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A SUPPORT PRINCIPLE FOR A DISCRETE INCLUSION WITH A VECTOR-VALUED CRITERION FUNCTION

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It has been shown by Boltianskii [1, 2] that, for a process described by a differential or discrete inclusion, a point which achieves a minimum of a given scalar-valued criterion function will satisfy the support principle [1, 2], if the process under consideration has local sections [1, 2]. The case where local sections do not exist has been studied by the author in [3]. It should be noted that the papers [1 — 3] deal with scalar minimization problems. In this paper, we shall prove the support principle for a discrete inclusion with a vector-valued criterion function. From the obtained results, it will be easy to derive the support principle for the case where local sections exist as well as for the case where they do not. Also, the discrete minimax problem [2] is included as a special case in our theorem. Our proof of the support principle is based on a general theory of inconsistency of a system of inclusions which is presented (without proofs) in the paragraph §2. It will be easy to see that our method can also be applied to discrete time-lag systems, and discrete distributed parameter systems [3].

§1. NOTATIONS AND DEFINITIONS

Let X, Y be two vector topological spaces; A, B two subsets of X; T set-valued mapping carrying points of A into non-empty subsets of Y, and k a positive integer. The following notations will be used:

R^k: k-dimensional Euclidean space.

$R^k_+ = \{x = (x_1, x_2, \ldots, x_k) \in R^k : x_i \geqslant 0,\ i = 1, 2, \ldots k\}$: nonnegative orthant in R^k.

$\mathring{R}^k_+ = \{x = (x_1, x_2, \ldots, x_k) \in R^k : x_i > 0,\ i = 1, 2, \ldots k\}$: positive orthant in R^k.

X^*: set of all linear continuous functionals defined on X.

$\langle x^*, x\rangle$: value of $x^* \in X^*$ at the point $x \in X$.

2^X: set of all non-empty subsets of X.

$V_X, V_X^1, \dots$: neighborhoods of $0 \in X$.

$int\ A$: interior of A.

$A \,\underline{*}\, B = \{x \in X : x + A \subset B\}$ (1).

$T(A) = \bigcup_{x \in A} T(x)$: range of T.

$c^T(y^*, x) = sup\ \{\langle y^*, y\rangle : y \in T(x)\}\ (y^* \in Y^*, x \in A)$.

$graph\ T = \{(x, y) : x \in A, y \in T(x)\}$: *graph of* T.

$int\, T$: mapping $x \mapsto int\ T(x) \quad (x \in A)$.

If $A = A' \times \Omega$, the direct product then, for fixed $\omega \in \Omega$, $T(.,\omega)$ is a mapping defined by $x' \mapsto T(x', \omega)\ (x' \in A')$.

Given k set-valued mappings $T_i : D \to 2^{Y_i}$, $i = 1, \dots, k$, we denote by $\prod_{i=1}^{k} T_i$ the mapping

$$x \mapsto \prod_{i=1}^{k} T_i(x) = T_1(x) \times T_2(x) \times \dots \times T_k(x)$$

Definition 1.1. Given a set I of indices and mappings $T_\varepsilon : A \to 2^{Y'}$, $\varepsilon \in I$, we shall say that the family $\{T_\varepsilon, \varepsilon \in I\}$ is uniformly upper semi-continuous (uniformly lower semi-continuous) if for every $\hat{x} \in A$ and every neighborhood V_Y there exists a neighborhood V_X such that

$$T_\varepsilon(x) \subset T_\varepsilon(\hat{x}) + V_Y$$

$$(T_\varepsilon(\hat{x}) \subset T_\varepsilon(x) + V_Y)$$

whenever $\varepsilon \in I$, $x \in (\hat{x} + V_X) \cap A$.

The family $\{T_\varepsilon, \varepsilon \in I\}$ is said to be uniformly continuous, if it is both uniformly upper and uniformly lower semi-continuous.

Definition 1.2. A mapping $T : A \to 2^Y$ is upper semi-continuous if the family $\{T_\varepsilon\}$ consisting of a single mapping $T_\varepsilon = T$ is uniformly upper semi-continuous. Similarly, one can define lower semi-continuity, as well as continuity, of T.

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(1) The operation $\underline{*}$ was first introduced by Pontryagin and plays an important role in the theory of linear differential games.

Definition 1.3. A mapping $T: A \to 2^Y$ is closed if the conditions $x_n \to \dot{x}$ $(x_n, x \in D)$, $y_n \to y$ and $y_n \in T(x_n)$ imply that $y \in T(x)$.

Definition 1.4. A mapping $T: A \to 2^Y$ is convex if its graph is a convex set in $X \times Y$.

§2. NECESSARY CONDITIONS FOR THE INCONSISTENCY OF A SYSTEM OF INCLUSIONS.

Let X, Y_i, $i = 1-3$, be vector topological spaces, D, E subsets in X, $T_i: D \to 2^{Y_i}$, $t_i: E \to 2^{Y_i}$ set-valued mappings. Let there be given a point $x_0 \in X$ and a single-valued mapping $g: X \times \mathring{R}^1_+ \times X \to X$.

Definition 2.1. The pair (E, t_1) will be called a $(g, \underline{*})$-tangent approximation at x_0 of the pair (D, T_1) if there exists a positive number a_1 and mappings $Q: (0, a_1) \times D \times E \to 2^{Y_1}$, $f_1: (0, a_1) \to \mathring{R}^1_+$ such that

a) The family

$$\{Q_{\varepsilon, x}, (\varepsilon, x) \in I\} \tag{2.1}$$

with $Q_{\varepsilon, x} = Q(\varepsilon, x, .)$, $I = (0, a_1) \times D$ is uniformly continuous.

b) For every $\varepsilon \in (0, a_1)$ the mapping $Q(\varepsilon, ., .)$ is closed.

c) For every $(\varepsilon, x, \xi) \in (0, a_1) \times D \times E$ the set $Q(\varepsilon, x, \xi)$ is convex and satisfies the condition

$$Q(\varepsilon, x, \xi) \subset \frac{T_1(x)}{f_1(\varepsilon)} \underline{*}\, t_1(\xi). \tag{2.2}$$

d) For every $\hat{x} \in E$ and every neighborhood V_{Y_1} there exist a number $\delta \in (0, a_1)$ and a neighborhood V_x such that

$$Q(\varepsilon, x, \hat{x}) \cap V_{Y_1} \neq \phi \tag{2.3}$$

whenever

$$0 < \varepsilon < \delta, \quad x \in g(x_0, \varepsilon, \hat{x} + V_x) \cap D. \tag{2.4}$$

Example 2.1. Let T_1, t_1 be mappinggs defined by

$$T_1(x) = F(x) + N, \quad t_1(x) = f(x) + N \tag{2.5}$$

where $F: D \to Y_1$, $f: E \to Y_1$ are single-valued continuous mappings, N is an arbitrary cone.

Assume that for every $\hat{x} \in E$ and every neighborhood V_{Y_1} there exist a number $\delta > 0$ and a neighborhood V_x such that

$$F(x) \in \varepsilon\, (f(\hat{x}) + V_{Y_1}) \tag{2.6}$$

whenever

$$0<\varepsilon<\delta, \qquad x \in \left[x_0+\varepsilon(\widehat{x}+V_X)\right] \cap D. \tag{2.7}$$

Then the pair (E, t_1) is a $(g_1, *)$-tangent approximation at x_0 of the pair (D, T_1), where

$$g_1(\xi, \varepsilon, x) = \xi + \varepsilon x. \tag{2.8}$$

T ·rove this, it suffices to take $f_1(\varepsilon) = \varepsilon$, $Q(\varepsilon, x, \xi) = \frac{1}{\varepsilon} F(x) - f(\xi)$.

Note that Condition (2.6) holds, provided that X and Y_1 are Banach spaces, $F(x) = \widetilde{F}(x) - \widetilde{F}(x_0)$, $f(x) = \widetilde{F}'_{x_0}(x)$, where $\widetilde{F}: X \to Y_1$ is a Frechet differentiable mapping and $\widetilde{F}'_{x_0}$, its Frechet derivative at x_0.

