A SUPPORT PRINCIPLE FOR A DISCRETE INCLUSION WITH A VECTOR-VALUED CRITERION FUNCTION

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It has been shown by Boltianskii [1, 2] that, for a process described by a differential or discrete inclusion, a point which achieves a minimum of a given scalar-valued criterion function will satisfy the support principle [1, 2], if the process under consideration has local sections [1, 2]. The case where local sections do not exist has been studied by the author in [3]. It should be noted that the papers [1 - 3] deal with scalar minimization problems. In this paper, we shall prove the support principle for a discrete inclusion with a vector-valued criterion function. From the obtained results, it will be easy to derive the support principle for the case where local sections exist as well as for the case where they do not. Also, the discrete minimax problem [2] is included as a special case in our theorem. Our proof of the support principle is based on a general theory of inconsistency of a system of inclusions which is presented (without proofs) in the paragraph §2. It will be easy to see that our method can also be applied to discrete time-lag systems, and discrete distributed parameter systems [3].

§1. NOTATIONS AND DEFINITIONS

Let $X$, $Y$ be two vector topological spaces; $A$, $B$ two subsets of $X$; $T$ set-valued mapping carrying points of $A$ into non-empty subsets of $Y$, and $k$ a positive integer. The following notations will be used:

$\mathbb{R}^k$: $k$-dimensional Euclidean space.

$\mathbb{R}^k_+ = \{ x = (x_1, x_2, \ldots, x_k) \in \mathbb{R}^k : x_i \geq 0, i = 1, 2, \ldots, k \}$: nonnegative orthant in $\mathbb{R}^k$.

$\hat{\mathbb{R}}^k_+ = \{ x = (x_1, x_2, \ldots, x_k) \in \mathbb{R}^k : x_i > 0, i = 1, 2, \ldots, k \}$: positive orthant in $\mathbb{R}^k$. 
\( \mathcal{X}^* \): set of all linear continuous functionals defined on \( \mathcal{X} \).

\( (x^*, x) \): value of \( x^* \in \mathcal{X}^* \) at the point \( x \in \mathcal{X} \).

\( 2^\mathcal{X} \): set of all non-empty subsets of \( \mathcal{X} \).

\( V_x, V_x^1, ... \): neighborhoods of \( 0 \in \mathcal{X} \).

\( \text{int } A \): interior of \( A \).

\( A \ast B = \{ x \in \mathcal{X} : x + A \subset B \} \) (1).

\( T(A) = \bigcup_{x \in A} T(x) \): range of \( T \).

\( c^T(y^*, x) = \sup \{(y^*, y) : y \in T(x)\} \) \((y^* \in Y^*, x \in A)\).

\( \text{graph } T = \{(x, y) : x \in A, y \in T(x)\} \): graph of \( T \).

\( \text{int } T \): mapping \( x \mapsto \text{int } T(x) \) \((x \in A)\).

If \( A = A' \times \Omega \), the direct product then, for fixed \( \omega \in \Omega \), \( T(\cdot, \omega) \) is a mapping defined by \( x' \mapsto T(x', \omega) \) \((x' \in A')\).

Given \( k \) set-valued mappings \( T_i : D \to 2^Y \), \( i = 1, \ldots, k \), we denote by \( \prod_{i=1}^k T_i \) the mapping

\[
x \mapsto \prod_{i=1}^k T_i(x) = T_1(x) \times T_2(x) \times \cdots \times T_k(x)
\]

**Definition 1.1.** Given a set \( I \) of indices and mappings \( T_\varepsilon : A \to 2^Y \), \( \varepsilon \in I \), we shall say that the family \( \{T_\varepsilon, \varepsilon \in I\} \) is uniformly upper semi-continuous (uniformly lower semi-continuous) if for every \( \widehat{x} \in A \) and every neighborhood \( V_Y \) there exists a neighborhood \( V_x \) such that

\[
T_\varepsilon(x) \subset T_\varepsilon(\widehat{x}) + V_Y
\]

\[
(T_\varepsilon(\widehat{x}) \subset T_\varepsilon(x) + V_Y)
\]

whenever \( \varepsilon \in I \), \( x \in (\widehat{x} + V_x) \cap A \).

The family \( \{T_\varepsilon, \varepsilon \in I\} \) is said to be uniformly continuous, if it is both uniformly upper and uniformly lower semi-continuous.

**Definition 1.2.** A mapping \( T : A \to 2^Y \) is upper semi-continuous if the family \( \{T_\varepsilon\} \) consisting of a single mapping \( T_\varepsilon = T \) is uniformly upper semi-continuous. Similarly, one can define lower semi-continuity, as well as continuity, of \( T \).

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(1) The operation \( \ast \) was first introduced by Pontryagin and plays an important role in the theory of linear differential games.
Definition 1.3. A mapping $T : A \rightarrow 2^Y$ is closed if the conditions $x_n \rightarrow x (x_n, x \in D), y_n \rightarrow y$ and $y_n \in T(x_n)$ imply that $y \in T(x)$.

Definition 1.4. A mapping $T : A \rightarrow 2^Y$ is convex if its graph is a convex set in $X \times Y$.

§ 2. NECESSARY CONDITIONS FOR THE INCONSISTENCY OF A SYSTEM OF INCLUSIONS.

Let $X, Y_i, i = 1 - 3$, be vector topological spaces, $D, E$ subsets in $X$, $T_i : D \rightarrow 2^{Y_i}, t_i : E \rightarrow 2^{Y_i}$ set-valued mappings. Let there be given a point $x_o \in X$ and a single-valued mapping $g : X \times R^1_+ \times X \rightarrow X$.

Definition 2.1. The pair $(E, t_i)$ will be called a $(g, \_\_)$-tangent approximation at $x_o$ of the pair $(D, T_i)$ if there exists a positive number $a_1$ and mappings $Q : (0, a_1) \times D \times E \rightarrow 2^{Y_1}, f_i : (0, a_1) \rightarrow R^1_+$ such that

a) The family

$$\{Q_{\varepsilon, x}, (\varepsilon, x) \in I\}$$

with $Q_{\varepsilon, x} = Q(\varepsilon, x, \_)$, $I = (0, a_1) \times D$ is uniformly continuous.

b) For every $\varepsilon \in (0, a_1)$ the mapping $Q(\varepsilon, \_\_)$ is closed.

c) For every $(\varepsilon, x, \xi) \in (0, a_1) \times D \times E$ the set $Q(\varepsilon, x, \xi)$ is convex and satisfies the condition

$$Q(\varepsilon, x, \xi) \subset \frac{T_i(x)}{f_i(\varepsilon)} \ast t_i(\xi).$$

(2.2)

d) For every $\hat{x} \in E$ and every neighborhood $V_{Y_1}$ there exist a number $\delta \in (0, a_1)$ and a neighborhood $V_x$ such that

$$Q(\varepsilon, x, \hat{x}) \cap V_{Y_1} \neq \emptyset$$

(2.3)

whenever

$$0 < \varepsilon < \delta, \ x \in g(x_o, \varepsilon, \hat{x} + V_x) \cap D.$$

(2.4)

Example 2.1. Let $T_1, t_1$ be mappngs defined by

$$T_1(x) = F(x) + N, \ t_1(x) = f(x) + N$$

(2.5)

where $F : D \rightarrow Y_1, f : E \rightarrow Y_1$ are single-valued continuous mappings, $N$ is an arbitrary cone.

Assume that for every $\hat{x} \in E$ and every neighborhood $V_{Y_1}$ there exist a number $\delta > 0$ and a neighborhood $V_x$ such that

$$F(x) \in \varepsilon (f(\hat{x}) + V_{Y_1})$$

(2.6)
whenever

\[ \delta < \varepsilon < \delta, \quad x \in \left[ x_0 + \varepsilon (\hat{x} + V_X) \right] \cap D. \tag{2.7} \]

Then the pair \((E, t_1)\) is a \((g_1, \varepsilon)\)-tangent approximation at \(x_o\) of the pair \((D, T_1)\), where

\[ g_1(\varepsilon, x, \varepsilon) = \frac{1}{\varepsilon} \frac{1}{\varepsilon} F(x) - f(\hat{x}). \tag{2.8} \]

To prove this, it suffices to take \(f_1(\varepsilon) = \varepsilon, Q(\varepsilon, x, \varepsilon) = \frac{1}{\varepsilon} F(x) - f(\hat{x})\).

Note that Condition (2.6) holds, provided that \(X \) and \(Y_1\) are Banach spaces, \(F(x) = \hat{F}(x) - \hat{F}(x_0), f(x) = \hat{F}_x'(x)\), where \(\hat{F} : X \to Y_1\) is a Frechet differentiable mapping and \(\hat{F}_x'\), its Frechet derivative at \(x_o\).

**Example 2.2.** Let the mappings \(T_1, t_1\) be defined as in Example 2.1., but let \(N\) be a closed convex set. Assume that for every \(\hat{x} \in E\) and every neighborhood \(V_{Y_1}\) there exist a number \(\delta > 0\) and a neighborhood \(V_X\) such that

\[ \left[ \frac{1}{\varepsilon} (F(x) + (1 - \varepsilon) N) - f(\hat{x}) \right] \cap V_{Y_1} \neq \emptyset \tag{2.9} \]

for all \((\varepsilon, x)\) satisfying (2.7). Then the assertion in the previous example is true. To prove this, choose \(a_1 = 1, f_1(\varepsilon) = \varepsilon, Q(\varepsilon, x, \varepsilon) = \frac{1}{\varepsilon} (F(x) + (1 - \varepsilon) N) - f(\hat{x})\).

