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ON THE CONTRACTION PRINCIPLE

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In the last 15 years many authors have generalized the Banach contraction principle in relation with several classes of generalized contractions. In the present paper we shall establish, in Part I, the inclusion relations between these classes, and a fixed point theorem for spaces with a family of pseudometrics, in Part II, some new results on fixed points for multivalued generalized contractions, then in Part III, a fixed point theorem of Krasnoselski type and finally, in Part IV, some fixed point theorems for non-expansive mappings in locally convex spaces.

I. GENERALIZED CONTRACTIONS AND FIXED POINT THEOREM.

A—INCLUSION RELATIONS BETWEEN CLASSES OF GENERALIZED CONTRACTIONS.

Let X be a metric space and T be a mapping in X. We recall the following definitions:

Definition 1 [2]. We say that T belongs to the class $\mathcal{B}$ iff there exists a number $\alpha \in [0, 1)$ such that $d(Tx, Ty) \leqslant \alpha d(x, y)$ for any $x, y \in X$.

Definition 2 [15]. $T \in \mathcal{R}$ iff there exists a non-increasing function $\alpha : (0, \infty) \to [0, 1)$ such that

$$d(Tx, Ty) \leqslant \alpha(d(x, y))d(x, y), \qquad (x \neq y). \tag{1}$$

Definition 3 [10]. $T \in \mathcal{K}$ iff there exists a function $\alpha(\xi, \eta)$ defined when $0 < \xi \leqslant \eta < \infty$ such that $\alpha(\xi, \eta) < 1$ and

$$d(Tx, Ty) \leqslant \alpha(\xi, \eta)d(x, y), \quad (\xi \leqslant d(x, y) \leqslant \eta). \tag{2}$$

Definition 4 [3]. $T \in \mathcal{V}$ iff there exists a upper-semicontinuous from the right function $\alpha : (0, \infty) \to [0, 1)$ satisfying (1).

Definition 5 [12]. $T \in \mathcal{M}$ iff $\forall \varepsilon > 0 \ \exists \delta > 0$ such that

$$\varepsilon \leqslant d(x, y) < \varepsilon + \delta \Rightarrow d(Tx, Ty) < \varepsilon. \tag{3}$$

Notation:

$$r(x, y) = \max \left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}[d(x, Ty) + d(y, Tx)] \right\}.$$

Definition 6 [21]. $T \in \mathcal{C}$ iff $\forall \varepsilon > 0\ \exists \delta > 0$ such that

$$r(x, y) < \varepsilon + \delta \Rightarrow d(Tx, Ty) < \varepsilon$$

and T is continuous.

It is known that if X is complete and T belongs to one of the above classes then T has a unique fixed point x^* and $T^n x \to x^*$ for every x in X [2, 3, 10, 15, 21].

Definition 7 [20]. $T \in \mathcal{S}$ iff there exists a function $\alpha : [0, \infty) \to [0, 1)$ such that

$$\sup \{\alpha(t) : t \in [a, b]\} < 1, \quad (0 < a \leqslant b < \infty), \tag{4}$$

and (1) is satisfied.

Theorem 1. The following inclusion relations are true

$$\mathcal{B} \subset \mathcal{R} \subseteq \mathcal{K} \subseteq \mathcal{S} \subseteq \mathcal{V} \subset \mathcal{M} \subset \mathcal{C}.$$

Proof.

1. *Proof of* $\mathcal{B} \subset \mathcal{R}$. Obviously $\mathcal{B} \subseteq \mathcal{R}$. To show that $\mathcal{B} \neq \mathcal{R}$ we shall prove the following fact.

Lemma 1. $T \in \mathcal{R}$ iff there exists a function $\alpha(\xi) < 1$ such that

$$d(Tx, Ty) \leqslant \alpha(\xi) d(x, y), \ (0 < \xi \leqslant d(x, y)). \tag{5}$$

Proof of Lemma 1. Obviously, if $T \in \mathcal{R}$ then (5) holds. Now let (5) hold we define

$$\tilde{\alpha}(t) = \inf\{\alpha(\xi) : \xi \leqslant t\}, \qquad (t > 0).$$

Evidently, $\tilde{\alpha}$ is non-increasing and $\tilde{\alpha}(t) < 1$. To verify (1) we must show that

$$d(Tx, Ty) \leqslant [\tilde{\alpha}(d(x, y)) + \varepsilon] d(x, y) \tag{6}$$

for every $\varepsilon > 0$.

Let $\varepsilon > 0$, by the definition of $\tilde{\alpha}$ there exists $\xi \leqslant d(x, y)$ such that $\alpha(\xi) \leqslant \tilde{\alpha}(d(x, y) + \varepsilon$.

From this fact and (5) we obtain (6).

Lemma 2. If X is compact and T satisfies

$$d(Tx, Ty) < d(x, y), \quad (x \neq y)$$

then $T \in \mathcal{R}$.

Proof of Lemma 2. We must show that there exists a function α satisfying the condition in Lemma 1. Put

$$\alpha(\xi) = \sup \left\{ \frac{d(Tx, Ty)}{d(x, y)} : x, y \in X, \ d(x, y) \geqslant \xi \right\}, \quad (\xi > 0).$$

We verify that $\alpha(\xi) < 1$. In fact, if $\alpha(\xi) = 1$ then there are two sequences x_n, y_n such that $\dfrac{d(Tx_n, Ty_n)}{d(x_n, y_n)} \to 1$.

Since X is compact. so is $\{(x, y) : d(x, y) \geqslant \xi\}$, and we may assume $x_n \to x^*$, $y_n \to y^*$. But then $d(Tx^*, Ty^*) = d(x^*, y^*)$ which contradicts the assumption.

Example 1. Let $X = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots, 0\right\}$, $T\left(\frac{1}{n}\right) = \frac{1}{n+1}$, $T(0) = 0$. It is easy to verify that T satisfies all the conditions in Lemma 2. Hence $T \in \mathcal{R}$.