Example 2.2. Let the mappings T_1, t_1 be defined as in Example 2.1., but let N be a closed convex set. Assume that for every $\widehat{x} \in E$ and every neighborhood V_{Y_1} there exist a number $\delta > 0$ and a neighborhood V_X such that

$$\left[\frac{1}{\varepsilon}(F(x) + (1-\varepsilon)N) - f(\widehat{x})\right] \cap V_{Y_1} \neq \phi \tag{2.9}$$

for all (ε, x) satisfying (2.7). Then the assertion in the previous example is true. To prove this, choose $a_1 = 1$, $f_1(\varepsilon) = \varepsilon$, $Q(\varepsilon, x, \xi) = \frac{1}{\varepsilon}(F(x) + (1-\varepsilon)N) - f(\xi)$.

Definition 2.2. A pair (E, t_2) will be called a g-tangent approximation at x_0 of the pair (D, T_2) if the following conditions are fulfilled:

a. The mapping T_2 is closed, and for every $x \in D$ the set $T_2(x)$ is convex.

b. There exist a number $a_2 > 0$ and a mapping $f_2 : (0, a_2) \to \mathring{R}^1_+$ such that for every $\widehat{x} \in E$ and every neighborhood V_{Y_2} there exist a number $\delta \in (0, a_2)$ and a neighborhood V_X such that

$$t_2(\widehat{x}) \subset \frac{T_2(x)}{f_2(\varepsilon)} + V_{Y_2}$$

for all (ε, x) satisfying the conditions (2.4).

Remark 2.1. Let (E, t_1) be a $(g, *)$-tangent approximation at x_0 of the pair (D, T_1). Suppose that T_1 is closed and, for every $x \in D$, the set $T_1(x)$ is convex. Then the pair (E, t_1) is also a g-tangent approximation at x_0 of the pair (D, T_1).

Example 2.3. Let the mappings T_1 and t_1 be defined as in Example 2.2. Then the pair (E, t_1) is a g_1-tangent approximation at x_0 of the pair $(D. T_1)$.

Definition 2.3. (see [1,2]). Given an arbitrary mapping $T: X \to 2^Y$, where X and Y are two normed spaces. we shall say that T has local sections if for every pair $(x_0, y_0) \in graph\ T$ there exists a smooth(1) function σ from a neighborhood of x_0 into Y such that $\sigma(x_0)=y_0$ and $\sigma(x) \in T(x)$. The function σ will be called a local section of T, corresponding to the pair (x_0, y_0).

Example 2.4. Let X, Y_2 be normed spaces, $T_2: X \to 2^{Y_2}$ a closed mapping such that

1. For every x the set $T_2(x)$ is convex and compact.
2. $0 \in T_2(x_0)$.
3. T_2 has local sections.

Suppose σ is a local section of T_2 corresponding to the pair $(x_0, 0)$ and σ'_{x_0} is its Frechet derivative at x_0. It can be shown that the pair (X, t_2), where $t_2(x) = T_2(x_0) + \sigma'_{x_0}(x)$ is a g_1-tangent approximation at x_0 of the pair (X, T_2).

Definition 2.4. A pair (E, t_3) will ba called a g-interior approximation at x_0 of the pair (D, T_3) if for every solution $\hat{x}$ of the system

$$x \in E,\ 0 \in t_3(x) \tag{2.10}$$

there exist a number $\delta > 0$ and a neighborhood V_x such that

$$0 \in T_3(x) \tag{2.11}$$

for all (ε, x) satisfying the conditions (2.4).

Example 2.5. Let X, Y_3 be Banach spaces, N a set in Y_3, $h: X \times Y_3 \to R^1$ a functional which is Frechet differentiable at $(x_0, y_0) \in X \times N$. Assume that $h(x_0, y_0) \leqslant 0$, $(h'_x, h'_y) \neq 0$, where h'_x and h'_y are the partial derivatives of f at (x_0, y_0). Assume further that $\tilde{D} \neq \phi$, $\tilde{N} \neq \phi$ where $\tilde{D}$ $(\tilde{N})$ is the set of all $x \in X (y \in Y_3)$ for each of which there exist a number $\delta > 0$ and a neighborhood $V_x (V_{Y_3})$ such that $x_0 + \varepsilon(x + V_x) \subset D$ $(y_0 + \varepsilon(y + V_{Y_3}) \subset N)$ whenever $0 < \varepsilon < \delta$.

Let us set

$$E = \tilde{D},$$

$$T_3(x) = \{y \in Y_3 : h(x, y) \leqslant 0\} - N,$$

$$t_3(x) = \begin{cases} Y_3 & \text{if } h(x_0, y_0) < 0 \\ -\tilde{N} + \{y \in Y_3 : h'_x(x) + h'_y(y) < 0\} & \text{if } h(x_0, y_0) = 0. \end{cases}$$

Under these assumptions it is readily verified that the pair (E, t_3) is a g_1-interior approximation at x_0 of the pair (D, T_3).

Example 2.6. Let X, Y_3 be normed spaces, $T_3: D \to 2^{Y_3}$ a lower semi-continuous mapping such that

1. For every x, $T_3(x)$ is a convex set with a non-empty interior.
2. $0 \in T_3(x_0)$.
3. T_3 has a local section σ, corresponding to the pair $(x_0, 0)$.

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(1) i. e σ is continuous together with its Frechet derivative in a neighborhood of x_0.

Let us set

$$t_3(x) = \begin{cases} int\, T_3(x_0) + \sigma'_{x_0}(x) & \text{if } 0 \overline{\in} int\, T_3(x_0), \\ Y_3 & \text{if } 0 \in int\, T_3(x_0), \end{cases}$$

where σ'_{x_0} is the Frechet derivative of σ at x_0. One can show that the pair (X, t_3) is a g_1-interior approximation at x_0 of the pair (D, T_3).

Definition 2.5. The pair (E, t_3) will be called a g-quasiinterior approximation at x_0 of the pair (D, T_3) if for every solution $\hat{x}$ of the system (2.10) and every neighborhood V_{Y_3} there exist numbers δ, $a_3 > 0$ $(\delta < a_3)$, a mapping $f_3 : (0, a_3) \to \overset{\circ 1}{R}_+$ and a neighborhood V_X such that

$$\left[\frac{1}{f_3(\varepsilon)} T_3(x) \underline{*}\, t_3(\hat{x})\right] \cap V_{Y_3} \neq \Phi \tag{2.11}$$

for all (ε, x) satisfying conditions (2.4).

Example 2.7. Suppose that the pair (E, t_3) is a g-quasiinterior approximation at x_0 of the pair (D, T_3). Suppose further that $int\, t_3(x) \neq \phi$ for all $x \in E$. Then the pair $(E, int\, t_3)$ is a g-interior approximation at x_0 of the pair (D, T_3).

Example 2.8. Let there be given a number $\alpha \in (0, 1]$, a twice differentiable mapping $F : R^n \to R^m$ a convex set $E \subset R^n$ and a convex cone $N \subset R^m$. Denote by $dF(x_0, \tilde{x})$ and $d^2F(x_0, \tilde{x}, \tilde{x})$ the first and second differentials of F at x_0.

Assume that

$$-F(x_0) \in N \underline{*} N, \qquad -dF(x_0, \tilde{x}) \in N \underline{*} N.$$

Then

1. The pair $(X, \hat{f}(\cdot, \alpha))$ is a $(g, \underline{*})$-tangent approximation at x_0 of the pair $(X, \hat{F})$, where

$$\hat{f}(x, \alpha) = f(x, \alpha) + N,$$

$$f(x, \alpha) = 2[F(x_0) + dF(x_0, \tilde{x})] + \alpha^2[d^2F(x_0, \tilde{x}, \tilde{x}) + dF(x_0, x)],$$

$$g(\xi, \varepsilon, x) = \xi + \varepsilon \tilde{x} + \frac{\varepsilon^2}{2} x,$$

$$\hat{F}(x) = F(x) + N.$$

2. The pair $(X, \hat{f}(\cdot, \alpha))$ is a g-tangent approximation at x_0 of the pair $(X, \hat{F})$ provided the cone N is closed.

3. The pair $(X, int\, \hat{f}(\cdot, \alpha))$ is a g-interior approximation at x_0 of the pair $(X, int\, \hat{F})$ if $int\, N \neq \phi$, $0 \overline{\in} int\, \hat{F}(x_0)$, $0 \overline{\in} int\, (dF(x_0, \tilde{x}) + N)$.