**Definition 2.2.** A pair \((E, t_2)\) will be called a \(g\)-tangent approximation at \(x_o\) of the pair \((D, T_2)\) if the following conditions are fulfilled:

a. The mapping \(T_2\) is closed, and for every \(x \in D\) the set \(T_2(x)\) is convex.

b. There exist a number \(a_2 > 0\) and a mapping \(f_2 : (0, a_2) \to \mathbb{R}_+\) such that for every \(\hat{x} \in E\) and every neighborhood \(V_{Y_2}\) there exist a number \(\delta \in (0, a_2)\) and a neighborhood \(V_X\) such that

\[ t_2(\hat{x}) < \frac{T_2(x)}{f_2(\varepsilon)} + V_{Y_2} \]

for all \((\varepsilon, x)\) satisfying the conditions (2.4).

**Remark 2.1.** Let \((E, t_1)\) be a \((g, \varepsilon)\)-tangent approximation at \(x_o\) of the pair \((D, T_1)\). Suppose that \(T_1\) is closed and, for every \(x \in D\), the set \(T_1(x)\) is convex. Then the pair \((E, t_1)\) is also a \(g\)-tangent approximation at \(x_o\) of the pair \((D, T_1)\).

**Example 2.3.** Let the mappings \(T_1\) and \(t_1\) be defined as in Example 2.2. Then the pair \((E, t_1)\) is a \(g_1\)-tangent approximation at \(x_o\) of the pair \((D, T_1)\).

**Definition 2.3.** (see [1,2]). Given an arbitrary mapping \(T : X \to 2^Y\), where \(X\) and \(Y\) are two normed spaces, we shall say that \(T\) has local sections if for
every pair \((x_o, y_o) \in \text{graph } T\) there exists a smooth (i) function \(\sigma\) from a neighborhood of \(x_o\) into \(Y\) such that \(\sigma(x_o) = y_o\) and \(\sigma(x) \in T(x)\). The function \(\sigma\) will be called a local section of \(T\), corresponding to the pair \((x_o, y_o)\).

**Example 2.4.** Let \(X, Y\) be normed spaces, \(T_2: X \to 2^Y\) a closed mapping such that

1. For every \(x\) the set \(T_2(x)\) is convex and compact.
2. \(0 \in T_2(x_o)\).
3. \(T_2\) has local sections.

Suppose \(\sigma\) is a local section of \(T_2\) corresponding to the pair \((x_o, 0)\) and \(\sigma_{x_o}'\) is its Frechet derivative at \(x_o\). It can be shown that the pair \((X, t_2)\), where \(t_2(x) = T_2(x) + \sigma_{x_o}'(x)\) is a \(g_1\)-tangent approximation at \(x_o\) of the pair \((X, T_2)\).

**Definition 2.4.** A pair \((E, t_3)\) will be called a \(g\)-interior approximation at \(x_o\) of the pair \((D, T_3)\) if for every solution \(\tilde{x}\) of the system

\[
x \in E, \quad 0 \in t_3(x)
\]

there exist a number \(\delta > 0\) and a neighborhood \(V_x\) such that

\[
0 \in T_3(x)
\]

for all \((\varepsilon, x)\) satisfying the conditions (2.4).

**Example 2.5.** Let \(X, Y\) be Banach spaces, \(N\) a set in \(Y_3\), \(h: X \times Y \to R\) a functional which is Frechet differentiable at \((x_o, y_o) \in X \times N\). Assume that

\[ h(x_o, y_o) < 0, \quad (h_x', h_y') \neq 0 \]

where \(h_x'\) and \(h_y'\) are the partial derivatives of \(f\) at \((x_o, y_o)\). Assume further that \(\tilde{D} \neq \phi, \tilde{N} \neq \phi\) where \(\tilde{D}\) (\(\tilde{N}\)) is the set of all \(x \in X(y \in Y_3)\) for each of which there exist a number \(\delta > 0\) and a neighborhood \(V_x(V_{y_3})\) such that \(x_o + \varepsilon(x + V_x) \subseteq D(y_o + \varepsilon(y + V_{y_3}) \subseteq N\) whenever \(0 < \varepsilon < \delta\).

Let us set

\[
E = \tilde{D},
\]

\[
T_3(x) = \{ y \in Y_3: h(x, y) < 0 \} - N,
\]

\[
t_3(x) = \begin{cases} Y_3 \\ - \tilde{N} + \{ y \in Y_3: h_x'(x) + h_y'(y) < 0 \} \end{cases} \quad \text{if } h(x_o, y_o) < 0
\]

Under these assumptions it is readily verified that the pair \((E, t_3)\) is a \(g_1\)-interior approximation at \(x_o\) of the pair \((D, T_3)\).

**Example 2.6.** Let \(X, Y\) be normed spaces, \(T_3: D \to 2^Y\) a lower semi-continuous mapping such that

1. For every \(x\), \(T_3(x)\) is a convex set with a non-empty interior.
2. \(0 \in T_3(x_o)\).
3. \(T_3\) has a local section \(\sigma\), corresponding to the pair \((x_o, 0)\).

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(1) i.e \(\sigma\) is continuous together with its Frechet derivative in a neighborhood of \(x_o\).
Let us set

\[ t_3(x) = \begin{cases} \text{int } T_3(x) + \sigma_{\chi_0}'(x) & \text{if } 0 \subseteq \text{int } T_3(x), \\ Y_3 & \text{if } 0 \subseteq \text{int } T_3(x), \end{cases} \]

where \( \sigma_{\chi_0}' \) is the Frechet derivative of \( \sigma \) at \( x_0 \). One can show that the pair \((X, t_3)\) is a \( g_1 \)-interior approximation at \( x_0 \) of the pair \((D, T_3)\).

**Definition 2.5.** The pair \((E, t_3)\) will be called a \( g \)-quasiinterior approximation at \( x_0 \) of the pair \((D, T_3)\) if for every solution \( \tilde{x} \) of the system (2.10) and every neighborhood \( V_{Y_3} \) there exist numbers \( \delta, a_3 > 0 \) \((\delta < a_3)\), a mapping \( f_3 : (0, a_3) \to R^1_+ \) and a neighborhood \( V_x \) such that

\[ \left[ \frac{1}{f_3(\varepsilon)} T_3(x) \ast t_3(\tilde{x}) \right] \cap V_{Y_3} \neq \emptyset \tag{2.11} \]

for all \((\varepsilon, x)\) satisfying conditions (2.4).

**Example 2.7.** Suppose that the pair \((E, t_3)\) is a \( g \)-quasiinterior approximation at \( x_0 \) of the pair \((D, T_3)\). Suppose further that \( \text{int } t_3(x) \neq \emptyset \) for all \( x \in E \). Then the pair \((E, \text{int } t_3)\) is a \( g \)-interior approximation at \( x_0 \) of the pair \((D, T_3)\).

**Example 2.8.** Let there be given a number \( \alpha \in (0, 1] \), a twice differentiable mapping \( F : R^n \to R^m \) a convex set \( E \subseteq R^n \) and a convex cone \( N \subseteq R^m \). Denote by \( dF(x_0, \tilde{x}) \) and \( d^2F(x_0, \tilde{x}, \tilde{x}) \) the first and second differentials of \( F \) at \( x_0 \).

Assume that

\[ -F(x_0) \subseteq N^\perp N, \quad -dF(x_0, \tilde{x}) \subseteq N^\perp N. \]

Then

1. The pair \((X, \hat{f}(\cdot, \alpha))\) is a \((g, \hat{\alpha})\)-tangent approximation at \( x_0 \) of the pair \((X, \hat{F})\), where

\[ \hat{f}(x, \alpha) = f(x, \alpha) + N, \]

\[ f(x, \alpha) = 2[F(x_0) + dF(x_0, \tilde{x})] + \alpha^2[d^2F(x_0, \tilde{x}, \tilde{x}) + dF(x_0, x)]. \]

\[ g(\xi, \varepsilon, x) = \xi + \varepsilon \tilde{x} + \frac{\varepsilon^2}{2} x, \]

\[ \hat{F}(x) = F(x) + N. \]

2. The pair \((X, \hat{f}(\cdot, \alpha))\) is a \( g \)-tangent approximation at \( x_0 \) of the pair \((X, \hat{F})\) provided the cone \( N \) is closed.