On the other hand, for $x_n = \frac{1}{n}$ we have

$$\frac{d(Tx_n, Tx_{n+1})}{d(x_n, x_{n+1})} \to 1, \quad \text{i.e. } T \notin \mathcal{B}.$$

This example shows that the proof of Theorem 2 (1. XVI) in [7] is erroneous.

2. *Proof of* $\mathcal{R} \subset \mathcal{K}$. We have only to put $\tilde{\alpha}(\xi, \eta) = \alpha(\xi)$ for all $\eta \geqslant \xi$, where α is the function in emma 1.

1. *Proof of* $\mathcal{K} \subseteq \mathcal{S}$. We remark that the function α in Defintion 3 must be assumed non-increasing in ξ and non-decreasing in η. In fact, if necessary, α can be replaced by

$$\alpha^*(\xi, \eta) = \inf\{\alpha(\xi', \eta') : \xi' \leqslant \xi \leqslant \eta \leqslant \eta'\}.$$

Put

$$\tilde{\alpha}(t) = \overline{\lim_{\substack{\xi \to t-0 \\ \eta \to t+0}}} \alpha(\xi, \eta), \qquad (t > 0).$$

It is easy to verify that $\sup\{\tilde{\alpha}(t) : t \in [a, b]\} = \alpha(a, b) < 1$, i.e (4) holds.

4. *Proof of* $\mathcal{S} \subseteq \mathcal{V}$. Put

$$\tilde{\alpha}(t) = \overline{\lim_{s \to t}} \alpha(s), \qquad (t > 0),$$

where α is the function mentioned in Definition 7. It is easy to verify that $\tilde{\alpha}$ is upper semicontinuous and $\alpha(t) \leqslant \tilde{\alpha}(t) < 1$.

5. *Proof of* $\mathcal{V} \subset \mathcal{M}$. To show that $\mathcal{V} \subseteq \mathcal{M}$ we put $\beta(t) = \alpha(t)t$, $(t > 0)$, where α is the function in Definition 4. Then $\beta(\varepsilon) < \varepsilon$ for every $\varepsilon > 0$. By the upper semicontinuity of β, $\forall \varepsilon > 0$, $\exists \delta > 0$ such that

$$\varepsilon \leqslant t < \varepsilon + \delta \Rightarrow \beta(t) < \varepsilon.$$

It follows that

$$\varepsilon \leqslant d(x, y) < \varepsilon + \delta \Rightarrow \beta(d(x, y)) < \varepsilon \Rightarrow d(Tx, Ty) < \varepsilon,$$

i.e. $T \in \mathcal{M}$.

To show that $\mathcal{V} \neq \mathcal{M}$ we shall use the following fact.

Lemma 3. Let $T \in \mathcal{V}$ and let the sequence $\{d(x_n, y_n)\}$ monotonically decrease to $t > 0$. Then

$$\overline{\lim_{n \to \infty}} \frac{d(Tx_n, Ty_n)}{d(x_n, y_n)} < 1. \tag{7}$$

Proof of Lemma 3. For each n we have

$$\frac{d(Tx_n, Ty_n)}{d(x_n, y_n)} \leqslant \alpha(d(x_n, y_n)). \tag{8}$$

Choose $\varepsilon > 0$ such that $\alpha(t) + 2\varepsilon < 1$, then there exists N such that

$$\alpha(d(x_n, y_n)) \leqslant \alpha(t) + \varepsilon < 1 - \varepsilon, \qquad (\forall n \geqslant N).$$

Since α is upper semicontinuous from the right, then (8) implies (7).

Example 2. Let H be a separable Hilbert space, $\{e_n\}$ be an orthonormal basis in H. Put $X = \{x_n\} \cup \{y_n\} \cup \{\theta\}$, where $x_n = \left(1 - \frac{1}{n}\right) e_n$, $y_n = 2e_n$, θ is the origin in H.

Setting $T\theta = \theta$, $Tx_n = \theta$, $Ty_n = x_n$ $(\forall n)$ we have

$$d(Tx_n, T\theta) = 0, \qquad d(x_n, \theta) = 1 - \frac{1}{n},$$

$$d(Tx_n, Tx_m) = 0, \qquad d(x_n, x_m) = \sqrt{\left(1 - \frac{1}{n}\right)^2 + \left(1 - \frac{1}{m}\right)^2},$$

$$d(Ty_n, T\theta) = 1 - \frac{1}{n}, \qquad d(y_n, \theta) = 2,$$

$$d(Tx_n, Ty_n) = 1 - \frac{1}{n}, \qquad d(x_n, y_n) = 1 + \frac{1}{n}, \tag{9}$$

$$d(Tx_n, Ty_m) = 1 - \frac{1}{m}, \qquad d(x_n, y_m) = \sqrt{\left(1 - \frac{1}{n}\right)^2 + 4},$$

$$d(Ty_n, Ty_m) = \sqrt{\left(1 - \frac{1}{n}\right)^2 + \left(1 - \frac{1}{m}\right)^2}, \quad d(y_n, y_m) = \sqrt{8}.$$

It is easy to verify that $T \in \mathcal{M}$, since for $\varepsilon \geqslant \sqrt{2}$ then δ may be chosen arbitrarily, if $1 \leqslant \varepsilon < \sqrt{2}$ then $\delta = \sqrt{8} - \varepsilon$ and finaly, if $0 < \varepsilon < 1$ then $\delta = 1 - \varepsilon$. On the other hand, $T \notin \mathcal{U}$ since (9) shows that (7) does not hold.

6. Proof of $\mathcal{M} \subset \mathcal{C}$. Obviously $\mathcal{M} \subseteq \mathcal{C}$ since (3) may be replaced by

$$d(x, y) < \varepsilon + \delta \Rightarrow d(Tx, Ty) < \varepsilon.$$

To prove $\mathcal{M} \neq \mathcal{C}$ we construct the following example.