Definition 2.6. A 4-tuple

$$(E, t_1, t_2, t_3) \tag{2-12}$$

will be called a g-approximation at x_0 of the 4-tuple

$$(D, T_1, T_2, T_3) \tag{2.13}$$

if the pair (E, t_1) is a $(g, \_)$-tangent approximation at x_0 of the pair (D, T_1), the pair (E, t_2) is a g-tangent approximation at x_0 of the pair (D, T_2), and the pair (E, t_3) is a g-interior approximation at x_0 of the pair (D, T_3).

Definition 2.7. We shall say that a g-approximation (2.12) is convex, if all the mappings t_i, $i = 1-3$, are convex.

Let us set

$$Y_{12} = Y_1 \times Y_2, \qquad t_{12} = t_1 \times t_2.$$

Definition 2.8. We shall say that a g-approximation (2.12) is regular, if there exist a neighborhood $V_{Y_{12}}$ and a continuous mapping $h: V_{Y_{12}} \to E$ such that

a. For all $\sigma \in V_{Y_{12}}$ we have

$$\sigma \in t_{12}(h(\sigma)),$$
$$0 \in t_3(h(\sigma)).$$

b. For every number $\delta > 0$ and every neighborhood V_x there exist a number $\varepsilon \in (0, \delta)$ and a continuous mapping $\zeta_\varepsilon : V_{Y_{12}} \to D$ such that

$$\zeta_\varepsilon(\sigma) \in g(x_0, \varepsilon, h(\sigma) + V_x)$$

for all $\sigma \in V_{Y_{12}}$.

Throughout the subsequent part of this paragraph, we shall assume that X, Y_3 are vector topological spaces, and $Y_i = R^{k_i}$, $i = 1, 2$.

The following theorem gives necessary conditions for the inconsistency of a given system of inclusions.

Theorem 2.1. Assume that the 4-tuple (2.12) is a regular g-approximation at x_0 of the 4-tuple (2.13). Then if the system

$$x \in D, \quad 0 \in T_i(x), \quad i = 1-3, \tag{2.14}$$

is inconsistent (i.e. has no solutions), so is the system:

$$x \in E, \quad 0 \in t_i(x), \quad i = 1-3. \tag{2.15}$$

Put
$$k = k_1 + k_2.$$

Definition 2.9. A set $E \subset X$ is said to be $(g, k+1)$-contingent to a set $D \subset X$ at a given point $x_0 \in X$ if for every system of $(k+1)$ elements $x_1, x_2, \ldots, x_{k+1}$ of E, every number $\delta > 0$ and every neighborhood V_x there exist a number $\varepsilon \in (0, \delta)$ and a continuous mapping $\eta_\varepsilon : P^k \to D$ such that

$$\eta_\varepsilon(\lambda) \in g(x_0, \varepsilon, \sum_{i=1}^{k+1} \lambda_i x_i + V_x)$$

for all $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{k+1}) \in P^k$, where

$$P^k = \left\{ \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{k+1}) : \lambda_i \geqslant 0,\ i = 1, 2, \ldots, k+1;\ \sum_{i=1}^{k+1} \lambda_i = 1 \right\}.$$

Example 2.9. Suppose that X, Y are normed spaces and $F: X \to 2^Y$ is a set-valued mapping. Suppose further that F has local sections, and that for every $x \in X$ the set $F(x)$ is convex. Consider a point $(x_0, y_0) \in graph F$ and define a mapping $f: X \to 2^Y$ by the formula $f(x) = F(x_0) - y_0 - \sigma'_{x_0}(x)$, where σ'_{x_0} is the Frechet derivative of the local section σ of F corresponding to the pair (x_0, y_0). If $E \subset X$ is $(g_1, k+1)$-contingent to $D \subset X$ at x_0, then $graph\ \widetilde{f}$ is also $(g_1, k+1)$-contingent to $graph\ \widetilde{F}$ at (x_0, y_0). Here $\widetilde{f}$ and $\widetilde{F}$ are the restrictions of f and F to E and D respectively.

Proof. Let $(x_i, y_i) \in graph\ \widetilde{f}$, $i = 1, 2, \ldots, k+1$, V_X, V_Y and $\delta > 0$ be given. Select a ball V'_Y such that $(k+3) V'_Y \subset V_Y$. Let σ_i be a local section of F, corresponding to the pair $(x_0, \widetilde{y}_i)$ with $\widetilde{y}_i = y_i + y_0 - \sigma'_{x_0}(x_i)$ and let σ'_i be the Frechet derivative of σ_i at x_0. We can find a positive number δ' such that

$$\varepsilon \sum_{i=1}^{k+1} \lambda_i \left[-\sigma'_{x_0}(x_i) + \sigma'_i \left(\sum_{n=1}^{k+1} \lambda_n x_n \right) \right] \in V'_Y$$

whenever

$$0 < \varepsilon < \delta', \quad \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{k+1}) \in P^k.$$

It follows from Definition 2.3. that there exist a positive number δ'' and a neighborhood $V'_X \subset V_X$ such that

$$\sigma_i(x) \in \sigma_i(x_0) - \varepsilon \left[\sigma'_i \left(\sum_{n=1}^{k+1} \lambda_n x_n \right) \right] + V'_Y \Big]$$

for all $i = 0, 1, \ldots, k+1$,

$$0 < \varepsilon < \delta'', \quad \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{k+1}) \in P^k;$$

$$x \in x_0 + \varepsilon \left(\sum_{i=1}^{k+1} \lambda_i x_i + V'_X \right),$$

where $\sigma_0 = \sigma$, $\sigma'_0 = \sigma'_{x_0}$.

Since E is $(g_1, k+1)$-contingent to D at x_0, we can find a positive number $\varepsilon < \min(\delta, \delta', \delta'', 1)$ and a mapping $\eta^1_\varepsilon: P^k \to D$ such that

$$\eta^1_\varepsilon(\lambda) \in x_0 + \varepsilon \left(\sum_{i=1}^{k+1} \lambda_i x_i + V'_X \right)$$

for all $\lambda \in P^k$. Now define a mapping $\eta_\varepsilon: P^k \to graph\ \widetilde{F}$ by setting $\eta_\varepsilon(\lambda) = (\eta^1_\varepsilon(\lambda), \eta^2_\varepsilon(\lambda))$ where

$$\eta^2_\varepsilon(\lambda) = (1-\varepsilon)\, \sigma(\eta^1_\varepsilon(\lambda)) + \varepsilon \sum_{i=1}^{k+1} \lambda_i \sigma_i(\eta^1_\varepsilon(\lambda)).$$

It is clear that the number ε and the mapping η_ε have the properties required in Definition 2.9.

Corollary 2.1. Assume that the 4-tuple (2.12) is a g-approximation at x_0 of the 4-tuple (2.13), where

1. E is $(g, k+1)$-contingent to D at x_0.

2. The mappings t_i $i = 1,2$, are convex and satisfy the condition $0 \in int\ t_{12}(E)$.

3. For every $x \in E$, $t_3(x)$ is a convex set with a non-empty interior.

Assume further that at least one of the following conditions is fulfilled.

1. The mapping t_3 is convex.

2. The mapping t_3 is of the form $t_3(x) = q(x) + N_3$, where N_3 is a cone in Y_3, $q: E \to 2^{Y_3}$ a lower semi-continuous mapping such that, for every $x \in Y$, $q(x)$ is a bounded set.

3. The mapping t_3 is lower semi-continuous, and Y_3 is a normed space.

Then if the system (2.14) is inconsistent, so is the system

$$x \in E, \quad 0 \in t_i(x), \quad i = 1,2, \quad 0 \in int\ t_3(x).$$

As an immediate consequence of Corollary 2.1 and Lemma 2.1 stated below, we have

Corollary 2.2. Assume that the 4-tuple (2.12) is a convex g-approximation at x_0 of the 4-tuple (2-13) where.

1. E is $(g, k+1)$-contingent to D at x_0.

2. For every $x \in E$, $t_3(x)$ has a non-empty interior.

Then if the system (2.14) is inconsistent, there exist linear continuous functionals $y_i^* \in Y_i^*$, $i = 1-3$, not all zero, such that, for all $x \in E$,

$$\sum_{i=1}^{3} c^{t_i}(y_i^*, x) \leqslant 0. \tag{2.16}$$

Remark 2.2. Corollary 2.2 includes as special cases the main results obtained by Halkin and Neustadt in [4] and [5] respectively.