3. The pair \((X, \text{int } \hat{f}(\cdot, \alpha))\) is a \( g \)-interior approximation at \( x_0 \) of the pair \((X, \text{int } \hat{F})\) if \( \text{int } N \neq \emptyset, 0 \subseteq \text{int } \hat{F}(x_0), 0 \subseteq \text{int } (dF(x_0, \tilde{x}) + N). \)

**Definition 2.6.** A 4-tuple

\[ (E, t_1, t_2, t_3) \tag{2.12} \]

will be called a \( g \)-approximation at \( x_0 \) of the 4-tuple

\[ (D, T_1, T_2, T_3) \tag{2.13} \]

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if the pair \((E, t_1)\) is a \((g, \ldots)\)-tangent approximation at \(x_o\) of the pair \((D, T_1)\), the pair \((E, t_2)\) is a \(g\)-tangent approximation at \(x_o\) of the pair \((D, T_2)\), and the pair \((E, t_3)\) is a \(g\)-interior approximation at \(x_o\) of the pair \((D, T_3)\).

**Definition 2.7.** We shall say that a \(g\)-approximation \((2.12)\) is convex, if all the mappings \(t_i, i = 1 - 3\), are convex.

Let us set
\[ Y_{12} = Y_1 \times Y_2, \quad t_{12} = t_1 \times t_2. \]

**Definition 2.8.** We shall say that a \(g\)-approximation \((2.12)\) is regular, if there exist a neighborhood \(V_{Y_{12}}\) and a continuous mapping \(h: V_{Y_{12}} \to E\) such that

a. For all \(\sigma \in V_{Y_{12}}\) we have
\[ \sigma \in t_{12}(h(\sigma)), \quad 0 \in t_3(h(\sigma)). \]

b. For every number \(\delta > 0\) and every neighborhood \(V_x\) there exist a number \(\epsilon \in (0, \delta)\) and a continuous mapping \(\xi_\epsilon: V_{Y_{12}} \to D\) such that
\[ \xi_\epsilon(\sigma) \in g(x_o, \epsilon, h(\sigma) + V_x) \]
for all \(\sigma \in V_{Y_{12}}\).

Throughout the subsequent part of this paragraph, we shall assume that \(X, Y_3\) are vector topological spaces, and \(Y_i = R^{k_i}, i = 1, 2\).

The following theorem gives necessary conditions for the inconsistency of a given system of inclusions.

**Theorem 2.1.** Assume that the 4-tuple \((2.12)\) is a regular \(g\)-approximation at \(x_o\) of the 4-tuple \((2.13)\). Then if the system
\[ x \in D, \quad 0 \in T_i(x), \quad i = 1 - 3, \]
is inconsistent (i.e. has no solutions), so is the system:
\[ x \in E, \quad 0 \in t_i(x), \quad i = 1 - 3. \]

Put
\[ k = k_1 + k_2. \]

**Definition 2.9.** A set \(E \subset X\) is said to be \((g, k + 1)\)-contingent to a set \(D \subset X\) at a given point \(x_o \in X\) if for every system of \((k + 1)\) elements \(x_1, x_2, \ldots, x_{k+1}\) of \(E\), every number \(\delta > 0\) and every neighborhood \(V_x\) there exist a number \(\epsilon \in (0, \delta)\) and a continuous mapping \(\eta_\epsilon: P^k \to D\) such that
\[ \eta_\epsilon(\lambda) \in g(x_o, \epsilon, \sum_{i=1}^{k+1} \lambda_i x_i + V_x) \]
for all \(\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{k+1}) \in P^k\), where
\[ P^k = \left\{ \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{k+1}) : \lambda_i \geq 0, \quad \sum_{i=1}^{k+1} \lambda_i = 1 \right\}. \]
Example 2.9. Suppose that $X$, $Y$ are normed spaces and $F : X \to 2^Y$ is a set-valued mapping. Suppose further that $F$ has local sections, and that for every $x \in X$ the set $F(x)$ is convex. Consider a point $(x_0, y_0) \in \text{graph} F$ and define a mapping $f : X \to 2^Y$ by the formula $f(x) = F(x_0) - y_0 - \sigma_{x_0}^{r}(x)$, where $\sigma_{x_0}^{r}$ is the Frechet derivative of the local section $\sigma$ of $F$ corresponding to the pair $(x_0, y_0)$. If $E \subset X$ is $(g_1, k+1)$-contingent to $D \subset X$ at $x_0$, then $\text{graph} \, \tilde{f}$ is also $(g_1, k+1)$-contingent to $\text{graph} \, \tilde{F}$ at $(x_0, y_0)$. Here $\tilde{f}$ and $\tilde{F}$ are the restrictions of $f$ and $F$ to $E$ and $D$ respectively.

Proof. Let $(x_i, y_i) \in \text{graph} \, \tilde{f}$, $i = 1, 2, ..., k+1$, $V_X, V_Y$ and $\delta > 0$ be given. Select a ball $V'_Y$ such that $(k+3) \, V'_Y \subset V_Y$. Let $\sigma_i$ be a local section of $F$, corresponding to the pair $(x_0, y_0)$ with $\tilde{y_i} = y_i + y_0 - \sigma_{x_0}^{r}(x_i)$ and let $\sigma'_i$ be the Frechet derivative of $\sigma_i$ at $x_0$. We can find a positive number $\delta'$ such that

$$\varepsilon \sum_{i=1}^{k+1} \lambda_i \left[ -\sigma_{x_0}^{r}(x_i) + \sigma'_i \left( \sum_{n=1}^{k+1} \lambda_n x_n \right) \right] \in V'_Y$$

whenever

$$0 < \varepsilon < \delta', \quad \lambda = (\lambda_1, \lambda_2, ..., \lambda_{k+1}) \in P^k.$$

It follows from Definition 2.23, that there exist a positive number $\delta''$ and a neighborhood $V'_x \subset V_X$ such that

$$\sigma_i(x) \in \sigma_i(x_0) - \varepsilon \left[ \sigma'_i \left( \sum_{n=1}^{k+1} \lambda_n x_n \right) + V'_Y \right]$$

for all $i = 0, 1, ..., k+1,

$$0 < \varepsilon < \delta'', \quad \lambda = (\lambda_1, \lambda_2, ..., \lambda_{k+1}) \in P^k;$$

$$x \in x_0 + \varepsilon \left( \sum_{i=1}^{k+1} \lambda_i x_i + V'_X \right),$$

where $\sigma = \sigma, \sigma'_0 = \sigma_{x_0}^{r}$.

Since $E$ is $(g_1, k+1)$-contingent to $D$ at $x_0$, we can find a positive number $\varepsilon < \min(\delta, \delta', \delta'', 1)$ and a mapping $\eta_e^1 : P^k \to D$ such that

$$\eta_e^1(\lambda) \in x_0 + \varepsilon \left( \sum_{i=1}^{k+1} \lambda_i x_i + V'_X \right)$$

for all $\lambda \in P^k$. Now define a mapping $\eta_e : P^k \to \text{graph} \, \tilde{F}$ by setting $\eta_e(\lambda) = (\eta_e^1(\lambda), \eta_e^2(\lambda))$ where

$$\eta_e^2(\lambda) = (1 - \varepsilon) \sigma(\eta_e^1(\lambda)) + \varepsilon \sum_{i=1}^{k+1} \lambda_i \sigma_i(\eta_e^1(\lambda)).$$

It is clear that the number $\varepsilon$ and the mapping $\eta_e$ have the properties required in Definition 2.9.
Corollary 2.1. Assume that the 4-tuple (2.12) is a $g$-approximation at $x_\ast$ of the 4-tuple (2.13), where

1. $E$ is $(g, k+1)$-contingent to $D$ at $x_\ast$.
2. The mappings $t_i$, $i = 1, 2$, are convex and satisfy the condition $0 \leq \text{int } t_{12}(E)$.
3. For every $x \in E$, $t_3(x)$ is a convex set with a non-empty interior.

Assume further that at least one of the following conditions is fulfilled.

1. The mapping $t_3$ is convex.
2. The mapping $t_3$ is of the form $t_3(x) = q(x) + N_3$, where $N_3$ is a cone in $Y_3$, $q : E \to 2^{Y_3}$ a lower semi-continuous mapping such that, for every $x \in Y$, $q(x)$ is a bounded set.
3. The mapping $t_3$ is lower semi-continuous, and $Y_3$ is a normed space.

Then if the system (2.14) is inconsistent, so is the system

\[ x \in E, \quad 0 \in t_i(x), \quad i = 1, 2, \quad 0 \in \text{int } t_3(x). \]

As an immediate consequence of Corollary 2.1 and Lemma 2.1 stated below, we have

Corollary 2.2. Assume that the 4-tuple (2.12) is a convex $g$-approximation at $x_\ast$ of the 4-tuple (2-13) where.

1. $E$ is $(g, k+1)$-contingent to $D$ at $x_\ast$.
2. For every $x \in E$, $t_3(x)$ has a non-empty interior.

Then if the system (2.14) is inconsistent, there exist linear continuous functionals $y_i^* \in \bar{Y}_i^*$, $i = 1 - 3$, not all zero, such that, for all $x \in E$,

\[ \sum_{i=1}^{3} c^{1i}(y_i^*, x) \leq 0. \tag{2.16} \]

Remark 2.2. Corollary 2.2 includes as special cases the main results obtained by Halkin and Neustadt in [4] and [5] respectively.

Lemma 2.1. Assume that, for every $x \in E$, $t_3(x)$ is a convex set with a non-empty interior. Assume further that $t_{123}(E)$ is convex, where $t_{123} = t_1 \times t_2 \times \text{int } t_3$.