Example 3. Let $X = \{x_0, x_1, y_0, y_1, z\}$ as in Fig. 1, where

$$d(x_0, y_0) = d(y_0, y_1) = d(x_1, y_1) = d(x_0, x_1) = 2,$$
$$d(z, x_1) = d(z, y_1) = 1.$$

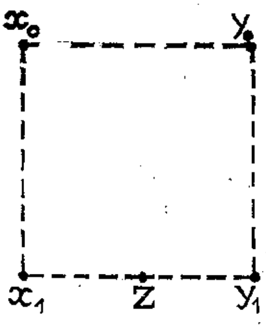

Fig. 1

Putting $Tx_0 = x_1$, $Ty_0 = y_1$, $Tx_1 = Ty_1 = Tz = z$, we have

$$d(Tx_0, Ty_0) = 2, \qquad r(x_0, y_0) = 2,$$
$$d(Tx_0, Tz) = d(Ty_0, Tz) = 1, \qquad r(x_0, z) = r(y_0, z) = \sqrt{5},$$
$$d(Tx_0, Tx_1) = d(Ty_0, Ty_1) = 1, \qquad r(x_0, x_1) = r(y_0, y_1) = 2,$$
$$d(Tx_0, Ty_1) = d(Ty_0, Tx_1) = 1, \qquad r(x_0, y_1) = r(y_0, x_1) = 2\sqrt{2},$$
$$c(Tx_1, Tz) = d(Ty_1, Tz) = 0, \qquad r(x_1, z) = r(y_1, z) = 1,$$
$$d(Tx_1, Ty_1) = 0. \qquad r(x_1, y_1) = 2.$$

It is easy to verify that $T \in \mathcal{C}$ since if $\varepsilon > 2$ then δ may be chosen arbitrarily, if $1 < \varepsilon \leqslant 2$ then $\delta < 2\sqrt{2} - \varepsilon$ and finally, if $0 < \varepsilon \leqslant 1$ then $\delta < 2 - \varepsilon$.

On the other hand, $T \in \mathcal{M}$ since it does not satisfy the following necessary condition

$$d(Tx, Ty) < d(x, y), \quad (x \neq y).$$

The proof of Theorem 1 is completed.

B — A FIXED POINT THEOREM FOR SPACES WITH A FAMILY OF PSEUDO-METRICS

Now, let $(X, \mathcal{D})$ be a space with a family of pseudometrics $\mathcal{D} = \{d_\alpha\}$. It is known that a space becomes such a space iff it is uniform [8]. A standard example of these spaces is a locally convex space. We assume that the family $\mathcal{D}$ has the following property

$$d_\alpha(x, y) = 0 \ (\forall d_\alpha \in \mathcal{D}) \Rightarrow x = y.$$

Then $(X, \mathcal{D})$ becomes a Hausdorff space. This class of spaces includes the class of probabilistic metric spaces [5], which in turn contains the class of metric spaces [17].

Let T be a mapping in X. Denote

$$r_\alpha(x, y) = max\{d_\alpha(x, y), d_\alpha(x, Tx), d_\alpha(y, Ty), \frac{1}{2}[d_\alpha(x, Ty) + d_\alpha(y, Tx)]\}.$$

Theorem 2. Let $(X, \mathcal{D})$ be a complete space with a family of pseudometrics and let T be a continuous mapping in X with the property: $\forall \varepsilon > 0$, $\forall d_\alpha \in \mathcal{D}$, $\exists \delta_\alpha = \delta_\alpha(\varepsilon) > 0$ such that

$$r_\alpha(x, y) < \varepsilon + \delta_\alpha \Rightarrow d_\alpha(Tx, Ty) < \varepsilon.$$

Then T has a unique fixed point x^* and $T^n x \to x^*$ $(\forall x \in X)$.

Proof. Let x_0 be an arbitrary element of X we construct the sequence $x_{n+1} = Tx_n$ $(n = 0, 1, 2, \ldots)$. In view of Theorem 1.1 in [21] which still holds for pseudometrics this sequence is Cauchy for each $d_\alpha \in \mathcal{D}$. Hence, by definition, it is Cauchy in X and, since X is complete, it converges to $x^* \in X$. Since T is continuous, $Tx^* = x^*$. Finally x^* is unique by the separateness of X. The proof of Theorem 2 is completed.

When $\mathcal{D} = \{d\}$, d is a unique metric in X we obtain Theorem 1.1 in [21] which generalizes the results in [2, 3, 4, 10, 12, 15].

Corollary 1. Let $(X, \mathcal{D})$ be a complete space with a family of pseudometrics and let T be a mapping in X with the property: $\forall \varepsilon > 0$ $\forall d_\alpha \in \mathcal{D}$, $\exists \delta > 0$ such that

$$\varepsilon \leqslant d_\alpha(x, y) < \varepsilon + \delta \Rightarrow d_\alpha(Tx, Ty) < \varepsilon.$$

Then T has a unique fixed point x^* and $T^n x \to x^*$ $(\forall x \in X)$.

Corollary 2 [5]. Let $(X, \mathcal{D})$ be a complete space with a family of pseudometrics and let T be a mapping with the property; $\forall d_\alpha \in \mathcal{D}$, $\exists k_\alpha \in (0, 1)$ such that

$$d_\alpha(Tx, Ty) \leqslant k_\alpha d_\alpha(x, y), \qquad (\forall x, y \in X).$$

Then T has a unique fixed point x^* and $T^n x \to x^*$ $(\forall x \in X)$.

This follows from Corollary 1 by setting $\delta_\alpha = \dfrac{\varepsilon - k_\alpha \varepsilon}{k_\alpha}$. In turn, Corollary 2 implies the following fact.

Corollary 3 [17]. Let $(X, \mathcal{F}, \Delta)$ be a complete Menger space, where Δ is a continuous function satisfying $\Delta(a, a) \geqslant a$ $(\forall a \in [0,1])$. Let T be a mapping in X with the property: $\exists k \in (0,1)$ such that

$$F_{Tx\,Ty}(kt) \geqslant F_{xy}(t). \quad (\forall x, y \in X: \ \forall t > 0).$$

Then T has a unique fixed point x^* and $T^n x \to x^*$ $(\forall x \in X)$.

