E(x) \cap (s_0 + U')$, i.e. $v = s_0 + u'$, with $v \in E(x)$, $u' \in U'$. On the other hand, as we have seen previously, $v \in E(x)$ implies $v = q + w$, with $w \in \frac{1}{2} B_\nu$ and $q \in Q_\nu \subset E(C) \cap X_\nu$. Setting $s = v_0 + \alpha(q - v_0)$, we have $s - v_0 = \alpha(q - v_0) \in \alpha(Q_\nu - v_0) \subset U$. Furthermore, $s = \alpha q + (1-\alpha) v_0 \in \overline{co}\ E(C) \cap x_\nu$ (because $0 < \alpha < 1$) and $s - v = v - v_0 + \alpha(q - v_0) = w_0 - u' + \alpha(-w_0 + u' - w) = (1-\alpha)(w_0 - u') - \alpha w$

$$= (1-\alpha) w_0 + \alpha\left(-w - \frac{1-\alpha}{\alpha} u'\right) \in (1-\alpha) B_\nu + \alpha\left(\frac{1}{2} B_\nu + \frac{1}{2} B_\nu\right) = B_\nu, \text{ i.e.}$$

$$s \in E(x) + B_\nu.$$

Thus, for every $x \in x_0 + V$, the set $E_\nu(x)$ meets $v_0 + U$ at least at one point s, proving the lower semi-continuity of E_ν.

Let now $C' = C + B_\nu$, $C'_\nu = C' \cap X_\nu$. For every $x \in C'_\nu$ denote by $\pi(x)$ the projection of x on C and set $\tilde{f}(x) = f(\pi(x))$, $\tilde{E}_\nu(x) = E_\nu(\pi(x))$. Since π is continuous (Lemma 3) the mapping f is upper semi-continuous, while $\tilde{E}_\nu$ is continuous at every $x \in C'_\nu$. If $x \in \partial_a C'_\nu \subset \partial_a C'$, then the element $j(\pi(x) - x) \in X^*$

defined as in Lemma 4 is simultaneously an outward normal of C' at x, of C at $\pi(x)$, and of the ball $\pi(x)+B_\nu$ at x. For every $u\in E(\pi(x))+B_\nu$, since $u=v+w$, with $w\in B_\nu$, $v\in E(\pi(x))\subset x+I_C(\pi(x))$, we have $\langle j(\pi(x)-x), u-x\rangle=\langle j(\pi(x)-x), v-\pi(x)\rangle+\langle j(\pi(x)-x), w+\pi(x)-x\rangle\leqslant 0$. Thus, the restriction of the functional $-j(\pi(x)-x)$ on X_ν is a common element to $N_{C'_\nu}(x)$ and $N_{\tilde{E}_\nu(x)}(x)$. Therefore, the mapping $x\to N_{\tilde{E}_\nu(x)}(x)\setminus\{0\}$ satisfies the inward* boundary condition on C'_ν and by appling Theorem 11, we obtain $x_\nu\in C'_\nu\cap\cap E_\nu(\pi(x_\nu))$ and $y_\nu\in f(\pi(x_\nu))$ satisfying

$$\forall x\in E_\nu(\pi(x_\nu))\ \langle y_\nu, x-x_\nu\rangle\geqslant 0 \tag{3.16}$$

Since x_ν are all contained in the compact set $\overline{co}\,E(C)$, we can assume $x_\nu\to\bar{x}$, and since $\rho(x_\nu, C)\leqslant\varepsilon_\nu\downarrow 0$, we have $\bar{x}\in C$. By Lemma 3, $\pi(x_\nu)\to\pi(\bar{x})=\bar{x}$ and from the closedness of $E:\bar{x}\in E(\bar{x})$.