Then, if the system

\[ x \in E, \quad 0 \in t_i(x), \quad i = 1, 2, \quad 0 \in \text{int } t_3(x) \]

is inconsistent, there exist linear continuous functionals $y_i^* \in \bar{Y}_i^*$, $i = 1 - 3$, not all zero, such that, for all $x \in E$, we have the inequality (1.16).

As a consequence of Corollary 2.2, we shall obtain a multiplier rule for an optimization problem with a vector-valued criterion function.

Definition 2.10. Let $M$ be a convex cone in vector topological space $Z$ ($M + Z$), $Q$ a subset in $X$, $S$ a singlevalued mapping from $Q$ into $Z$. A point
$x_0 \in Q$ is said to be $M$-extremal (for $S$) if, for every $x \in Q$, the inclusion $S(x) - S(x_0) \subseteq M$ implies that $S(x) - S(x_0) \subseteq M$.

We shall state necessary conditions that every $M$-extremal point must satisfy provided certain hypotheses are fulfilled. It should be pointed out that the Theorem 2.2 stated below differs from known results (see, for example, [5,7]) in that the cone $M$ may have an empty interior and the constraints under consideration can be given by set-valued mappings.

Assume that

(a). $Q$ is the set of all solutions of the system (2.14),

$$Z = \prod_{i=1}^{3} Z_i, \quad S = \prod_{i=1}^{3} S_i, \quad M = \prod_{i=1}^{3} M_i$$

where

$$Z_i = R^{m_i}, \quad i = 1, 2$$

$Z_3$ is a vector topological space, $S_i : D \rightarrow Z_i$, $i = 1 - 3$, are single-valued mappings, $M_i \subset Z_i$, $i = 1 - 3$, are convex cones, and $M_3$ has a non-empty interior.

(c) If $M_3 = Z_3$, then $M_1 \neq [0]$ and $M_1 \cap (-M_1) = [0]$

(d) There exist a set $E \subset X$ and mappings

$$g: X \times \mathbb{R}_+ \times X \rightarrow X, \quad i_1: E \rightarrow 2^{Y_i}, \quad s_i: E \rightarrow Z_i, \quad i = 1 - 3,$$

such that

(d1). $E$ is $(g, k + 1)$-contingent to $D$ at $x_0$,

where

$$k = k_1 + k_2 + m'_1 + m'_2,$$

$$m'_i = \begin{cases} m_i & \text{if } M_i \neq Z_i, \\ 0 & \text{if } M_i = Z_i. \end{cases}$$

(d2). The 4-tuple (2.12) is a convex $g$-approximation at $x_0$ of the 4-tuple (2.13).

(d3). $\text{int } t_g(x) \neq \emptyset$ for every $x \in E$.

(d4). Mappings $s_i$, $i = 1 - 3$, are $(M_1 \cup M_i)$-convex and the 4-tuple $(E, \tilde{s}_1, \tilde{s}_2, \tilde{s}_3)$ is a $g$-approximation at $x_0$ of the 4-tuple $(D, \tilde{S}_1, \tilde{S}_2, \tilde{S}_3)$, where the mappings $\tilde{s}_i, \tilde{S}_i$ are defined by the formulae

$$\tilde{s}_i(x) = s_i(x) + \tilde{M}_i,$$

$$\tilde{S}_i(x) = S_i(x) - S_i(x_0) + \tilde{M}_i,$$

$$\tilde{M}_i = \begin{cases} M_1 & \text{if } i = 1, \ M_3 \neq Z_3; \\ M_1 \setminus [0] & \text{if } i = 1, \ M_2 = Z_3; \\ M_2 & \text{if } i = 2, \\ \text{int } M_3 & \text{if } i = 3. \end{cases}$$
Recall that a single-valued mappings from a set $E \subset X$ into a linear space $Y$ is called $N$-convex if $E$ is convex and if, also,

$$as(x_1) + (1 - a) s(x_2) \leq sl(x_1 + (1 - a)x_2) + N$$

for all $a \in [0, 1]$ and $x_i \in E$, $i = 1, 2$. Here $N$ is a given convex cone in $Y$.

**Theorem 2.2.** Assume that conditions (a) — (d) hold. If $x_o$ is an $M$-extremal point, then there exist linear continuous functionals $y_i^* \in Y_i^*$, $z_i \in M_i^-$, $i = 1 - 3$, not all zero, such that

$$\sum_{i=1}^{3} \left\{ c^{T_i}(y_i^*, x) + (z_i^*, s_i(x)) \right\} \leq 0 \quad (2.17)$$

for all $x \in E$. Furthermore, we have

$$c^{T_i}(y_i^*, x_o) = 0, \quad i = 1 - 3,$$

if we can find a point $x_o \in E$ such that

$$t_i(x_o) = T_i(x_o), \quad s_i(x_o) = 0, \quad i = 1 - 3.$$

Recall that

$$\hat{M}_i^- = \{ z_i^* \in Z_i^* : (z_i^*, z) \leq 0 \text{ for all } z \in M_i \}.$$

**Theorem 2.3.** Assume that conditions (a) — (c) hold. Assume further that there are an arbitrary set $\Omega$ and mappings $\hat{t}_i: E \times \Omega \rightarrow 2^Y_i$, $\hat{s}_i: E \times \Omega \rightarrow Y_i$, $i = 1 - 3$, such that

1. $0 \in int q_{12}(E, \omega)$ for every fixed $\omega \in \Omega$ where

$$q_{12}(x, \omega) = \prod_{i=1}^{2} q_i(x, \omega),$$

$$q_i(x, \omega) = \begin{cases} \hat{t}_i(x, \omega) \times \{ \hat{s}_i(x, \omega) + \hat{M}_i \} & \text{if } M_i \neq Z_i, \\
\hat{t}_i(x, \omega) & \text{if } M_i = Z_i. \end{cases}$$

2. $q(E \times \Omega)$ is a convex set, where

$$q = q_1 \times q_2 \times \text{int } q_3.$$

3. For every fixed $\omega \in \Omega$ we can find a mapping $g_\omega: X \times \hat{H}_i^1 \times X \rightarrow X$ such that the conditions (d1) — (d4) hold if $g$, $t_i$ and $s_i$ are replaced by, $g_\omega$, $\hat{t}_i(., \omega)$ and $\hat{s}_i(., \omega)$, respectively.

Finally, assume that $x_o$ is an $M$-extremal point. Then there exist linear continuous functionals $y_i^* \in Y_i^*$, $z_i^* \in \hat{M}_i^-$, $i = 1 - 3$, not all zero, such that

$$\sum_{i=1}^{3} \left\{ c^{T_i}(y_i^*, \hat{x}) + (z_i^*, \hat{s}_i(\hat{x})) \right\} \leq 0$$

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for all $\tilde{x} = (x, \omega) \in E \times \Omega$. Furthermore, conditions (2.17) are also fulfilled, if we can find a point $(\tilde{\xi}_0, \omega_0) \in E \times \Omega$ such that

$$
\tilde{t}_i(\tilde{\xi}_0, \omega_0) = T_i(x_0), \quad \tilde{s}_i(\tilde{\xi}_0, \omega_0) = 0, \quad i = 1 - 3.
$$

§3. A MULTIPLIER RULE FOR CONSTRAINED OPTIMIZATION PROBLEMS WITH VECTOR-VALUED CRITERIA IN FINITE-DIMENSIONAL SPACES.

The purpose of this paragraph is to prove a multiplier rule for a constrained optimization problem, from which the support principle in discrete inclusions follows directly.

Assume that

$$
Q = \{ x \in X : 0 \in F_i(x), \quad i = 1, 2 \},
$$

where $F_i, \ i = 1, 2$, are set-valued mappings from $X = \mathbb{R}^n$ into $\mathbb{R}^k_i$.

Assume further that

(1) The mapping $F_1$ is such that:

(a) For every $x \in X$, $F_1(x)$ is a convex compact subset with a non-empty interior.

(b) Functions $c F_1(\psi, x)$ and $\frac{\partial}{\partial x} c F_1(\psi, x)$ are continuous in all of their variables.

(2) The mapping $F_2$ has local sections, and, for every $x \in X$, the set $F_2(x)$ is convex.

(3) $M = M' \times M''$, where $M' \subset \mathbb{R}^{m_1}$ is an arbitrary convex cone, and $M'' = \mathbb{R}^{m_2}_{+}$.

(4) $S = (S', S'')$, where $S': X \rightarrow \mathbb{R}^{m_1}$ is a differentiable mapping, and $S'': X \rightarrow \mathbb{R}^{m_2}$ is defined by the formulae

$$
S''(x) = (S''_1(x), S''_2(x), \ldots, S''_{m_2}(x)),
$$

$$
S''_i(x) = \max_{\alpha \in \Delta_i} h_i(x, \alpha), \quad i = 1, 2, \ldots, m_2
$$

$(\Delta_i$ is a compact topological space, $h_i(x, \alpha)$ is a function which is defined and is continuous together with its partial derivatives $\frac{\partial}{\partial x^j} h_i(x, \alpha), \ j = 1, 2, \ldots, n, \ on \ the \ direct \ product \ X \times \Delta_i$).