Note that from Theorem 2 one can derive many results in probabilistic metric spaces for the class of mappings mentioned in Theorem 1. For example, it is possible to formulate a theorem analogous to Theorem 2.1 in [6].

II – MULTIVALUED GENERALIZED CONTRACTIONS

In this part X denotes a metric space, $CL(X)$ - the family of all nonempty closed subsets of X, D – the Hausdorff distance in $CL(X)$.

Theorem 3. Let X be a complete metric space, and let S, T be two mappings from X into $CL(X)$. Assume that there exist non-decreasing functions $\alpha_i : [0, \infty) \to [0,1)$, $i = 1,\dots,5$ such that

$$\alpha_1(t) + \alpha_2(t) + 2\alpha_3(t) + \alpha_5(t) < 1, \quad \alpha_1(t) + \alpha_2(t) + 2\alpha_4(t) + \alpha_5(t) < 1,$$

$(\forall t \geqslant 0)$ and for any $x, y \in X$

$$D(Sx, Ty) \leqslant a_1 d(x, Sx) + a_2 d(y, Ty) + a_3 d(x, Ty) + a_4 d(y, Sx) + a_5 d(x, y), \qquad (10)$$

where $a_i = \alpha_i(d(x, y))$, $i = 1,\dots,5$.

Then the fixed point set of S, and the fixed point set of T are both nonempty, and they coincide.

Proof.

Let $x_0 \in X$, $r > d(x_0, Sx_0)$. There exists $x_1 \in Sx_0$ such that $d(x_0, x_1) < r$. Denote $t_0 = d(x_0, x_1)$, from (10) we get

$$d(x_1, Tx_1) \leqslant D(Sx_0, Tx_1) \leqslant \alpha_1(t_0)\, d(x_0, Sx_0) + \alpha_2(t_0)\, d(x_1, Tx_1) + \alpha_3(t_0)\, d(x_0, Tx_1) + \alpha_5(t_0)\, d(x_0, x_1).$$

Since $x_1 \in Sx_0$ and $d(x_0, Tx_1) \leqslant d(x_0, x_1) + d(x_1, Tx_1)$ we have

$$d(x_1, Tx_1) \leqslant M d(x_0, x_1),$$

where

$$M = \frac{\alpha_1(t_0) + \alpha_3(t_0) + \alpha_5(t_0)}{1 - \alpha_2(t_0) - \alpha_3(t_0)} < 1.$$

If $M = 0$ or $d(x_0, x_1) = 0$ then $x_1 \in Tx_1$. Otherwise we have

$$d(x_1, Tx_1) < \min\{d(x_0, x_1), Mr\}.$$

Then there exists $x_2 \in Tx_1$ such that

$$d(x_1, x_2) < \min\{d(x_0, x_1), Mr\}.$$

Analogously, there exists $x_3 \in Sx_2$ such that

$$d(x_2, x_3) < \min\{d(x_1, x_2), NMr\},$$

where

$$N = \frac{\alpha_2(t_1) + \alpha_4(t_1) + \alpha_5(t_1)}{1 - \alpha_1(t_1) - \alpha_4(t_1)}, \; t_1 = d(x_1, x_2).$$

By induction we obtain a sequence $\{x_n\}$ with the following properties:

(i) $$x_n \in \begin{cases} Tx_{n-1} & \text{if } n \text{ is even}, \\ Sx_{n-1} & \text{if } n \text{ is odd}. \end{cases}$$

(ii) $d(x_n, x_{n+1}) < \min\{d(x_{n-1}, x_n), c^n r\}$, where $c = \max\{M, N\}$.

Since $c < 1$, from (ii) it follows that $\{x_n\}$ is Cauchy. Since X is complete, $x_n \to x^* \in X$. We now show that $x^* \in Sx^*$. In fact,

$$d(x^*, Sx^*) \leqslant d(x^*, x_{2n}) + d(x_{2n}, Sx^*) \leqslant d(x^*, x_{2n}) + D(Tx_{2n-1}, Sx^*) \leqslant$$
$$\leqslant d(x^*, x_{2n}) + a_{1n} d(x^*, Sx^*) + a_{2n} d(x_{2n-1}, x_{2n}) + a_{3n} d(x^*, x_{2n}) +$$
$$+ a_{4n} d(x_{2n-1}, Sx^*) + a_{5n} d(x^*, x_{2n-1}),$$

where $a_{in} = \alpha_i(d(x_{2n-1}, x^*))$, $i = 1, \dots, 5$.

Since $x_n \to x^*$, for n large enough we have

$$d(x^*, Sx^*) \leqslant d(x^*, x_{2n}) + \alpha_1(1)\, d(x^*, Sx^*) + \alpha_2(1)\, d(x_{2n-1}, x_{2n}) +$$
$$+ \alpha_3(1)\, d(x^*, x_{2n}) + \alpha_4(1)\, d(x_{2n-1}, Sx^*) + \alpha_5(1)\, d(x_{2n-1}, x^*).$$

Letting $n \to \infty$ we get

$$d(x^*, Sx^*) \leqslant [\alpha_1(1) + \alpha_4(1)]\, d(x^*, Sx^*),$$

hence $x^* \in Sx^*$.

Now we show that $x \in Sx$ iff $x \in Tx$. In fact, let $x \in Sx$. From (10) we have

$$d(x, Tx) \leqslant D(Sx, Tx) \leqslant [\alpha_2(0) + \alpha_3(0)]\, d(x, Tx),$$

i.e $x \in Tx$. Similarly, $x \in Tx$ implies $x \in Sx$. The proof of Theorem 3 is complete.

Corollary. Let X be a complete metric space, and let T be a mapping from X into $GL(X)$. Assume there exist nondecreasing functions $\alpha_i : (0, \infty) \to (0, 1)$ such that $\sum_{i=1}^{5} \alpha_i(t) < 1$, $(\forall t > 0)$ and for any $x, y \in X$, $x \neq y$

$$D(Tx, Ty) \leqslant a_1 d(x, Tx)+a_2 d(y, Ty)+a_3 d(x, Ty)+a_4 d(y, Tx)+a_5 d(x, y), \quad (11)$$

where $a_i = \alpha_i(d(x, y))$, $\quad i = 1, \dots, 5$.