Lemma 2.1. Assume that, for every $x \in E$, $t_3(x)$ is a convex set with a non-empty interior. Assume further that $t_{123}(E)$ is convex, where $t_{123}=t_1\times t_2 \times int\ t_3$ Then, if the system

$$x \in E, \quad 0 \in t_i(x), \ i = 1,2, \ 0 \in int\ t_3(x)$$

is inconsistent, there exist linear continuous functionals $y_i^* \in Y_i^*$, $i = 1-3$. not all zero, such that, for all $x \in E$, we have the inequality (1.16).

As a consequence of Corollary 2.2, we shall obtain a multiplier rule for an optimization problem with a vector-valued criterion function.

Definition 2.10. Let M be a convex cone in vector topological space Z $(M \neq Z)$, Q a subset in X, S a singlevalued mapping from Q into Z. A point

$x_0 \in Q$ is said to be M-extremal (for S) if, for every $x \in Q$, the inclusion $S(x_0) - S(x) \in M$ implies that $S(x) - S(x_0) \in M$.

We shall state necessary conditions that every M-extremal point must satisfy provided certain hypotheses are fulfilled. It should be pointed out that the Theorem 2.2 stated below differs from known results (see, for example, [5-7]) in that the cone M may have an empty interior and the constraints under consideration can be given by set-valued mappings.

Assume that

(a). Q is the set of all solutions of the system (2.14).

(b). $$Z = \prod_{i=1}^{3} Z_i\ , \qquad S = \prod_{i=1}^{3} S_i\ , \qquad M = \prod_{i=1}^{3} M_i$$

where

$$Z_i = R^{m_i}, \qquad i = 1,2$$

Z_3 is a vector topological space, $S_i : D \to Z_i$, $i = 1-3$, are single-valued mappings, $M_i \subset Z_i$, $i = 1-3$, are convex cones, and M_3 has a non-empty interior.

(c) If $M_3 = Z_3$, then $M_1 \neq \{0\}$ and $M_1 \cap (-M_1) = \{0\}$

(d) There exist a set $E \subset X$ and mappings

$$g : X \times \mathring{R}^1_+ \times X \to X, \quad t_i : E \to 2^{Y_i}, \quad s_i : E \to Z_i\ , \quad i = 1-3,$$

such that

(d1). E is $(g, k+1)$-contingent to D at x_0,

where $$k = k_1 + k_2 + m_1' + m_2',$$

$$m_i' = \begin{cases} m_i & \text{if } M_i \neq Z_i\,, \\ 0 & \text{if } M_i = Z_i. \end{cases}$$

(d2). The 4-tuple (2.12) is a convex g-approximation at x_0 of the 4-tuple (2.13).

(d3). *int* $t_3(x) \neq \phi$ for every $x \in E$.

(d4). Mappings s_i, $i = 1-3$, are $(M_i \,\underline{*}\, M_i)$-convex and the 4-tuple $(E, \tilde{s}_1, \tilde{s}_2, \tilde{s}_3)$ is a g-approximation at x_0 of the 4-tuple $(D, \tilde{S}_1, \tilde{S}_2, \tilde{S}_3)$, where the mappings $\tilde{s}_i$, $\tilde{S}_i$ are defined by the formulae

$$\tilde{s}_i(x) = s_i(x) + \tilde{M}_i\,,$$

$$\tilde{S}_i(x) = S_i(x) - S_i(x_0) + \tilde{M}_i\,,$$

$$\tilde{M}_i = \begin{cases} M_1 & \text{if} \quad i = 1, \quad M_3 \neq Z_3; \\ M_1 \setminus \{0\} & \text{if} \quad i = 1, \quad M_3 = Z_3; \\ M_2 & \text{if} \quad i = 2, \\ \text{int } M_3 & \text{if} \quad i = 3. \end{cases}$$

Recall that a single-valued mappings from a set $E \subset X$ into a linear space Y is called N-convex if E is convex and if, also,

$$\alpha s(x_1) + (1 - \alpha)\, s(x_2) \in s(\alpha x_1 + (1 - \alpha)x_2) + N$$

for all $\alpha \in [0, 1]$ and $x_i \in E$, $i = 1, 2$. Here N is a given convex cone in Y.

Theorem 2.2. Assume that conditions (a) — (d) hold. If x_0 is an M-extremal point, then there exist linear continuous functionals $y_i^* \in Y_i^*$, $z_i \in M_i^-$, $i = 1-3$, not all zero, such that

$$\sum_{i=1}^{3} \{ c^{t_i}(y_i^*, x) + \langle z_i^*, s_i(x)\rangle\} \leqslant 0 \tag{2.17}$$

for all $x \in E$. Furthermore, we have

$$c^{T_i}(y_i^*, x_0) = 0, \qquad i = 1-3,$$

if we can find a point $\xi_0 \in E$ such that

$$t_i(\xi_0) = T_i(x_0), \quad s_i(\xi_0) = 0, \quad i = 1-3.$$

Recall that

$$M_i^- = \{ z_i^* \in Z_i^* : \langle z_i^*, z\rangle \leqslant 0 \text{ for all } z \in M_i\}.$$

Theorem 2.3. Assume that conditions (a)—(c) hold. Assume further that there are an arbitrary set Ω and mappings $\widehat{t}_i$: $E\times\Omega \to 2^{Y_i}$, $\widehat{s}_i$; $E \times \Omega \to Y_i$, $i = 1-3$, such that

1. $0 \in int\, q_{12}(E, \omega)$ for every fixed $\omega \in \Omega$ where

$$q_{12}(x, \omega) = \prod_{i=1}^{2} q_i(x, \omega),$$

$$q_i(x, \omega) = \begin{cases} \widehat{t}_i(x, \omega) \times \{\widehat{s}_i(x, \omega) + \widetilde{M}_i\} & \text{if } M_i \neq Z_i, \\ \widehat{t}_i(x, \omega) & \text{if } M_i = Z_i. \end{cases}$$

2. $q(E\times\Omega)$ is a convex set, where

$$q = q_1 \times q_2 \times int\ q_3.$$

3. For every fixed $\omega \in \Omega$ we can find a mapping $g_\omega : X \times \mathring{R}_+^1 \times X \to X$ such that the conditions (d1)—(d4) hold if g, t_i and s_i are replaced by, g_ω, $\widehat{t}_i(.,\omega)$ and $\widehat{s}_i(.,\omega)$, respectively.

Finally, assume that x_0 is an M-extremal point. Then there exist linear continuous functionals $y_i^* \in Y_i^*$, $z_i^* \in M_i^-$, $i = 1-3$, not all zero, such that

$$\sum_{i=1}^{3} \{c^{\widehat{t}_i}(y_i^*, \widehat{x}) + \langle z_i^*, \widehat{s}_i(\widehat{x})\rangle\} \leqslant 0$$

for all $\widehat{x} = (x, \omega) \in E \times \Omega$. Furthermore, conditions (2.17) are also fulfilled, if we can find a point $(\xi_0, \omega_0) \in E \times \Omega$ such that

$$\widehat{T}_i(\xi_0, \omega_0) = T_i(x_0), \ \widehat{s}_i(\xi_0, \omega_0) = 0, \qquad i = 1-3.$$

§3. A MULTIPLIER RULE FOR CONSTRAINED OPTIMIZATION PROBLEMS WITH VECTOR-VALUED CRITERIA IN FINITE-DIMENSIONAL SPACES.

The purpose of this paragraph is to prove a multiplier rule for a contrained optimization problem, from which the support principle in discrete inclusions follows directly.

Assume that

$$Q = \{x \in X : 0 \in F_i(x), \quad i = 1, 2\},$$

where F_i, $i = 1, 2$, are set-valued mappings from $X = R^n$ into R^{k_i}.

Assume further that

(1) The mapping F_1 is such that:

(a) For every $x \in X$, $F_1(x)$ is a convex compact subset with a non-empty interior.

(b) Functions $c^{F_1}(\psi, x)$ and $\frac{\partial}{\partial x} c^{F_1}(\psi, x)$ are continuous in all of their variables.