Let

$$
M' = \{ \psi \in \mathbb{R}^{m_1} : \langle \psi, y \rangle \leq 0 \ for \ all \ y \in M' \},
$$

$$
\Delta_i(x_0) = \{ \tilde{x} \in \Delta_i : h_i(x_0, \tilde{x}) = \max_{\alpha \in \Delta_i} h_i(x_0, \alpha) \}.
$$

It is clear that $\Delta_i(x_0) \neq \phi$. 75
Theorem 3.1. Assume that the conditions (1) – (4) hold. Let \( x_0 \) be a point in \( Q \), \( \sigma_2 \) a local section of \( F_2 \) corresponding to the pair \( (x_0, 0) \). If \( x_0 \) is an \( M \)-extremal point for \( S \), then there exist vectors \( \psi_i \in R^{k_1}, i = 1, 2, \psi \in M' \); numbers \( \nu^j_i \leq 0, j = 1, 2, ..., n + 1, i = 1, 2, ..., m_2 \); and points \( \alpha^j_i \in \Delta_i(x_0), j = 1, 2, ..., n + 1; i = 1, 2, ..., m_2 \), such that

(a) Not all of the quantities \( \psi_i, i = 1, 2, \psi \) and \( \nu^j_i, j = 1, 2, ..., n + 1, i = 1, 2, ..., m_2 \) are zero.

(b) \( c^{F_1}(\psi_1, x_0) = 0, i = 1, 2 \) \hspace{1cm} (3.1)

(c) \( \frac{\partial}{\partial x} H(x_0) = 0 \) \hspace{1cm} (3.2)

where

\[
H(x) = c^{F_1}(\psi_1, x) + \langle \psi_2, \sigma_2(x) \rangle + \langle \psi, S'(x) \rangle + \sum_{j=1}^{n+1} \sum_{i=1}^{m_2} \nu^j_i h_i(x_0, \alpha^j_i).
\]

Proof. By using the separation theorem, we can show that \( 0 \in F_1(x) \) if and only if

\[
f_1(x) = \max_{\alpha_0 \in \Delta_0} \left[ -c^{F_1}(\alpha_0, x) \right] \leq 0,
\]

where

\[
\Delta_0 = \{ \alpha \in R^{k_1}: \| \alpha \| = 1 \}.
\]

Denote by \( D \) the graph of the mapping \( F_2 \):

\[
D = \text{graph} F_2 = \{ \eta = (x, \xi) \in R^a \times R^{k_2}: x \in R^a, \xi \in F_2(x) \}.
\]

It is clear that \( x \in Q \) if and only if

\[
\eta = (x, \xi) \in D, \quad 0 \in T_i(\eta), \quad i = 2, 3, \hspace{1cm} (3.3)
\]

where \( T_2(\eta) = \xi, \quad T_3(\eta) = f_1(x) + R^1_+ \).

Let

\[
S_1(\eta) = S'(x), \quad S_2(\eta) = S''(x), \quad S(\eta) = (S_1(\eta), S_2(\eta)).
\]

Then, by the \( M \)-extremality of \( x_0 \), we see that \( \eta_0 = (x_0, 0) \) is an \( M \)-extremal point for the mapping \( S(\eta) \) subject to the constraints (3.3).

The example 2.9 shows that, whatever the natural \( k \) may be, the set \( E \) defined by

\[
E = \{ \eta = (x, \xi) : x \in R^a, \xi \in F_2(x_0) + \frac{\partial \sigma_2(x_0)}{\partial x} x \}
\]

is \( (g_1, k + 1) \)-contingent to \( D \) at point \( \eta_0 \) where the function \( g_1 \) is defined by

\[
g_1(\eta_0, \varepsilon, \eta) = \eta_0 + \varepsilon \eta.
\]

It is easy to verify that the pair \( (E, t_2) \), where \( t_2(\eta) = \xi \), is a \( g_1 \)-tangent approximation at \( \eta_0 \) of the pair \( (D, T_2) \).
Define a set-valued mapping $t_3$ as follows:

$$t_3(\eta) = \begin{cases} \bar{R}^{-1}_+ + \max_{\alpha \in \Delta_0(x_0)} \left\{ -\frac{\partial c^F_1(x_0, x_0)}{\partial x} x \right\} & \text{if } f_1(x_0) = 0, \\ R^+ & \text{if } f_1(x_0) < 0, \end{cases}$$

where $\Delta_0(x_0) = \{ \bar{\alpha} \in \Delta_0: -c^F_1(\bar{\alpha}, x_0) = \max_{\alpha \in \Delta_0} (-c^F_1(x_0, x_0)) \}.$

According to Lemma 3.1, to be proved below, the pair $(E, t_3)$ is a $g_1$-interior approximation at $x_0$ of the pair $(D, T_3).$ Let us set

$$M_1 = M', \quad M_3 = M''$$

$$p_1(\eta) = \max \left\{ \frac{\partial h_i(x_0, \alpha)}{\partial x} x: \alpha \in \Delta_i(x_0) \right\},$$

$$s_3(\eta) = (p_1(\eta), p_2(\eta), \ldots, p_{m_2}(\eta)),$$

$$s_1(\eta) = \frac{\partial S'(x_0)}{\partial x} x.$$

It is not hard to verify that all the conditions for the applicability of Theorem 2.2 are satisfied(1). Hence we can find vectors $\psi_2 \in R^{k_2}, \psi' \in M'$ and numbers $\beta_0^i, \beta_1^i, \ldots, \beta_{m_2}^i$, such that

1. $(\psi_2, \psi', \beta_0^i, \beta_1^i, \beta_{m_2}^i) \neq 0.$

2. For every $\eta \in E$ we have

$$\langle \psi_2, \xi \rangle + \langle \psi', \frac{\partial S'(x_0)}{\partial x} x \rangle + \sup_{\gamma \in t_3(\eta)} \beta_0^i \gamma + \sum_{i=1}^{m_2} \beta_1^i \max_{\alpha \in \Delta_i(x_0)} \frac{\partial h_i(x_0, \alpha)}{\partial x} x \leq 0. \quad (3.4)$$

From the last inequality it follows that $\beta_0^i \leq 0$ and that, furthermore, $\beta_0^i = 0$ if $f_1(x_0) < 0.$ Hence, the inequality (3.4) can be rewritten in the form

$$q(x, \xi) \leq 0 \quad (3.5)$$

where

$$q(x, \xi) = \langle \psi_2, \xi \rangle + \beta_0^i \max_{\alpha \in \Delta_0(x_0)} \left\{ -\frac{\partial c^F_1(x_0, x_0)}{\partial x} x \right\} +$$

$$+ \langle \psi', \frac{\partial S'(x_0)}{\partial x} x \rangle + \sum_{i=1}^{m_2} \beta_1^i \max_{\alpha \in \Delta_i(x_0)} \frac{\partial h_i(x_0, \alpha)}{\partial x} x.$$

Since the inequality (3.4) is fulfilled for all $(x, \xi) \in E,$ the system

$$(x, \xi) \in R^n \times R^{k_2}, \quad 0 \leq q(x, \xi) - \bar{R}^{-1}_+, \quad 0 \leq -\xi + F_2(x_0) + \frac{\partial S_2(x_0)}{\partial x} x$$

(1) In this case $T_1, S_2, M_2$ are absent.
is inconsistent. Therefore, by the separation theorem, there exist a scalar \( \phi \geq 0 \) and a vector \( \varphi \in R^{k_2} \) not both zero, such that

\[
\mu q(x, \xi) - \langle \varphi, \xi \rangle + c_{F_2}(\varphi, x_o) + \left( \varphi, \frac{\partial \sigma_2(x_o)}{\partial x} x \right) \leq 0
\]  

(3.6)

for all \((x, \xi) \in R^n \times R^{k_2}\). It follows from (3.6) that

\[
\mu \psi_2 = \varphi
\]  

(3.7)

Setting \((x, \xi) = 0\) in (3.6) and taking into account the fact that \(0 \in F_2(x_o)\), we obtain

\[
c_{F_2}(\mu \psi_2, x_o) = 0.
\]  

(3.8)

From (3.6) - (3.8), we have

\[
\langle a, x \rangle + \min_{\alpha \in \Delta(x_o)} \langle b(\alpha), x \rangle \leq 0
\]  

(3.9)

for all \(x \in R^n\) where

\[
b(\alpha) = -\mu \beta_o \frac{\delta c_{F_1}(\alpha_o, x_o)}{\delta x} + \sum_{i=1}^{m_2} \mu_i \frac{\delta h_i(x_o, \alpha_i)}{\delta x},
\]

\[
\alpha = (\alpha_i, x_1, ..., x_{m_2}),
\]

\[
\Delta(x_o) = \Delta_o(x_o) \times \Delta_1(x_o) \times \cdots \times \Delta_{m_2}(x_o),
\]

\[
a = \left[ \frac{\partial \sigma_2(x_o)}{\partial x} \right]^\top \mu \psi_2 + \left[ \frac{\partial S'(x_o)}{\partial x} \right]^\top \mu \psi'.
\]

From (3.9) it is not difficult to show that

\[-a \notin \text{co} \{ b(\alpha) : \alpha \in \Delta(x_o) \}.
\]