Then T has a fixed point. If, in addition, T is singlevalued then this fixed point is unique.

Proof. Define

$$\beta_1 = \beta_2 = \frac{\alpha_1+\alpha_2}{2}, \quad \beta_3 = \beta_4 = \frac{\alpha_3+\alpha_4}{2}, \quad \beta_5 = \alpha_5,$$

and put $\beta_i(0) = 0$, $i = 1, \dots, 5$. It is easy to see that (10) holds for $S = T$ and $a_i = \beta_i(d(x, y))$. Since β_i are nondecreasing and

$$\beta_1(t)+\beta_2(t)+2\beta_3(t)+\beta_5(t) = \beta_1(t)+\beta_2(t)+2\beta_4(t)+\beta_5(t) = \sum_{i=1}^{5} \alpha_i(t) < 1$$

$(\forall t \geqslant 0)$, it suffices to apply Theorem 3.

Let T be singlevalued and x^*, y^*-its fixed points. Then from (11) we obtain

$$d(x^*, y^*) = d(Tx^*, Ty^*) \leqslant (a_3^* + a_4^* + a_5^*) d(x^*, y^*),$$

where $a_i^* = \alpha_i(d(x^*, y^*))$, a contradiction. This completes the proof of the Corollary.

Note that this corollary is related with a result of Alesina, Massa, Roux [1]. For singlevalued mappings Wong [24] has proved an analogous Theorem, where α_i were assumed upper semicontinuous from the right.

Theorem 4. Let X be a complete metric space, and let S, T be two mappings from X into $CL(X)$. Assume, there exists a nondecreasing function $\alpha : [0, \infty) \to [0, 1)$ such that for any $x, y \in X$

$$D(Sx, Ty) \leqslant \alpha(d(x, y)) r(x, y), \quad (12)$$

where $r(x, y)$ is defined as in Part I.

Then the conclusion of Theorem 3 holds.

Proof. Let $x_0 \in X$, $r > d(x_0, Sx_0)$. There exists $x_1 \in Sx_0$ such that $d(x_0, x_1) = t_0 < r$. From (12) we get

$$d(x_1, Tx_1) \leqslant D(Sx_0, Tx_1) \leqslant \alpha(t_0) \max \left\{ d(x_0, x_1), d(x_1, Tx_1), \frac{1}{2} d(x_0, Tx_1) \right\} \leqslant$$

$$\leqslant \alpha(t_0) \max \left\{ d(x_0, x_1), d(x_1, Tx_1), \frac{1}{2} [d(x_0, x_1)+d(x_1, Tx_1)] \right\} = \alpha(t_0) d(x_0, x_1).$$

Hence

$$d(x_1, Tx_1) < \min \{ d(x_0, x_1), \alpha(t_0) r \}.$$

There exists $x_2 \in Tx_1$ so that

$$d(x_1, x_2) < \min \{ d(x_0, x_1), \alpha(t_0) r \}.$$

By induction we get a sequence $\{x_n\}$ with the properties (i) (ii) mentioned in the proof of Theorem 3, where $c = \alpha(t_0)$. Thus, $\{x_n\}$ is Cauchy, so $x_n \to x^* \in X$.

We now show that $x^* \in Sx^*$. Indeed, we have

$$d(x^*, Sx^*) < d(x^*, x_{2n}) + D(Tx_{2n-1}, Sx^*) \leqslant d(x^*, x_{2n}) + \alpha(d(x_{2n-1}, x^*)) \max \{d(x_{2n-1}, x^*), d(x_{2n-1}, x_{2n}), d(x^*, Sx^*), \frac{1}{2}[d(x_{2n-1}, x^*) + d(x^*, Sx^*) + d(x^*, x_{2n})]\}.$$

Assume that $d(x^*, Sx^*) = 2\varepsilon > 0$. For n large enough we have

$$\max \{d(x_{2n-1}, x^*), d(x_{2n-1}, x_{2n}), d(x^*, x_{2n})\} < \varepsilon.$$

Now we obtain

$$d(x^*, Sx^*) \leqslant d(x^*, x_{2n}) + \alpha(\varepsilon)\, d(x^*, Sx^*).$$

Letting $n \to \infty$ we get a contradiction:

$$d(x^*, Sx^*) \leqslant \alpha(\varepsilon)\, d(x^*, Sx^*) < d(x^*, Sx^*).$$

Thus, $x^* \in Sx^*$. As in the proof of Theorem 3, it is easy to show that the two fixed point sets of S and T coincide. This completes the proof of Theorem 4.

Theorem 5. Let X be a complete metric space and T be a continuous mapping in X. Assume that there exists a upper semicontinuous from the right function $\alpha : (0, \infty) \to [0,1)$ such that for any $x, y \in X$, $x \neq y$

$$d(Tx, Ty) \leqslant \alpha(d(x, y))\, r(x, y). \tag{13}$$

Then T has a unique fixed point x^* and $T^n x \to x^*$ $(\forall x \in X)$.

Proof. Let $x_0 \in X$, $x_{n+1} = Tx_n$ $(n = 0, 1, 2, \ldots)$. We may assume that $x_{n+1} \neq x_n$ $(\forall n)$. From (13) we have

$$d(x_{n+1}, x_{n+2}) \leqslant \alpha_n d(x_n, x_{n+1}). \tag{14}$$

where $\alpha_n = \alpha(d(x_n, x_{n+1}))$. Denoting $c_n = d(x_n, x_{n+1})$ we obtain a decreasing sequence which converges to $p \geqslant 0$.

If $p > 0$ then $\alpha_n \leqslant \alpha(p) + \varepsilon < 1$ for n large enough by the assumption on α. From this and (14) we get $p = 0$.