(2) The mapping F_2 has local sections, and, for every $x \in X$, the set $F_2(x)$ is convex.

(3) $M = M' \times M''$, where $M' \subset R^{m_1}$ is an arbitrary convex cone, and $M'' = R^{m_2}_+$.

(4) $S = (S', S'')$, where $S' : X \to R^{m_1}$ is a differentiable mapping, and S'': $X \to R^{m_2}$ is defined by the formulae

$$S''(x) = (S''_1(x), S''_2(x), \ldots, S''_{m_2}(x)),$$

$$S''_i(x) = \max_{\alpha \in \Delta_i} h_i(x, \alpha), \ i = 1, 2, \ldots, m_2$$

(Δ_i is a compact topological space, $h_i(x, \alpha)$ is a function which is defined and is continuous together with its partial derivatives $\frac{\partial}{\partial x^j} h_i(x, \alpha)$, $j = 1, 2, \ldots, n$, on the direct product $X \times \Delta_i$).

Let

$$M'^- = \{\psi \in R^{m_1} : \langle \psi, y \rangle \leqslant 0 \text{ for all } y \in M'\},$$

$$\Delta_i(x_0) = \{\widetilde{\alpha} \in \Delta_i : h_i(x_0, \widetilde{\alpha}) = \max_{\alpha \in \Delta_i} h_i(x_0, \alpha)\},$$

It is clear that $\Delta_i(x_0) \neq \phi$.

Theorem 3.1. Assume that the conditions (1) — (4) hold. Let x_o be a point in Q, σ_2 a local section of F_2 corresponding to the pair $(x_o, 0)$. If x_o is an M-extremal point for S, then there exist vectors $\psi_i \in R^{k_i}$, $i = 1, 2$. $\psi \in M'^-$; numbers $\mu_i^j \leqslant 0$, $j = 1, 2, \dots, n+1$, $i = 1, 2, \dots, m_2$; and points $\alpha_i^j \in \Delta_i(x_o)$, $j = 1, 2, \dots, n+1$; $i = 1, 2, \dots m_2$. such that

(a) Not all of the quantities ψ_i, $i = 1, 2$, ψ and μ_i^j, $j = 1, 2, \dots, n+1$, $i = 1, 2, \dots, m_2$; are zero.

(b) $c^{F_i}(\psi_i, x_o) = 0$, $i = 1, 2$ (3.1)

c) $\frac{\partial}{\partial x} H(x_o) = 0,$ (3.2)

where

$$H(x) = c^{F_1}(\psi_1, x) + \langle \psi_2, \sigma_2, (x) \rangle + \langle \psi, S'(x) \rangle + \sum_{j=1}^{n+1} \sum_{i=1}^{m_2} \mu_i^j h_i(x_o, \alpha_i^j).$$

Proof. By using the separation theorem, we can show that $0 \in F_1(x)$ if and only if

$$f_1(x) = \max_{\alpha_o \in \Delta_o} \{-c^{F_1}(\alpha_o, x)\} \leqslant 0,$$

where

$$\Delta_o = \{\alpha \in R^{k_1} : \| \alpha \| = 1\}.$$

Denote by D the graph of the mapping F_2:

$$D = graph F_2 = \{\eta = (x, \xi) \in R^n \times R^{k_2} : x \in R^n, \quad \xi \in F_2(x)\}.$$

It is clear that $x \in Q$ if and only if

$$\eta = (x, \xi) \in D, \quad 0 \in T_i(\eta), \quad i = 2, 3, \tag{3.3}$$

where $T_2(\eta) = \xi$, $T_3(\eta) = f_1(x) + R^1_+$.

Let

$$S_1(\eta) = S'(x), \; S_3(\eta) = S''(x), \; S(\eta) = (S_1(\eta), \; S_3(\eta)).$$

Then, by the M-extremality of x_o, we see that $\eta_o = (x_o, 0)$ is an M-extremal point for the mapping $S(\eta)$ subject to the constraints (3.3).

The example 2.9 shows that, whatever the natural k may be, the set E defined by

$$E = \left\{ \eta = (x, \xi) : x \in R^n, \xi \in F_2(x_o) + \frac{\partial \sigma_2(x_o)}{\partial x} x \right\}$$

is $(g_1, k+1)$-contingent to D at point η_o where the function g_1 is defined by

$$g_1(\eta_o, \varepsilon, \eta) = \eta_o + \varepsilon\eta.$$

It is easy to verify that the pair (E, l_2), where $l_2(\eta) = \xi$, is a g_1-tangent approximation at η_o of the pair (D, T_2).

Define a set-valued mapping t_3 as follows:

$$t_3(\eta) = \begin{cases} \mathring{R}^1_+ + \max\limits_{\alpha_0 \in \Delta_0(x_0)} \left\{ -\dfrac{\partial c^{F_1}(\alpha_0, x_0)}{\partial x} x \right\} & \text{if } f_1(x_0) = 0, \\ R^1 & \text{if } f_1(x_0) < 0, \end{cases}$$

where $\Delta_0(x_0) = \{ \tilde{\alpha} \in \Delta_0 : -c^{F_1}(\tilde{\alpha}_0, x_0) = \max\limits_{\alpha_0 \in \Delta_0} (-c^{F_1}(\alpha_0, x_0)) \}$.

According to Lemma 3.1. to be proved below, the pair (E, t_3) is a g_1-interior approximation at x_0 of the pair (D, T_3). Let us set

$$M_1 = M', \quad M_3 = M'',$$

$$p_i(\eta) = \max \left\{ \frac{\partial h_i(x_0, \alpha)}{\partial x} x : \alpha \in \Delta_i(x_0) \right\},$$

$$s_3(\eta) = (p_1(\eta), p_2(\eta), \dots, p_{m_2}(\eta)),$$

$$s_1(\eta) = \frac{\partial S'(x_0)}{\partial x} x .$$

It is not hard to verify that all the conditions for the applicability of Theorem 2.2 are satisfied(1). Hence we can find vectors $\psi'_2 \in R^{k_2}$, $\psi' \in M'^-$ and numbers β'_0, $\beta'_i \leqslant 0$, $i = 1, 2, \dots, m_2$, such that

1. $$(\psi'_2, \psi', \beta'_0, \beta'_1, \beta'_{m_2}) \neq 0.$$

2. For every $\eta \in E$ we have

$$\langle \psi'_2, \xi \rangle + \langle \psi', \frac{\partial S'(x_0)}{\partial x} x \rangle + \sup_{\gamma \in t_3(\eta)} \beta'_0 \gamma + \sum_{i=1}^{m_2} \beta'_i \max_{\alpha \in \Delta_i(x_0)} \frac{\partial h_i(x_0, \alpha)}{\partial x} x \leqslant 0. \quad (3.4)$$

From the last inequality it follows that $\beta'_0 \leqslant 0$ and that, furthermore, $\beta'_0 = 0$ if $f_1(x_0) < 0$. Hence, the inequality (3.4) can be rewritten in the form

$$q(x, \xi) \leqslant 0 \quad (3.5)$$

where

$$q(x, \xi) = \langle \psi'_2, \xi \rangle + \beta'_0 \max_{\alpha \in \Delta_0(x_0)} \left\{ -\frac{\partial c^{F_1}(\alpha, x_0)}{\partial x} x \right\} + \left\langle \psi', \frac{\partial S'(x_0)}{\partial x} x \right\rangle + \sum_{i=1}^{m_2} \beta'_i \max_{\alpha \in \Delta_i(x_0)} \frac{\partial h_i(x_0, \alpha)}{\partial x} x.$$

Since the inequality (3.4) is fulfilled for all $(x, \xi) \in E$, the system

$$(x, \xi) \in R^n \times R^{k_2}, \quad 0 \in q(x, \xi) - \mathring{R}^1_+, \quad 0 \in -\xi + F_2(x_0) + \frac{\partial \sigma_2(x_0)}{\partial x} x$$

(1) In this case T_1, S_2, M_2 are absent.