Hence, we can find points \(a= (\alpha_0^j, \alpha_1^j, ..., \alpha_{m_2}^j) \in \Delta(x_o), \ j = 1, 2, ..., n+1\)

and nonnegative scalars \(v_j, j = 1, 2, ..., n+1\), with \(\sum_{j=1}^{n+1} v_j = 1\), such that

\[
a = -\sum_{j=1}^{n+1} v_j b(a^j),
\]

i.e.

\[
\left[ \frac{\partial \sigma_2(x_o)}{\partial x} \right]^\top \psi_2 + \left[ \frac{\partial S'(x_o)}{\partial x} \right]^\top \psi + \sum_{j=1}^{n+1} \mu_o^j \frac{\delta c_{F_1}(\alpha_j^o, x_o)}{\delta x} + \sum_{i=1}^{m_2} \sum_{j=1}^{n+1} \mu_i^j \frac{\delta h_i(x_o, \alpha_i^j)}{\delta x} = 0
\]  

(3.10)

where

\[
\psi_2 = \mu \psi_2', \ \psi = \mu \psi',
\]

\[
\mu_o^j = -\mu \beta_o^j v_i, \ \mu_i^j = \mu \beta_i^j v_i, \ j = 1, 2, ..., n+1; \ i = 1, 2, ..., m_2.
\]

(1) If \(A\) is a matrix, then its transpose is denoted by \(A^\top\)
Let us note that not all of the quantities $\psi_i$, $\psi_j$, $\mu^j_0$ and $\mu^i_j$, $j = 1, 2, ..., n + 1; i = 1, 2, ..., m_2$, are zero.

Furthermore, we have

$$-\mu^j_0 c^{F_1}(\alpha^j_0, x_o) = 0$$

(3.11)

for every $j = 1, 2, ..., n + 1$.

Putting

$$\psi_1 = \sum_{j=1}^{n+1} \mu^j_0 \alpha^j_0,$$

we obtain from (3.11) that $c^{F_1}(\psi_1, x_o) = 0$.

Lemma 3.2. to be proved below and the inequality (3.10) show that the relation (3.2) holds.

To conclude the proof of our theorem it remains to verify that condition (a) is also fulfilled. To this end, it suffices to prove that relation $\psi_1 = 0$ implies that $\mu^j_0 = 0$ for every $j = 1, 2, ..., n + 1$. Indeed, by hypothesis and from relation (3.11), we have

$$c^{F_1}(-\mu^j_0 \alpha^j_0, x_o) = c^{F_1}\left(\sum_{i=1,2,...,n+1}^{n+1} \mu^i_0 \alpha^i_0, x_o\right) = 0$$

for all $j = 1, 2, ..., n + 1$. Consequently, for every fixed $j$ the linear function $\langle \mu^j_0 \alpha^j_0, y \rangle$ of the variable $y$ is constant on the set $F_1(x_o)$. Hence, $\mu^j_0 \alpha^j_0 = 0$ since $F_1(x_o)$ has a non-empty interior. From the last relation and the fact that $\alpha^j_0 \neq 0$, we have $\mu^j_0 = 0, j = 1, 2, ..., n + 1$. This completes the proof of the Theorem.

**Remark 3.1.** Let $c^{F_2}(\psi_2, x)$ be a differentiable function, where $\psi_2$ is the vector mentioned in Theorem 3.1. Then, as in [1, 2] we can show that

$$\left[ \frac{\partial \sigma_2(x_o)}{\partial x} \right]^{\text{T}} \psi_2 = \frac{\partial c^{F_2}(\psi_2, x_o)}{\partial x}.$$

In fact, from the definition of a local section and the condition (b) of Theorem 3.1 we conclude that

$$\xi(x) \leq \xi(x_o) = 0$$

for all $x$ in the domain of function $\sigma_2$ where

$$\xi(x) = \langle \psi_2, \sigma_2(x) \rangle - c^{F_2}(\psi_2, x).$$

So, the element $x_o$ achieves a local maximum for $\xi$, hence the desired relation
Remark 3.2. As in [2], we shall say that the family of functions
\[ \{ h_i(\alpha, x), \alpha_i \in \Lambda_i, i = 1, 2, \ldots, m_2 \} \]  
(3.12)
is nondegenerate at \( x \) if there is a point \( \hat{x} \) in \( X \) such that
\[ \frac{\partial h_i(\alpha, x)}{\partial x} \hat{x} < 0 \]
for all \( \alpha_i \in \Lambda_i(x) \), \( i = 1, 2, \ldots, m_2 \). It is easy to see from conditions (a) and (c) of Theorem 3.1 that if the family (3.12) is nondegenerate, then \( (\psi_1, \psi_2, \psi) \neq 0 \).

Let us now assume that
(3') \( \tilde{M} = M' \times M'' \times N''' \) where \( M' \) and \( M'' \) are the same cones as in condition (3), and \( M''' = R^{m_3} \).

(4') \( \tilde{S} = (S', S'', S''') \) where \( S' \) and \( S'' \) are the same mappings as in condition (4), and \( S''' \) is a mapping defined by the formulae
\[ S'''(x) = (l_1(x), l_2(x), \ldots, l_{m_3}(x)), \]
\[ l_i(x) = \max \left( l^1_i(x), l^2_i(x), \ldots, l^{q_i}_i(x) \right), \quad i = 1, 2, \ldots, m_3. \]

Here \( l^1_i : R^n \rightarrow R^1 \) is a given smooth function.

Theorem 3.2. Assume that conditions (1), (2), (3') and (4') hold. Let \( x \) be a point in \( Q \), \( \sigma \) a local section of \( F_2 \), corresponding to the pair \( (x, 0) \). If \( x \) is an \( M \)-extremal point for the mapping \( S \), then there exist vectors, \( \psi_i \in R^{k_i} \), \( i = 1, 2 \); \( \psi \in M' \); numbers \( \mu_i^j \leq 0 \), \( j = 1, 2, \ldots, n + 1 \); \( i = 1, 2, \ldots, m_2 \); \( v_i^j \leq 0 \), \( j = 1, 2, \ldots, q_i \); \( i = 1, 2, \ldots, m_3 \), and points \( \alpha_i^j \in \Lambda_i(x), j = 1, 2, \ldots, n + 1 \); \( i = 1, 2, \ldots, m_2 \) such that
(a) Not all of the quantities \( \psi_i, i = 1, 2 \); \( \psi \); \( \mu_i^j, j = 1, 2, \ldots, n + 1 \); \( i = 1, 2, \ldots, m_2 \) and \( v_i^j, i = 1, 2, \ldots, q_i \); \( i = 1, 2, \ldots, m_3 \), are zero.
(b) \( c_{F_i} (\psi, x) = 0, i = 1, 2. \)
(c) \( \frac{\partial H(x)}{\partial x} = 0, \)
where
\[ H(x) = c_{F_1} (\psi, x) + \langle \psi, S'(x) \rangle + \langle \psi_2, \sigma(x) \rangle + \]
\[ + \sum_{j=1}^{n+1} \sum_{i=1}^{m_2} \mu_i^j h_i(x, \alpha_i^j) + \sum_{i=1}^{m_3} \sum_{j=1}^{q_i} v_i^j l_i^j(x). \]
(d) \( v_i^j \left[ l_i^j(x) - l_i(x) \right] = 0, j = 1, 2, \ldots, q_i \); \( i = 1, 2, \ldots, m_3. \)
\textbf{Proof.} We introduce the space $R^n + m_3$ of the variable $\eta = (x, \xi)$ where $x \subseteq R^n, \xi = (\xi_1, \xi_2, ..., \xi_{m_3}) \subseteq R^{m_3}$. Let us set

$$l_i^j(\eta) = (l_i^1(x) - \xi_i, l_i^2(x) - \xi_i, ..., l_i^{q_i}(x) - \xi_i), i = 1, 2, ..., m_3,$$

$$F_2^j(\eta) = \prod_{i=1}^{m_3} (l_i^j(\eta) + R^1_{+}).$$

Note that mapping $F_2^j$ has local sections. Indeed, let $(\eta, \xi)$ be an arbitrary point in its graph. Choose $\sigma(x) = l'(\eta) + \xi - l'(\eta)$ with $l'(\eta) = (l_1'(\eta), l_2'(\eta), ..., l_{m_3}'(\eta))$.

It is obvious that $\sigma$ is a local section of $F_2^j$ corresponding to the given pair.

Consider now the mappings

$$\tilde{F}_1(\eta) = F_1(x), \quad \tilde{F}_2(\eta) = F_2(x) \times F_2^j(\eta),$$

$$\tilde{S}'(\eta) = (S'(x), \xi_1, \xi_2, ..., \xi_{m_3}), \quad \tilde{S}''(\eta) = S''(x),$$

$$\tilde{S}(\eta) = (\tilde{S}'(\eta), \tilde{S}''(\eta)).$$

and the cone $\tilde{M} = \tilde{M}' \times \tilde{M}''$ where $\tilde{M}' = M' \times M''$, $\tilde{M}'' = M''$. It is easy to see that $x_o$ is a $\tilde{M}$-extremal point for the mapping $\tilde{S}$ subject to constraints (3.1) if and only if $\eta_o = (x_o, \xi_o)$ is a $\tilde{M}$-extremal point for the mapping $\tilde{S}$ subject to

$$0 \in \tilde{F}_i(\eta), i = 1, 2,$$

where $\xi_o$ is the vector with coordinates $l_1(x_o), l_2(x_o), ..., l_{m_3}(x_o)$.