Using the method of Wong [24] we now prove that $\{x_n\}$ is Cauchy. Assume the contrary, that $\exists \varepsilon > 0\ \forall n\ \exists p_n > q_n > n$ such that

$$d(x_{p_n}, x_{q_n}) \geqslant \varepsilon, \quad d(x_{p_n-1}, x_{q_n}) < \varepsilon.$$

Denoting $b_n = d(x_{p_n}, x_{q_n})$ we obtain

$$\varepsilon \leqslant b_n \leqslant d(x_{p_n}, x_{p_n-1}) + d(x_{p_n-1}, x_{q_n}) < d(x_{p_n}, x_{p_n-1}) + \varepsilon.$$

Hence the sequence b_n converges to ε from the right. From (13) we obtain

$$d(x_{p_n+1}, x_{q_n+1}) = d(Tx_{p_n}, Tx_{q_n}) \leqslant \alpha(b_n) \max \Big\{b_n, d(x_{p_n}, x_{p_n+1}), d(x_{q_n}, x_{q_n+1}), \frac{1}{2}\Big[d(x_{p_n}, x_{q_n+1}) + d(x_{q_n}, x_{p_n+1})\Big]\Big\} \leqslant \alpha(b_n)\Big[b_n + d(x_{p_n}, x_{p_n+1}) + d(x_{q_n}, x_{q_n+1})\Big]$$

Letting $n \to \infty$ we obtain a contradiction: $\varepsilon \leqslant \alpha(\varepsilon)\varepsilon < \varepsilon$. Thus, $\{x_n\}$ is Cauchy. The rest of the proof is obvious.

Note that this theorem generalized a result of Boyd and Wong [3].

In the sequel $B = B(x_0, R)$ denotes the open ball in X with center at x_0 and radius R. The two following local forms of the contraction principle generalize the results of Nadler [12] and Reich [16].

Theorem 6. Let X be a complete metric space and let T be a mapping from B into $CL(X)$. Assume that

1) There exist non decreasing functions $\alpha_i = (0, \infty) \to [0, 1)$, $i = 1, \ldots, 5$, such that $\sum \alpha_i(t) < 1$ $(\forall t > 0)$ and (11) holds for any $x, y \in B$, $x \neq y$,

2) $d(x_0, Tx_0) < (1 - c)R$ where

$$c = \frac{\alpha_1(s) + \alpha_2(s) + \alpha_3(s) + \alpha_4(s) + 2\alpha_5(s)}{2 - \alpha_1(s) - \alpha_2(s) - \alpha_3(s) - \alpha_4(s)} \, .$$

$$s = \sup \{ d(x_0, x) : x \in Tx_0 \cap B \}.$$

Then for each r satisfying

$$d(x_0, Tx_0) < r < (1 - c)R$$

there exists a fixed point x^* of T such that

$$d(x_0, x^*) \leqslant \frac{r}{1-c} \, .$$

Proof. Denoting

$$\beta_1 = \beta_2 = \frac{\alpha_1 + \alpha_2}{2}, \quad \beta_3 = \beta_4 = \frac{\alpha_3 + \alpha_4}{2}, \quad \beta_5 = \alpha_5,$$

we have

$$c = \frac{\beta_1(s) + \beta_3(s) + \beta_5(s)}{1 - \beta_2(s) - \beta_3(s)} = \frac{\beta_2(s) + \beta_4(s) + \beta_5(s)}{1 - \beta_1(s) - \beta_4(s)} \, .$$

Repeating the proof of Theorem 3 we obtain a sequence $\{x_n\}$ with the poperties (i), (ii). Noting that $s \geqslant d(x_0, x_1)$ and that the condition (ii) implies

$$d(x_0, x_n) \leqslant \frac{r}{1-c} \, ,$$

we have $\{x_n\} \subset B$. The rest of the proof is clear.

Theorem 7. The conclusion of Theorem 6 still holds if condition 1) is replaced by

1) There exists a nondecreasing function $\alpha: (0, \infty) \to [0, 1)$ such that

$$D(Tx, Ty) \leqslant \alpha(d(x, y))r(x, y)$$

for any $x, y \in B$, $x \neq y$, and if in condition 2) $c = \alpha(s)$.

Proof. The proof of this theorem is analogous to the proof of Theorems 4 and 6.

III – A FIXED POINT THEOREM OF KRASNOSELSKI TYPE.

Krasnoselski [9] has proved a fixed point theorem for mappings of the form $T+S$, where $T \in \mathcal{B}$ and S is a compact mapping in a Banach spase. Recently, Sehgal and Singh [18] extended this result to the case in which T satisfies (1) with a continuous function α. The following fact shows that these results can be both derived from a theorem of Sadovski [19] on condensing mappings. Recall that a mapping T is called condensing if for each bounded nonprecompact subset A we have $\Psi(TA) < \Psi(A)$, where Ψ is the Kuratowski or the Hausdorff measure of noncompactness.

Proposition 1. Let a mapping T satisfy (1) with a upper semi-continuous function α. Then T is condensing.

Proof. Let Ψ be the Kuratowski measure of noncompactness, and $\Psi(A) = a$. There exists $\eta > 0$ such that $\alpha(a) \leqslant 1 - 2\eta$. By the upper semi-continuity of α there is $\varepsilon_0 > 0$ such that

$$\alpha(t) \leqslant \alpha(a) + \eta, \quad (a - \varepsilon_0 \leqslant t \leqslant a + \varepsilon_0).$$

We may assume $\varepsilon_0 < a$.

Let ε be arbitrary in $(0, \varepsilon_0)$. By the definition of Ψ there is a cover $\{A_1, \dots, A_n\}$ of A such that $\delta(A_i) \leqslant a + \varepsilon$, $i = 1, \dots, n$, where $\delta(.)$ denotes the diameter of a set.