Let x be an arbitrary element of $E(\bar{x})$. For every $\lambda\in[0,1]$ and every $y^\lambda\in f(x^\lambda)$, where $x^\lambda=\lambda x+(1-\lambda)\bar{x}\in E(\bar{x})$, we have, since f is monotonous, $\langle y^\lambda-y_\nu, x^\lambda-\pi(x_\nu)\rangle\geqslant 0$, i.e.

$$\langle y^\lambda, x^\lambda-\pi(x_\nu)\rangle\geqslant\langle y_\nu, x^\lambda-\pi(x_\nu)\rangle \tag{3.17}$$

Using the continuity of E, we can find a sequence $x_\nu^\lambda\to x^\lambda$ such that $x_\nu^\lambda\in E(\pi(x_\nu))$, and hence, as has been seen previuosly, $x_\nu^\lambda=q_\nu^\lambda+\frac{1}{2}B_\nu$ with $q_\nu^\lambda\in E_\nu(\pi(x_\nu))$. By (3.16):

$$\langle y_\nu, q_\nu^\lambda-x_\nu\rangle\geqslant 0 \tag{3.18}$$

But

$$\langle y^\lambda, x^\lambda-\pi(x_\nu)\rangle\geqslant\langle y_\nu, x^\lambda-\pi(x_\nu)\rangle=\langle y_\nu, q_\nu^\lambda-x_\nu\rangle+$$

$$+\langle y_\nu, x_\nu-\pi(x_\nu)\rangle+\langle y_\nu, x_\nu^\lambda-q_\nu^\lambda\rangle+\langle y_\nu, x^\lambda-x_\nu^\lambda\rangle,$$

therefore, by taking the limit as $\nu\to\infty$ and noting that the last three terms of the just written sum tend to 0 (because y_ν is bounded), we have $\langle y^\lambda, x^\lambda-\bar{x}\rangle\geqslant 0$. Since $x^\lambda-\bar{x}=\lambda(x-\bar{x})$, we then deduce $\langle y^\lambda, x-\bar{x}\rangle\geqslant 0$. But f being upper semi-continuous on the segment $[x,\bar{x}]$, the set $\bigcup_{0\leqslant\lambda\leqslant 1} f(x^\lambda)$ is compact (in the weak topology). Hence one can select a subsequence along which $x^\lambda\to\bar{x}$ and $y^\lambda\xrightarrow[w]{} y\in f(\bar{x})$. Thus for every $x\in E(\bar{x})$ we have found $y\in f(\bar{x})$ such that

$$\langle y, x-\bar{x}\rangle\geqslant 0,$$

i. e. $\inf_{x \in E(\bar{x})} \sup_{y \in f(\bar{x})} \langle y, x - \bar{x} \rangle \geqslant 0$ and hence, by the minimax theorem, $\sup_{y \in f(\bar{x})} \inf_{x \in E(\bar{x})} \langle y, x - \bar{x} \rangle \geqslant 0$. □

Theorem 14. Let C be a convex closed set in a reflexive Banach space X; let f be a multivalued mapping from C into $Y = X^*$, such that $f(x)$ is nonempty, convex, compact for erery $x \in C$; let E be multivalued mapping from C into *aff* C which is continuous in the weak topology and such that the set $E(C)$ is compact, $f(x)$ is nonempty, convex, compact for every $x \in C$ and $E(x) - x \subset I_C(x)$ for every $x \in \partial_a C$. Assume that:

1) f is upper semi-continuous in the weak topology of X^* and the finite topology of X,

2) f is monotonous and bounded,

3) either of the following conditions holds:

a) C is bounded,

b) There are a point $a \in \bigcap_{x \in C} E(x)$ and a bounded set K such that

$$(\forall x \in C \setminus K) \qquad \inf_{y \in f(x)} \langle y, x - a \rangle > 0 \tag{3.19}$$

Then one can find $\bar{x} \in E(\bar{x}) \cap C$ and $\bar{y} \in f(\bar{x})$ satisfying

$$(\forall x \in E(\bar{x})) \qquad \langle \bar{y}, x - \bar{x} \rangle \geqslant 0 \tag{3.20}$$