\textbf{Applying Theorem 3.1 to the point $\eta_o$, we obtain Theorem 3.2.}

Let $h: X \times \Delta \rightarrow R^1$ be a function, which is defined and is continuous together with its partial derivatives $\frac{\partial}{\partial x_i} h(x, \alpha), i = 1, 2, ..., n$, on the direct product $X \times \Delta$ where $X = R^n$ and $\Delta$ is a compact topological space. Let

$$H(x) = \max_{\alpha \subseteq \Delta} h(x, \alpha), H(x_o) \leq 0.$$ 

Denote by $\Delta(x_o)$ the set of all points $\tilde{\alpha} \subseteq \Delta,$

which achieve a maximum for the function $h(x_o, \alpha)$ of the variable $\alpha \subseteq \Delta$. It is easy to verify that the function

$$c(x) = \max_{\alpha \subseteq \Delta(x_o)} \frac{\partial h(x_o, \alpha)}{\partial x_i} \cdot x$$

is convex and continuous. Let us set

$$h(x) = \begin{cases} c(x) + R^1_+ & \text{if } H(x_o) = 0, \\ R^1_+ & \text{if } H(x_o) < 0, \end{cases}$$

$$T(x) = H(x) + R^1_+.$$
Lemma 3.1. The pair \((X, t)\) is a \(g_1\)-interior approximation at \(x_0\) of the pair \((X, T)\) where \(g_1\) is defined by the formula \(g_1(\xi, \varepsilon, x) = \xi + \varepsilon x\).

Proof. If \(H(x_0) < 0\), then the assertion of the Lemma follows from the continuity of the function \(H(x)\). Now suppose that \(H(x_0) = 0\). Let \(\widehat{x}\) be an arbitrary point satisfying the condition \(0 \leq c(\widehat{x}) + R_{+1}\) i.e.,

\[
\frac{\partial h(x_0, \alpha)}{\partial x} \widehat{x} < -q \quad (q < 0)
\]

for all \(\alpha \in \Delta(x_0)\). Since the function \(\frac{\partial h(x_0, \alpha)}{\partial x}\) of the variable \(\alpha \in \Delta\) is continuous, we can find an open set \(A \subset \Delta\) such that \(\Delta(x_0) \subset A\) and, for all \(\alpha \in A\), the inequality (3.13) is satisfied. In view of the continuity of the function \(K(x) = \max_{\alpha \in \Delta \setminus A} h(x, \alpha)\) and the condition \(K(x_0) < H(x_0) = 0\), there is a neighborhood (a ball) \(V_{\varepsilon}^x\) such that

\[
K(x) < 0
\]

whenever \(x \in x_0 + V_{\varepsilon}^x\).

Consider now the function

\[
\zeta(\varepsilon, \alpha, x) = \frac{\partial h(x_0, \alpha)}{\partial x} x + \left[ \frac{\partial h(x_0 + \varepsilon(\widehat{x} + x), \alpha)}{\partial x} - \frac{\partial h(x_0, \alpha)}{\partial x} \right] (\widehat{x} + x)
\]

where \(\theta = \theta(\varepsilon, \alpha, x)\) is a function with the range contained in the interval \([0, 1]\). It is clear that \(\zeta(\varepsilon, \alpha, x)\) converges to zero uniformly in \(\alpha \in \Delta\) when \(\varepsilon\) and \(x\) converge to zero. Taking into account this fact and the relation

\[
\frac{1}{\varepsilon} \left[ h(x_0 + \varepsilon(\widehat{x} + x), \alpha) - h(x_0, \alpha) \right] - \frac{\partial h(x_0, \alpha)}{\partial x} \widehat{x} + \zeta(\varepsilon, \alpha, x),
\]

we see that there exist a real positive number \(\delta\) and a neighborhood \(V_{\varepsilon}^x\) such that

\[
h(x_0 + \varepsilon(\widehat{x} + x), \alpha) < h(x_0, \alpha) - \frac{q}{2} \varepsilon
\]

whenever \(\alpha \in A, 0 < \varepsilon < \delta, x \in V_{\varepsilon}^x\). Without loss of generality, we may assume that \(\varepsilon(\widehat{x} + V_{\varepsilon}^x) \subset V_{\varepsilon}^x\) for all \(\varepsilon \in (0, \delta)\). Conditions (3.14) and (3.15) show that, for \(0 < \varepsilon < \delta, x \in g_1(x_0, \varepsilon, \widehat{x} + V_{\varepsilon}^x)\) we have \(H(x) < 0\), i.e., \(0 \in T(x)\). This completes the proof of the Lemma.

Lemma 3.2. Let \(T\) be a set-valued mapping from \(R^n\) into \(R^m\) such that, for every \(\psi \in R^m\), the function

\[
c^T(\psi, x) = \sup_{y \in T(x)} \langle \psi, y \rangle
\]

has partial derivatives with respect to \(x\).
Furthermore, let \( I \) be a finite set of indices, \( \lambda_i, i \in I \), nonnegative real numbers, and \( \psi_i \in \mathbb{R}^n, i \in I \), vectors such that the maximum of the linear functions \( \langle \psi_i, y \rangle, i \in I \) and \( \langle \sum_{i \in I} \lambda_i \psi_i, y \rangle \) of the variable \( y \in T(x_o) \) is equal to zero. Then we have
\[
\frac{\partial}{\partial x} c^T \left( \sum_{i \in I} \lambda_i \psi_i, x_o \right) = \sum_{i \in I} \lambda_i \frac{\partial}{\partial x} c^T(\psi_i, x_o).
\]

**Proof.** Denote by \( e_j \) the \( j \)-th coordinate of the vector \( x \in \mathbb{R}^n \), and by \( e^j \) the \( j \)-th unit vector, i.e.
\[
e^j = (0, \ldots, 0, 1, 0, \ldots, 0).
\]

Obviously,
\[
\sum_{i \in I} \lambda_i c^T(\psi_i, x_o) = c^T \left( \sum_{i \in I} \lambda_i \psi_i, x_o \right).
\]
Hence, for all \( x \in \mathbb{R}^n \) we have
\[
\sum_{i \in I} \lambda_i [c^T(\psi_i, x_o + \alpha e^j) - c^T(\psi_i, x_o)] \geq 0,
\]
\[
\geq c^T \left( \sum_{i \in I} \lambda_i \psi_i, x_o + \alpha e^j \right) - c^T \left( \sum_{i \in I} \lambda_i \psi_i, x_o \right). \tag{3.16}
\]
Dividing (3.16) by \( \alpha > 0 \), then making \( \alpha \to +0 \), we obtain
\[
\frac{\partial}{\partial x^j} \sum_{i \in I} \lambda_i c^T(\psi_i, x_o) \geq \frac{\partial}{\partial x^j} c^T \left( \sum_{i \in I} \lambda_i \psi_i, x_o \right). \tag{3.17}
\]
To prove the converse inequality, it suffices to divide (3.16) by \( \alpha < 0 \) and to let \( \alpha \to -0 \). The proof of the Lemma is complete.

**Remark 3.3.** Theorem 3.2 is also valid for the case where the mappings \( S'' \) and \( S''' \) are absent, provided that the cone \( M' \) satisfies the following conditions:
\[
M' \neq \{0\}, \quad M' \cap (-M') = \{0\}.
\]

§4. THE SUPPORT PRINCIPLE FOR A DISCRETE INCLUSION

Consider now an optimal process described by a system of discrete inclusions
\[
x(k+1) \in A^k(x(k)), \quad k = 0, 1, \ldots, K-1, \tag{4.1}
\]
with restricted phase coordinates
\[
0 \in B^k(x(k)), \quad k = 0, 1, \ldots, K, \tag{4.2}
\]
where \( A^k \) and \( B^k \) are set-valued mappings from \( \mathbb{R}^n \) into \( \mathbb{R}^n \) and \( \mathbb{R}^{nk} \) respectively, and \( K \) is a fixed positive integer.
Let $S([x(k)])$ be a single-valued mapping from $R^{n(k+1)}$ into $R^m$, $M$ a convex cone in $R^m$ ($M \neq R^m$).

A sequence
\[ \{x(k), \ k = 0, 1, ..., K \} \]  
(4.3)
satisfying the constraints (4.1), (4.2) is said to be an admissible trajectory. An admissible trajectory
\[ \{x_0(k), \ k = 0, 1, ..., K \} \]  
(4.4)
is said to be optimal if, for any admissible trajectory (1.3), the condition $S([x_0(k)]) - S([x(k)]) \subseteq M$ implies that $S([x(k)]) - S([x_0(k)]) \subseteq M$.

Assume that
\[
A^k(x) = A^k_1(x) \cap A^k_2(x), \\
B^k(x) = B^k_1(x) \cap B^k_2(x)
\]
where the set-valued mappings $A^k_1$ and $B^k_1$ satisfy the following conditions:

1. The mappings $A^k_2$ and $B^k_2$ have local sections; $A^k_2(x)$ and $B^k_2(x)$ are convex sets for every $x \in R^n$.