We shall prove that there is a number $q < 1$ such that $\Psi(TA) \leqslant q(a + \varepsilon)$. Fix $i \in \{1, \dots, n\}$ and let x, y be arbitrary in A_i. If $a - \varepsilon_0 \leqslant d(x, y) \leqslant a + \varepsilon$ then $d(Tx, Ty) \leqslant (1 - \eta)(a + \varepsilon)$ since $\alpha(d(x, y)) \leqslant \alpha(a) + \eta \leqslant 1 - \eta$. If $d(x, y) < a - \varepsilon_0$ then $d(Tx, Ty) \leqslant \left(1 - \frac{\varepsilon_0}{2a}\right)(a + \varepsilon)$ since $\varepsilon < a$. Setting $q = \max\left\{1 - \eta, 1 - \frac{\varepsilon_0}{2a}\right\}$ we obtain

$$\delta(TA_i) \leqslant q(a + \varepsilon), \quad (i = 1, \dots, n).$$

Since $\{TA_1, \dots, TA_n\}$ covers TA we get $\Psi(TA) \leqslant q(a + \varepsilon)$. Since ε is arbitrary, $\Psi(TA) \leqslant qa < \Psi(A)$. The argument is analogous for the Hausdorff measure of noncompactness and so the proof of the proposition 1 is completed.

Now, following the method of Krasnoselski [10] we shall prove a fixed point theorem for mappings of the form $T + S$, where $T \in \mathcal{M}$.

Lemma 4. Let X be a Banach space, D be an open set in X, $T: D \to X$ be a mapping of the class $\mathcal{M}$. Then $I - T$ is a homeomorphism from D onto $(I - T)D$.

Proof. Let x_0 be an arbitrary element in D. There is $\varepsilon > 0$ such that $S(x_0, 2\varepsilon) \subset D$, where $S(x_0, \rho)$ is the closed ball with center at x_0 and radius ρ. Put $f_0 = x_0 - Tx_0$ and define the mapping $F_0 x = Tx + f_0 (\forall x \in D)$.

Since $T \in \mathcal{M}$ there exists $\delta > 0$ such that

$$\| x - y \| < \varepsilon + \delta \Rightarrow \| Tx - Ty \| < \varepsilon.$$

Put $\delta' = \min\{\varepsilon, \delta\}$, $r \in (\varepsilon, \varepsilon + \delta')$, $x \in S(x_0, r)$. Then, since $\| x - x_0 \| < \varepsilon + \delta$ and $T \in \mathcal{M}$ we have

$$\| F_0 x - x_0 \| = \| F_0 x - F_0 x_0 \| = \| Tx - Tx_0 \| < \varepsilon.$$

Thus, F_0 maps $S(x_0, r)$ into $S(x_0, \varepsilon)$. If $\| f - f_0 \| \leqslant r - \varepsilon$ then. $Fx = Tx + f$ maps $S(x_0, r)$ into itself. Since $F \in \mathcal{M}$ there exists a unique $x \in S(x_0, r)$ such that $x = Tx + f$, i.e. $(I - T)x = f$. This shows that the mapping $(I - T)^{-1}$ exists. and is continuous on $(I - T)D$. This concludes the proof of Lemma 4.

Let $\{T(\lambda) \colon \lambda \in [0,1]\}$ be a family of mappings in $\mathcal{M}$. Assume that $T(\lambda)x$ is continuous in λ for each fixed x and that the number δ mentioned in Definition 5 does not depend on λ. In this case the family $\{T(\lambda)\}$ is called uniform.

Lemma 4 shows that $[I - T(\lambda)]D$ is open and that for each $\lambda \in [0, 1]$ and $f \in [I - T(\lambda)]D$ the equation $x = T(\lambda)x + f$ has a unique solution $x = R(\lambda, f)$.

Lemma 5. Let $\{T(\lambda)\}$ be uniform. Then the mapping $R(\lambda, f)$ is continuous.

Proof. Let $x_0 \in D$ and let $\varepsilon > 0$ be such that $S(x_0, 2\varepsilon) \subset D$. Choose $\delta > 0$ so that $\| x - y \| < \varepsilon + \delta \Rightarrow \| T(\lambda)x - T(\lambda)y \| < \varepsilon$, $(\forall \lambda \in [0, 1])$. Put $\delta' = \min\{\varepsilon, \delta\}$, $r \in (\varepsilon, \varepsilon + \delta')$, $\lambda_0 \in [0, 1]$. Define $f_0 = x_0 - T(\lambda_0)x_0$ and $F_0 x = T(\lambda_0)x + f_0$. It is easy to see that F_0 maps $S(x_0, r)$ into $S(x_0, \varepsilon)$. Choose $\eta > 0$ so that $\eta \leqslant \frac{r - \varepsilon}{2}$ and

$$| \lambda - \lambda_0 | \leqslant \eta \Rightarrow \| T(\lambda)x_0 - T(\lambda_0)x_0 \| \leqslant \frac{r - \varepsilon}{2}.$$

Let (λ, f) be an arbitrary point in the η-neighbourhood of (λ_0, f_0) and define $Fx = T(\lambda)x + f$. Since for each x in $S(x_0, r)$ we have

$$\| Fx - x_0 \| = \| T(\lambda)x + f - T(\lambda_0)x_0 - f_0 \| \leqslant \| T(\lambda)x - T(\lambda)x_0 \| + \\ + \| T(\lambda)x_0 - T(\lambda_0)x_0 \| + \| f - f_0 \| \leqslant \varepsilon + \frac{r - \varepsilon}{2} + \frac{r - \varepsilon}{2} = r,$$

F maps $S(x_0, r)$ into itself. Sine $F \in \mathcal{M}$ the equation $x = T(\lambda)x + f$ has a unique solution in $S(x_0, r)$. Thus, $R(\lambda, f)$ is continuous, completing the proof of Lemma 5.

Theorem 8. Let D be an open bounded convex set in a Banach space X and let T, S be two mappings from $\overline{D}$ (the closure of D) into X, Assume $T \in \mathcal{M}$, S is compact and such that $(T + S)x \in \overline{D}$ for every x in ∂D (the boundary of D) Then $T + S$ has a fixed point in $\overline{D}$.

Proof. Consider the vector field $\Phi x = x - Tx - Sx$ on $\overline{D}$ and put $F(u, v) = u - Tu - Sv$ $(u, v \in \overline{D})$. Obviously Φ is F-partial invertible [10]. We may assume that Φ is nondegenerate on δD and hence a quasidegree $\gamma^*(\Phi, F; D)$ is defined. We must only show that $\gamma^*(\Phi, F; D) \neq 0$.