is inconsistent. Therefore, by the separation theorem, there exist a scalar $\varphi \geqslant 0$ and a vector $\varphi \in R^{k_2}$ not both zero, such that

$$\mu q(x, \xi) - \langle \varphi, \xi \rangle + c^{F_2}(\varphi, x_0) + \left\langle \varphi, \frac{\partial \sigma_2(x_0)}{\partial x} x \right\rangle \leqslant 0 \tag{3.6}$$

for all $(x, \xi) \in R^n \times R^{k_2}$. It follows from (3.6) that

$$\mu \psi_2' = \varphi. \tag{3.7}$$

Setting $(x, \xi) = 0$ in (3.6) and taking into account the fact that $0 \in F_2(x_0)$, we obtain

$$c^{F_2}(\mu \psi_2', x_0) = 0. \tag{3.8}$$

From (3.6) — (3.8), we have

$$\langle a, x \rangle + \min_{\alpha \in \Delta(x_0)} \langle b(\alpha), x \rangle \leqslant 0 \tag{3.9}$$

for all $x \in R^n$ where

$$b(\alpha) = -\mu \beta_0' \frac{\partial c^{F_1}(\alpha_0, x_0)}{\partial x} + \sum_{i=1}^{m_2} \mu \beta_i' \frac{\partial h_i(x_0, \alpha_i)}{\partial x},$$

$\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_{m2})$,

$\Delta(x_0) = \Delta_0(x_0) \times \Delta_1(x_0) \times \ldots \times \Delta_{m2}(x_0)$,

$$a = \left[\frac{\partial \sigma_2(x_0)}{\partial x}\right]^\tau \mu \psi_2' + \left[\frac{\partial S'(x_0)}{\partial x}\right]^\tau \mu \psi' \text{ (1)}.$$

From (3.9) it is not difficult to show that

$$-a \in \text{co}\{b(\alpha) : \alpha \in \Delta(x_0)\}.$$

Hence, we can find points $\alpha^j = \left(\alpha_0^j, \alpha_1^j, \ldots, \alpha_{m_2}^j\right) \in \Delta(x_0)$, $j = 1,2,\ldots, n+1$ and nonnegative scalars ν_j, $j = 1,2,\ldots, n+1$, with $\sum_{j=1}^{n+1} \nu_j = 1$, such that

$$a = -\sum_{j=1}^{n+1} \nu_j b(\alpha^j),$$

i.e.

$$\left[\frac{\partial \sigma_2(x_0)}{\partial x}\right]^\tau \psi_2 + \left[\frac{\partial S'(x_0)}{\partial x}\right]^\tau \psi + \sum_{j=1}^{n+1} \mu_0^j \frac{\partial c^{F_1}(\partial_0^j, x_0)}{\partial x} + \sum_{i=1}^{m_2} \sum_{j=1}^{n+1} \mu_i^j \frac{\partial h_i\left(x_0, \alpha_i^j\right)}{\partial x} = 0 \tag{3.10}$$

where

$$\psi_2 = \mu \psi_2', \quad \psi = \mu \psi',$$

$$\mu_0^j = -\mu \beta_0' \nu^j, \quad \mu_i^j = \mu \beta_j' \nu^j, \quad j = 1,2,\ldots, n+1;\ i = 1,2,\ldots m_2.$$

(1) If A is a matrix, then its transpose is denoted by A^τ

Let us note that not all of the quantities ψ, ψ_2, μ_o^j and μ_j^i, $j = 1, 2, \ldots, n+1$; $i = 1, 2, \ldots, m_2$, are zero.

Furthermore, we have

$$-\mu_o^j\, c^{F_1}(\alpha_o^j, x_o) = 0 \tag{3.11}$$

for every $j = 1, 2, \ldots, n+1$.

Putting

$$\psi_1 = \sum_{j=1}^{n+1} \mu_o^j\, \alpha_o^j,$$

we obtain from (3.11) that $c^{F_1}(\psi_1, x_o) = 0$.

Lemma 3.2. to be proved below and the inequality (3.10) show that the relation (3.2) holds.

To conclude the proof of our theorem it remains to verify that condition (a) is also fulfilled. To this end, it suffices to prove that relation $\psi_1 = 0$ implies that $\mu_o^j = 0$ for every $j = 1, 2, \ldots, n+1$. Indeed, by hypothesis and from relation (3.11), we have

$$c^{F_1}(-\mu_o^j\, \alpha_o^j, x_o) = c^{F_1}\Big(\sum_{\substack{i=1,2,\ldots,n+1\\ i\neq j}} \mu_o^i\, \alpha_o^i, x_o\Big) = 0$$

for all $j = 1, 2, \ldots, n+1$. Consequently, for every fixed j the linear function $\langle \mu_o^j\, \alpha_o^j, y\rangle$ of the variable y is constant on the set $F_1(x_o)$. Hence, $\mu_o^j\, \alpha_o^j = 0$ since $F_1(x_o)$ has a non-empty interior. From the last relation and the fact that $\alpha_o^j \neq 0$, we have $\mu_o^j = 0$, $j = 1, 2, \ldots, n+1$. This completes the proof of the Theorem.

Remark 3.1. Let $c^{F_2}(\psi_2, x)$ be a differentiable function, where ψ_2 is the vector mentioned in Theorem 3.1. Then, as in [1, 2] we can show that

$$\left[\frac{\partial \sigma_2(x_o)}{\partial x}\right]^{\tau} \psi_2 = \frac{\partial c^{F_2}(\psi_2, x_o)}{\partial x}.$$

In fact, from the definition of a local section and the condition (b) of Theorem 3.1 we conclude that

$$\zeta(x) \leqslant \zeta(x_o) = 0$$

for all x in the domain of function σ_2 where

$$\zeta(x) = \langle \psi_2, \sigma_2(x)\rangle - c^{F_2}(\psi_2, x).$$

So, the element x_o achieves a local maximum for ζ, hence the desired relation

Remark 3.2. As in [2], we shall say that the family of functions

$$\{h_i(x, \alpha_i), \alpha_i \in \Delta_i, i = 1, 2, \dots, m_2\} \tag{3.12}$$

is nondegenerate at x_o if there is a point $\widehat{x}$ in X such that

$$\frac{\partial h_i(x_o, \alpha_i)}{\partial x} \widehat{x} < 0$$

for all $\alpha_i \in \Delta_i(x_o)$, $i = 1, 2, \dots, m_2$. It is easy to see from conditions (a) and (c) of Theorem 3.1 that if the family (3.12) is nondegenerate, then $(\psi_1, \psi_2, \psi) \neq 0$.

Let us now assume that

(3') $\widehat{M} = M' \times M'' \times N'''$ where M' and M'' are the same cones as in condition (3), and $M''' = R_+^{m_3}$.

(4') $\widehat{S} = (S', S'', S''')$ where S' and S'' are the same mappings as in condition (4), and S''' is a mapping defined by the formulae

$$S'''(x) = (l_1(x), l_2(x), \dots, l_{m_3}(x)),$$

$$l_i(x) = max\left(l_i^1(x), l_i^2(x), \dots, l_i^{q_i}(x)\right), i = 1, 2, \dots m_3.$$

Here $l_i^j : R^n \to R^1$ is a given smooth function.

Theorem 3.2. Assume that conditions (1), (2), (3') and (4') hold. Let x_o be a point in Q, σ_2 a local section of F_2, corresponding to the pair $(x_o, 0)$. If x_o is an M-extremal point for the mapping S, then there exist vectors, $\psi_i \in R^{k_i}$, $i = 1, 2$; $\psi \in M'^-$; numbers $\mu_i^j \leqslant 0$, $j = 1, 2, \dots, n+1$; $i = 1, 2, \dots m_2$; $v_i^j \leqslant 0$, $j = 1, 2, \dots, q_i$; $i = 1, 2, \dots, m_3$, and points $\alpha_i^j \in \Delta_i(x_o)$, $j = 1, 2, \dots, n+1$; $i = 1, 2, \dots, m_2$ such that.

(a) Not all of the quantities ψ_i, $i = 1, 2$; ψ; μ_i^j, $j = 1, 2, \dots, n+1$; $i = 1, 2, \dots, m_2$ and v_i^j, $i = 1, 2, \dots, q_i$; $i = 1, 2, \dots, m_3$, are zero.

(b) $c^{F_i}(\psi_i, x_o) = 0$, $i = 1, 2$.