Proof. Construct the subspaces X_ν and the mappings E_ν as in the proof of Theorem 12, and set $E'_\nu = C' \cap X_\nu$, $C' = C + B_\nu$ (B_ν bieng a ball around 0, of radius ε_ν), $\tilde{f}(x) = f(\pi(x))$, $\tilde{E}_\nu = E_\nu(\pi(x))$ for every $x \in C'_\nu$. Since π is weakly continuous by Lemma 3, the mapping $\tilde{f}$ (from C'_ν into X^*_ν) is upper semi-continuous, while the mapping $\tilde{E}_\nu$ (from C'_ν into X_ν) is continuous. Also it is easily seen that $\tilde{E}(x) - x \subset I_{C'_\nu}(x)$ for every $x \in \partial_a C'_\nu$. Therefore, by Theorems 10 and 11, there are $x_\nu \in E_\nu(\pi(x_\nu)) \cap C'_\nu$ and $y_\nu \in f(\pi(x_\nu))$ satisfying (3.16) (we can assume $a \in Q_\nu$ for every ν, i. e $a \in \bigcap_x E_\nu(x)$). The proof can then be completed just in the same way as that of Theorem 13 (here we have not $\pi(x_\nu) \to \pi(\bar{x}) = \bar{x}$, but only $\pi(x_\nu) \xrightarrow[w]{} \bar{x}$; however, this does not matter; since E is continuous in the weak topology, for every $x^\lambda \in E(\bar{x})$ there is a sequence $x^\lambda_\nu \xrightarrow[w]{} x^\lambda$ such that $x^\lambda_\nu \in E(\pi(x_\nu))$ and $E(C)$ being compact, one can assume $x^\lambda_\nu \to x^\lambda$). □

Theorem 15. Let C be a convex closed set in arcflexive Banach space X, let f be a multivalued mapping from C into $Y = X^*$, such that $f(x)$ is nonempty, convex, weakly compact, and let E be a multivalued mapping from C into C, such that $E(x)$ is nonempty, convex, closed and bounded for every $x \in C$ and E is continuous in the weak topology, Assume:

1) f is upper semi-continuous in the weak topology of X^* and the finite topology of X;

2) f is monotonous and $f(C)$ is contained in a strongly compact set in X^*.

Then one can find $\bar{x} \in E(\bar{x})$ and $\bar{y} \in f(\bar{x})$ satisfying

$$(\forall x \in E(\bar{x})) \qquad \langle \bar{y}, x - \bar{x} \rangle \geqslant 0. \tag{3.21}$$

Proof. Let $\{V_\nu, \nu \in N\}$ be a local base of convex, closed, balanced neighbourhoods, in the weak topology of $0 \in X$. Since $E(C)$ is weakly compact, there exists for every ν a finite subset A_ν of $E(C)$ such that $E(C) \subset \bigcup_{q \in A_\nu} \left(q + \frac{1}{2} V_\nu\right)$. Let X_ν denote the linear hull of A_ν (assuming, as we may without loss of generality, that $0 \in E(C)$ and $0 \in A_\nu$ for every ν); let $C_\nu = C \cap X_\nu$, $E_\nu(x) = (E(x) + V_\nu) \cap C_\nu$.

Clearly $X_\nu = aff\ C_\nu$ and, as in the proof of Theorem 13, it is easily verified that every $E_\nu(x)$ ($x \in C_\nu$) is nonempty, convex, compact, and that the mapping E_ν is continuous.

Therefore, by Theorem 11 one can find $x_\nu \in E_\nu(x_\nu)$ and $y_\nu \in f(x_\nu)$ satisfying

$$(\forall x \in E_\nu(x_\nu)) \qquad \langle y_\nu, x - x_\nu \rangle \geqslant 0 \tag{3.22}$$

Since $\{x_\nu\} \subset C$, we can assume $x_\nu \xrightarrow[W]{} \bar{x} \in E(\bar{x})$ and (3.21) follows from (3.22) by an argument analogous to that used in the last part of the proof of Theorem 13 (the only difference is that here $\pi(x_\nu) = x_\nu$, $x_\nu^\lambda - q_\nu^\lambda \xrightarrow[W]{} 0$ and we should use the following theorem (see [8], theorem 3 (1.IX)): if $u_\nu \in X$, $u_\nu \xrightarrow[W]{} 0$, if M is strongly compact in X^*, then $\langle y, u_\nu \rangle \to 0$ uniformly with respect to $y \in M$. hence, since $f(C)$ is contained in a strongly compact set of X^*, $\langle y_\nu, x_\nu^\lambda - q_\nu^\lambda \rangle \to 0$, etc...) $\square$.

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