Consider the family of vector fields

$$\Phi(\lambda)x = x - \lambda Tx - \lambda Sx - (1-\lambda)x_0, \ (0 \leqslant \lambda \leqslant 1, \ x \in \overline{D}),$$

where x_0 is a fixed arbitrary point in D. In view of the boundary condition $T + S)\partial D \subset \overline{D}$, $\Phi(\lambda)$ is nondegenerate on ∂D for $0 \leqslant \lambda < 1$. By assumption the field $\Phi(1) = \Phi$ is nondegenerate too. The family $\{\lambda T\}$ is uniform and the mapping $S(\lambda, x) = \lambda Sx + (1-\lambda)x_0$ is compact. Thus, $\Phi(\lambda)$ defines a quasi-homotopy from $\Phi(0)x = x - x_0$ to $\Phi(1)x = \Phi x$. From this $\Upsilon^*(\Phi, F; D) = 1$, which concludes the proof of Theorem 8.

Note that the mentioned boundary condition can be replaced by a weaker one: there exists $x_0 \in D$ such that $(T + S)x - x_0 \neq c(x - x_0)$ for every $x \in \partial D$ and $c > 1$.

IV — NONEXPANSIVE MAPPINGS IN LOCALLY CONVEX SPACES

Let X be a locally convex space with a family of prenorms $\mathcal{P}$ generating the topology in X.

Definition 8. [11, 22] A mapping $T: X \to X$ is called strictly contractive if $\forall p \in \mathcal{P} \ \exists k = k(p) \in [0, 1)$ such that

$$p(Tx - Ty) \leqslant kp(x - y), \quad \forall x, y \in X.$$

or nonexpansive if $\forall p \in \mathcal{P}, \forall x, y \in X$

$$p(Tx - Ty) \leqslant p(x - y).$$

Recall that a set S in a vector space X is called star-shaped at $\overline{x} \in S$ if

$$\lambda x + (1-\lambda)\overline{x} \in S, \ (\forall \lambda \in [0, 1], \ x \in S).$$

Lemma 6. Let X be a complete Hausdorff locally convex space, S be a closed bounded star-shaped set in X, $T: S \to X$ be a nonexpansive mapping satisfying the boundary condition $T(\partial S) \subset S$. Then there exists a sequence $\{x_n\} \subset S$ such that $x_n - Tx_n \to 0$.

Proof. Let $\{k_n\}$ be a sequence of numbers tending to 1 $(n \to \infty)$. Define

$$T_n x = k_n Tx + (1 - k_n)\overline{x}, \ x \in S,$$

where $\overline{x}$ is the point at which S is star-shaped.

Obviously T_n is strictly contractive and $T_n(\partial S) \subset S$ for each n, hence, by a recent result of Sehgal and Singh [18] each T_n has a fixed point x_n. Then

$$x_n = k_n Tx_n + (1 - k_n)\overline{x}.$$

Consequently,

$$x_n - Tx_n = \left(\frac{1}{k_n} - 1\right)(\overline{x} - x_n), \ \forall n,$$

whence $\forall p \in \mathcal{P}$

$$p(x_n - Tx_n) = \left(\frac{1}{k_n} - 1\right)p(\overline{x} - x_n), \ \forall n.$$

Since S is bounded, we get

$$p(x_n - Tx_n) \to 0, \qquad \forall p \in \mathcal{P},$$

as $n \to \infty$. i. e $x_n - Tx_n \to 0$. The proof of Lemma 6 is completed.

Theorem 9 . Let X be a complete Hausdorff loeally convex space, let S be a compact star-shaped set of X, let $T: S \to X$ be a nonexpansive mapping satisfying the boundary condition: $T(\partial S) \subset S$. Then T has a fixed point.

Proof.

By Lemma 6 there exists a sequence $\{x_n\}$ such that $x_n - Tx_n \to 0$. By the compactness of S we may assume that $x_n \to x^* \in S$. Since T is continuous, $Tx_n \to Tx^*$. Hence $x^* = Tx^*$, which concludes the proof of Theorem 9.

Nole that this theorem generalizes a result of Taylor [22].

Definition 9. We say that a locally convex space X satisfies the Opial condition if for every x in X and every net $\{x_\nu\}$ weakly converging to x there exists $p \in \mathcal{P}$ such that

$$\underline{\lim}_{\nu}\, p(x_\nu - y) > \underline{\lim}_{\nu}\, p(x_\nu - x)$$

for each $y \neq x$.

Theorem 10. Let X be a complete Hausdorff locally convex space satisfying the Opial condition, let S be a weakly compact star-shaped set in X, let $T: S \to X$ be a nonexpansive mapping satisfying the boundary condition: $T(\partial S) \subset S$. Then T has a fixed point.

Proof.

By Lemma 6 there exists a sequence $\{x_n\}$ such that $z_n = x_n - Tx_n \to 0$. By the weak compactness of S. there exists a subnet $\{x_\nu\}$ of $\{x_n\}$ which weakly converges to $x^* \in S$. Since T is nonexpansive, $\forall p \in \mathcal{P}$ we have

$$p(Tx_\nu - Tx^*) \leqslant p(x_\nu - x^*).$$

Hence, $\forall p \in \mathcal{P}$

$$\underline{\lim}_{\nu}\, p(x_\nu - x^*) \geqslant \underline{\lim}_{\nu}\, p(Tx_\nu - Tx^*) \geqslant \underline{\lim}_{\nu}\, p(x_\nu - z_\nu - Tx^*).$$

Since $z_\nu \to 0$ we have

$$\underline{\lim}_{\nu}\, p(x_\nu - x^*) \geqslant \underline{\lim}_{\nu}\, p(x_\nu - Tx^*), \quad \forall p \in \mathcal{P}.$$

Consequently $Tx^* = x^*$ because X satisfies the Opial condition. The proof of Theorem 10 is completed.

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