(c) $\dfrac{\partial H(x_o)}{\partial x} = 0$,

where

$$H(x) = c^{F_1}(\psi_1, x) + \langle \psi, S'(x) \rangle + \langle \psi_2, \sigma(x) \rangle +$$

$$+ \sum_{j=1}^{n+1} \sum_{i=1}^{m_2} \mu_i^j h_i(x_o, \alpha_i^j) + \sum_{i=1}^{m_3} \sum_{j=1}^{q_i} v_i^j l_i^j(x).$$

(d) $v_i^j \left[l_i^j(x_o) - l_i(x_o)\right] = 0$, $j = 1, 2, \dots q_i$; $i = 1, 2, \dots, m_3$.

Proof. We introduce the space R^{n+m_3} of the variable $\eta = (x, \xi)$ where $x \in R^n$, $\xi = (\xi_1, \xi_2, \dots, \xi_{m_3}) \in R^{m_3}$. Let us set

$$l'_i(\eta) = (l_i^1(x) - \xi_i, l_i^2(x) - \xi_i, \dots, l_i^{q_i}(x) - \xi_i), \ i = 1, 2, \dots m_3,$$

$$F'_2(\eta) = \prod_{i=1}^{m_3} (l'_i(\eta) + R_+^{q_i}).$$

Note that mapping F'_2 has local sections. Indeed, let $(\bar{\eta}, \bar{\zeta})$ be an arbitrary point in its graph. Choose $\sigma(x) = l'(\eta) + \bar{\zeta} - l'(\bar{\eta})$ with $l'(\eta) = (l'_1(\eta), l'_2(\eta), \dots, l'_{m_3}(\eta))$. It is obvious that σ is a local section of F'_2 corresponding to the given pair.

Consider now the mappings

$$\tilde{F}_1(\eta) = F_1(x), \ \tilde{F}_2(\eta) = F_2(x) \times F'_2(\eta),$$

$$\tilde{S}'(\eta) = (S'(x), \xi_1, \xi_2, \dots, \xi_{m_3}), \quad \tilde{S}''(\eta) = S''(x),$$

$$\tilde{S}(\eta) = (\tilde{S}'(\eta), \tilde{S}''(\eta)),$$

and the cone $\tilde{M} = \tilde{M}' \times \tilde{M}''$ where $\tilde{M}' = M' \times M'''$, $\tilde{M}'' = M''$. It is easy to see that x_0 is a $\hat{M}$-extremal point for the mapping $\hat{S}$ subject to constraints (3.1) if and only if $\eta_0 = (x_0, \xi_0)$ is a $\tilde{M}$-extremal point for the mapping $\tilde{S}$ subject to

$$0 \in \tilde{F}_i(\eta), \ i = 1, 2,$$

where ξ_0 is the vector with coordinates $l_1(x_0), l_2(x_0), \dots, l_{m_3}(x_0)$.

Applying Theorem 3.1 to the point η_0, we obtain Theorem 3.2.

Let $h: X \times \Delta \to R^1$ be a function, which is defined and is continuous together with its partial derivatives $\frac{\partial}{\partial x_i} h(x, \alpha)$, $i = 1, 2, \dots, n$, on the direct product $X \times \Delta$ where $X = R^n$ and Δ is a compact topological space. Let $H(x) = \max_{\alpha \in \Delta} h(x, \alpha)$, $H(x_0) \leqslant 0$. Denote by $\Delta(x_0)$ the set of all points $\tilde{\alpha} \in \Delta$, which achieve a maximum for the function $h(x_0, \alpha)$ of the variable $\alpha \in \Delta$. It is easy to verify that the function

$$c(x) = \max_{\alpha \in \Delta(x_0)} \frac{\partial h(x_0, \alpha)}{\partial x} \cdot x$$

is convex and continuous. Let us set

$$l(x) = \begin{cases} c(x) + \overset{\circ}{R}{}^1_+ & \text{if } H(x_0) = 0, \\ R^1 & \text{if } H(x_0) < 0, \end{cases}$$

$$T(x) = H(x) + \overset{\circ}{R}{}^1_+.$$

Lemma 3.1. The pair (X, t) is a g_1-interior approximation at x_0 of the pair (X, T) where g_1 is defined by the formula $g_1(\xi, \varepsilon, x) = \xi + \varepsilon x$.

Proof. If $H(x_0) < 0$, then the assertion of the Lemma follows from the continuity of the function $H(x)$. Now suppose that $H(x_0)=0$. Let $\widehat{x}$ be an arbitrary point satisfying the condition $0 \in c(\widehat{x}) + \overset{\circ}{R}{}^1_+$ i. e.,

$$\frac{\partial h(x_0, \alpha)}{\partial x} \widehat{x} < -q \quad (q < 0) \tag{3.13}$$

for all $\alpha \in \Delta(x_0)$. Since the function $\frac{\partial h(x_0, \alpha)}{\partial x}$ of the variable $\alpha \in \Delta$ is continuous, we can find an open set $A \subset \Delta$ such that $\Delta(x_0) \subset \Delta$ and, for all $\alpha \in A$, the inequality (3.13) is satisfied. In view of the continuity of the function $K(x) = \max_{\alpha \in \Delta \setminus A} h(x, \alpha)$ and the condition $K(x_0) < H(x_0) = 0$, there is a neighborhood (a ball) V'_x such that

$$K(x) < 0 \tag{3.14}$$

whenever $x \in x_0 + V'_x$.

Consider now the function

$$\zeta(\varepsilon, \alpha, x) = \frac{\partial h(x_0, \alpha)}{\partial x} x + \left[\frac{\partial h(x_0 + \varepsilon\theta(\widehat{x} + x), \alpha)}{\partial x} - \frac{\partial h(x_0, \alpha)}{\partial x}\right](\widehat{x} + x)$$

where $\theta = \theta(\varepsilon, \alpha, x)$ is a function with the range contained in the interval $[0, 1]$. It is clear that $\zeta(\varepsilon, \alpha, x)$ converges to zero uniformly in $\alpha \in \Delta$ when ε and x converge to zero. Taking into account this fact and the relation

$$\frac{1}{\varepsilon}\left[h(x_0 + \varepsilon(\widehat{x} + x), \alpha) - h(x_0, \alpha)\right] - \frac{\partial h(x_0, \alpha)}{\partial x} \widehat{x} + \zeta(\varepsilon, \alpha, x),$$

we see that there exist a real positive number δ and a neighborhood V_x such that

$$h(x_0 + \varepsilon(\widehat{x} + x), \alpha) < h(x_0, \alpha) - \frac{q}{2}\varepsilon \tag{3.15}$$

whenever $\alpha \in A$, $0 < \varepsilon < \delta$, $x \in V_x$. Without loss of generality, we may assume that $\varepsilon(\widehat{x} + V_x) \subset V'_x$ for all $\varepsilon \in (0, \delta)$. Conditions (3.14) and (3.15) show that, for $0 < \varepsilon < \delta$, $x \in g_1(x_0, \varepsilon, \widehat{x} + V_x)$ we have $H(x) < 0$, i. e. $0 \in T(x)$. This completes the proof of the Lemma.

Lemma 3.2. Let T be a set-valued mapping from R^n into R^m such that, for every $\psi \in R^m$, the function

$$c^T(\psi, x) = \sup_{y \in T(x)} \langle \psi, y \rangle$$

has partial derivatives with respect to x.

Furthermore, let I be a finite set of indices. $\lambda_i, i \in I$, nonnegative real numbers, and $\psi_i \in R^m$, $i \in I$, vectors such that the maxium of the linear functions $\langle \psi_i, y \rangle$, $i \in I$, and $\langle \sum_{i \in I} \lambda_i \psi_i, y \rangle$ of the variable $y \in T(x_0)$ is equal to zero. Then we have

$$\frac{\partial}{\partial x} c^T \left(\sum_{i \in I} \lambda_i \psi_i, x_0 \right) = \sum_{i \in I} \lambda_i \frac{\partial}{\partial x} c^T(\psi_i, x_0).$$

Proof. Denote by x^j the j-th coordinate of the vector $x \in R^n$, and by e^j the j-th unit vector, i.e.

$$e^j = (0, \dots, 0, \underset{(j)}{1}, 0, \dots, 0).$$

Obviously,

$$\sum_{i \in I} \lambda_i c^T(\psi_i, x_0) = c^T \left(\sum_{i \in I} \lambda_i \psi_i, x_0 \right).$$

Hence, for all $\alpha \in R^1$ we have

$$\sum_{i \in I} \lambda_i [c^T(\psi_i, x_0 + \alpha e^j) - c^T(\psi_i, x_0)] \geqslant$$
$$\geqslant c^T \left(\sum_{i \in I} \lambda_i \psi_i, x_0 + \alpha e^j \right) - c^T \left(\sum_{i \in I} \lambda_i \psi_i, x_0 \right). \quad (3.16)$$

Dividing (3.16) by $\alpha > 0$, then making $\alpha \to +0$, we obtain ,

$$\frac{\partial}{\partial x^j} \sum_{i \in I} \lambda_i c^T(\psi_i, x_0) \geqslant \frac{\partial}{\partial x^j} c^T \left(\sum_{i \in I} \lambda_i \psi_i, x_0 \right). \quad (3.17)$$

To prove the converse inequality, it suffices to divide (3.16) by $\alpha < 0$ and to let $\alpha \to -0$. The proof of the Lemma is complete.