2. For every $x$, $A^k_1(x)$ and $B^k_1(x)$ are convex compact sets with non-empty interiors.

3. The functions $c^k_1(\psi, x), c^k_2(\varphi, x)$ are continuous (in their variables) together with their partial derivatives (with respect to $x$).

Let $\Delta_i$ be a compact topological space, $l^i(k, x)$ a smooth function, $h_i(k, x, \alpha)$ a scalar-valued function, which is defined and is continuous together with its partial derivatives $\frac{\partial}{\partial x_j} h_i(k, x, \alpha), j = 1, 2, ..., n$, on the direct product $X \times \Delta_i$.

Define the mapping $S^n: R^{n(k+1)} \rightarrow R^{m_2}$ and $S^{n'}: R^{n(k+1)} \rightarrow R^{m_3}$ by the formulae
\[
S^n = \left( S^n_1, S^n_2, ..., S^n_{m_2} \right), \quad S^{n'} = \left( S^{n'}_1, S^{n'}_2, ..., S^{n'}_{m_3} \right),
\]
\[
S^n_1([x(k)]) = \max_{\alpha \in \Delta_i} \sum_{k=0}^K h_i(k, x(k), \alpha),
\]
\[
S^{n'}_1([x(k)]) = \max_{j=1, 2, ..., q_i} \sum_{k=0}^K l^i_j(k, x(k)).
\]

Assume that
1. \(S = (S', S'', S''')\) where \[S'(\{x(k)\}) = \sum_{k=0}^{K} Q(k, x(k))\]

\((Q(k, x)\) is a smooth function from \(\mathbb{R}^n\) into \(\mathbb{R}^{m_1}\)).

2. \(M = M' \times R_{+}^{n2} \times R_{+}^{n3}\) where \(M'\) is a given convex cone in \(\mathbb{R}^{n1}\).

Denote by \(\Delta_i(\{\tilde{x}(k)\})\) the set of all points \(\tilde{\alpha} \in \Delta_i\), at which the function \(\sum_{i=0}^{k} h_i(k, \tilde{x}(k), \alpha)\) of the variable \(\alpha \in \Delta_i\) attains its maximum.

**Theorem 4.1** Let (4.4) be an admissible trajectory; \(\sigma^k_2\) and \(\xi^k_2\) local sections of the mappings \(A^k_2\) and \(B^k_2\) corresponding to the pairs \((\tilde{x}(k), \tilde{x}(k+1))\) and \((\tilde{x}(k), 0)\) respectively. If the trajectory (4.4) is optimal, then there exist vectors \(\psi_i(k) \in \mathbb{R}^n\) \((i = 1, 2, k = 0, 1, ..., K - 1)\), \(\varphi_i(k) \in \mathbb{R}^n\) \((i = 1, 2; k = 0, 1, ..., K)\), \(\psi \in M'\), numbers \(\mu_i^j < 0\) \((j = 1, 2, ..., n + 1; i = 1, 2, ..., m_2)\), \(\nu_i^j < 0\) \((j = 1, 2, ..., q_i; i = 1, 2, ..., m_3)\) and points \(\alpha_i^j \in \Delta_i(\{\tilde{x}(k)\})\) \((j = 1, 2, ..., n + 1; i = 1, 2, ..., m_2)\), such that

(a') Not all of the quantities \(\psi_i(k) \in \mathbb{R}^n\) \((i = 1, 2, k = 0, 1, ..., K - 1)\), \(\varphi_i(k) \in \mathbb{R}^n\) \((i = 1, 2; k = 0, 1, ..., K)\), \(\psi \in \mathbb{R}^n\), \(\mu_i^j < 0\) \((j = 1, 2, ..., n + 1; i = 1, 2, ..., m_2)\), \(\nu_i^j < 0\) \((j = 1, 2, ..., q_i; i = 1, 2, ..., m_3)\) are zero.

(b') \(c^A_i \psi_i(k) \tilde{x}(k)) = \psi_i(k) \tilde{x}(k+1)\), \(i = 1, 2; k = 0, 1, ..., K - 1\), \(c^B_i \varphi_i(k) \tilde{x}(k)) = 0\), \(i = 1, 2, k = 0, 1, ..., K\).

(c') \(\nu_i^j \sum_{k=0}^{K} h_i(k, \tilde{x}(k)) - S''(\{\tilde{x}(k)\}) = 0\), \(j = 1, 2, ..., q_i; i = 1, 2, ..., m_3\).

(d') \(\psi_1(k-1) + \varphi_2(k-1) = \frac{\partial H(\psi_1(k), \psi_2(k), \varphi_1(k), \varphi_2(k), x)}{\partial x} \bigg|_{x = \tilde{x}(k)}^{(1)}\) \(k = 0, 1, ..., K\), where

\[H(\psi_1(k), \psi_2(k), \varphi_1(k), \varphi_2(k), x) = \psi_2(k, \sigma^k_2(x)) + \psi_2(k, \xi^k_2(x)) + \]
\[+ c^A_{i1} \psi_1(k, x) + c^B_{i1} (\psi_1(k), x) + \psi_1(k, \varphi_2(k, x)) + \]
\[+ \sum_{j=1}^{n+1} \sum_{i=1}^{m_2} \mu_i^j h_i(k, x, \alpha_i^j) + \sum_{i=1}^{m_3} \sum_{j=1}^{q_i} \nu_i^j(k, x).\]

(1) We set \(\psi_1(k) = 0\) if \(k = -1, K\).
Proof. We introduce the space $R^{n(k+1)}$ of the variable $\tilde{x} = (x(0), x(1), ..., x(K))$ and define the mappings $T_i, i = 1, 2$, by the formulae

$$T_i(\tilde{x}) = \prod_{k=0}^{K} [A_i^k(x(k)) - x(k+1)] \times \prod_{k=0}^{K} B_i^k(x(k)).$$

It is clear that Theorem 4.1 is an immediate consequence of Theorem 3.2.

Remark 4.1. Let functions $c^A_2(\psi_2(k), x)$ and $c^B_2(\varphi_2(k), x)$ be differentiable (with respect to $x$), where $\psi_2(k)$ and $\varphi_2(k)$ are the vectors mentioned in Theorem 4.1. Then the condition (d') can be replaced by

$$(d'') \frac{\partial H'(\psi_1(k), \psi_2(k), \varphi_1(k), \varphi_2(k), x)}{\partial x} \bigg|_{x=x(k)} = \psi_1(k-1) + \psi_2(k-1),$$

where $k = 0, 1, ..., K$.

$$H'(\psi_1(k), \psi_2(k), \varphi_1(k), \varphi_2(k), x) = \sum_{i=1}^{2} \left[ c^A_i(\psi_i(k), x) + c^B_i(\varphi_i(k), x) \right] +$$

$$+ \langle \psi, Q(k, x) \rangle + \sum_{j=1}^{n+1} \sum_{i=1}^{m_2} \mu_i^j h_i(k, x, \alpha_1^j) + \sum_{i=1}^{m_2} v_i^j 1_{i}^j(k, x).$$

Définition 4.1. We shall say that the trajectory (4.4) satisfies a support principle if there exist vectors $\psi_i(k)$ ($i = 1, 2; k = 0, 1, ..., K-1$), $\varphi_i(k)$ ($i = 1, 2; k = 0, 1, ..., K$), $\psi \in M'$; numbers $\mu_i^j \leq 0$ ($j = 1, 2, ..., n+1$; $i = 1, 2, ..., m_2$), $\nu_i^j (j = 1, 2, ..., q_i; l = 1, 2, ..., m_2)$; and points $\alpha_1^j \in \Delta_i(\bar{x}(k))$ ($j = 1, 2, ..., n+1$; $i = 1, 2, ..., m_2$) such that conditions (a') — (c'), (d'') hold.

From Theorem 4.1 and Remark 4.1 we have

Theorem 4.2. Assume, in addition to the already made assumptions, that the functions $c^A_2(\psi_2(k), x)$ and $c^B_2(\varphi_2(k), x)$ are differentiable with respect to $x$, where $\psi_2(k)$ and $\varphi_2(k)$ are the vectors mentioned in Theorem 4.1. Then the optimal trajectory (4.4) satisfies the support principle.

Remark 4.2. It follows from Theorem 4.2 that the support principle holds for a process without local sections [3] (i.e. for the case where mappings $A_2^k$ and $B_2^k$ are absent), as well as for a process with local sections [1,2] (i.e. for the case where mappings $A_1^k$ and $B_1^k$ are absent). It should be noted that the set $A_2^k(x)$ and $B_2^k$ are not assumed to be compact.

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Remark 4.3. From Theorem 3.2, we can also deduce the support principle for discrete time-lag processes, as well as for discrete distributed parameter systems [3].

Remark 4.4. Theorems 4.1 and 4.2 are also valid for the case where the mappings $S''$ and $S'''$ are absent, provided that the cone $M'$ is such that $M' \neq \{0\}$ and $M' \cap (-M') = \{0\}$.

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