Remark 3.3. Theorem 3.2 is also valid for the case where the mappings S'' and S''' are absent, provided that the cone M' satisfies the following conditions:

$$M' \neq \{0\}, \quad M' \cup (-M') = \{0\}.$$

§4. THE SUPPORT PRINCIPLE FOR A DISCRETE INCLUSION

Consider now an optimal process described by a system of discrete inclusions

$$x(k+1) \in A^k(x(k)), \quad k = 0, 1, \dots, K-1, \quad (4.1)$$

with restricted phase coordinates

$$0 \in B^k(x(k)), \quad k = 0, 1, \dots, K, \quad (4.2)$$

where A^k and B^k are set-valued mappings from R^n into R^n and R^{n_k} respectively, and K is a fixed positive integer.

Let $S(\{x(k)\})$ be a single-valued mapping from $R^{n(k+1)}$ into R^m, M a convex cone in R^m $(M \neq R^m)$.

A sequence

$$\{x(k), \quad k = 0,1, \ldots, K\} \tag{4.3}$$

satisfying the constraints (4.1), (4.2) is said to be an admissible trajectory. An admissible trajectory

$$\{\overset{\circ}{x}(k), \; k = 0,1, \ldots, K\} \tag{4.4}$$

is said to be optimal if, for any admissible trajectory (4.3), the condition $S(\{\overset{\circ}{x}(k)\}) - S(\{x(k)\}) \in M$ implies that $S(\{x(k)\}) - S(\{\overset{\circ}{x}(k)\}) \in M$.

Assume that

$$A^k(x) = A_1^k(x) \cap A_2^k(x),$$

$$B^k(x) = B_1^k(x) \cap B_2^k(x)$$

where the set-valued mappings A_i^k and B_i^k satisfy the following conditions:

1. The mappings A_2^k and B_2^k have local sections; $A_2^k(x)$ and $B_2^k(x)$ are convex sets for every $x \in R^n$.

2. For every x, $A_1^k(x)$ and $B_1^k(x)$ are convex compact sets with non-empty interiors.

3. The functions $c^{A_1^k}(\psi, x)$, $c^{B_1^k}(\varphi, x)$ are continuous (in their variables) together with their partial derivatives (with respect to x).

Let Δ_i be a compact topological space, $l_i^j(k, x)$ a smooth function, $h_i(k, x, \alpha)$ a scalar-valued function, which is defined and is continuous together with its partial derivatives $\frac{\partial}{\partial x^j} h_i(k, x, \alpha)$, $j = 1, 2, \ldots, n$, on the direct product $X \times \Delta_i$.

Define the mapping $S'' : R^{n(k+1)} \to R^{m_2}$ and $S''' : R^{n(k+1)} \to R^{m_3}$ by the formulae

$$S'' = \left(S_1'', S_2'', \ldots, S_{m_2}''\right), \quad S''' = \left(S_1''', S_2''', \ldots, S_{m_3}'''\right),$$

$$S_i''(\{x(k)\}) = \max_{\alpha \in \Delta_i} \sum_{k=0}^{K} h_i(k, x(k), \alpha),$$

$$S_i'''(\{x(k)\}) = \max_{j=1,2,\ldots,q_i} \sum_{k=0}^{K} l_i^j(k, x(k)).$$

Assume that

1. $S = (S', S'', S''')$ where

$$S'(\{x(k)\}) = \sum_{k=0}^{K} Q(k, x(k))$$

($Q(k, x)$ is a smooth function from R^n into R^{m_1}).

2. $M = M' \times R_+^{m_2} \times R_+^{m_3}$ where M' is a given convex cone in R^{m_1}.

Denote by $\Delta_i(\{\overset{\circ}{x}(k)\})$ the set of all points $\hat{\alpha} \in \Delta_i$, at which the function $\sum_{i=0}^{k} h_i(k, \overset{\circ}{x}(k), \alpha)$ of the variable $\alpha \in \Delta_i$ attains its maximum.

Theorem4.1 Let (4.4) be an admissible trajectory; σ_2^k and ζ_2^k local sections of the mappings A_2^k and B_2^k corresponding to the pairs $(\overset{\circ}{x}(k), \overset{\circ}{x}(k+1))$ and $(\overset{\circ}{x}(k), 0)$ respectively. If the trajectory (4.4) is optimal, then there exist vectors $\psi_i(k) \in R^n$ ($i = 1, 2$, $k = 0, 1, \ldots, K-1$), $\varphi_i(k)$ ($i = 1, 2$; $k = 0, 1, \ldots, K$), $\psi \in M'^{-}$), numbers $\mu_i^j \leqslant 0$ ($j = 1, 2, \ldots, n+1$; $i = 1, 2, \ldots, m_2$), $\nu_i^j \leqslant 0$ ($j = 1, 2, \ldots, q_i$, $i = 1, 2, \ldots, m_3$) and points $\alpha_i^j \in \Delta_i(\{\overset{\circ}{x}(k)\})$ ($j = 1, 2, \ldots, n+1$; $i = 1, 2, \ldots, m_2$), such that

(a') Not all of the quantities $\psi_i(k)$ ($i = 1, 2$, $k = 0, 1, \ldots, K-1$), $\varphi_i(k)$ ($i = 1, 2$; $k = 0, 1, \ldots, K$), ψ, μ_i^j ($j = 1, 2, \ldots, n+1$; $i = 1, 2, \ldots, m_2$), ν_i^j ($j = 1, 2, \ldots, q_i$; $i = 1, 2 \ldots, m_3$) are zero.

(b') $c^{A_i^k}(\psi_i(k), \overset{\circ}{x}(k)) = \langle \psi_i(k), \overset{\circ}{x}(k+1) \rangle$, $i = 1, 2$; $k = 0, 1, \ldots, K-1$,

$c^{B_i^k}(\varphi_i(k), \overset{\circ}{x}(k)) = 0$, $i = 1, 2$, $k = 0, 1, \ldots, K$.

(c') $\nu_i^j\left(\sum_{k=0}^{K} l_i^j(k, \overset{\circ}{x}(k)) - S_i'''(\{\overset{\circ}{x}(k)\}\right) = 0$, $j = 1, 2, \ldots, q_i$; $i = 1, 2, \ldots m_3$.

(d') $\psi_1(k-1) + \varphi_2(k-1) = \left.\dfrac{\partial H(\psi_1(k), \psi_2(k), \varphi_1(k), \varphi_2(k), x)}{\partial x}\right|_{x = \overset{\circ}{x}(k)}$ (1)

$k = 0, 1, \ldots, K$, where

$$H(\psi_1(k), \psi_2(k), \varphi_1(k), \varphi_2(k), x) = \langle \psi_2(k), \sigma_2^k(x) \rangle + \langle \varphi_2(k), \zeta_2^k(x) \rangle +$$

$$+ c^{A_1^k}(\psi_1(k), x) + c^{B_1^k}(\varphi_1(k), x) + \langle \psi, Q(k, x) \rangle +$$

$$+ \sum_{j=1}^{n+1} \sum_{j=1}^{m_2} \mu_i^j h_i(k, x, \alpha_i^j) + \sum_{i=1}^{m_3} \sum_{j=1}^{q_i} \nu_i^j l_i^j(k, x).$$

(1) We set $\psi_i(k) = 0$ if $k = -1, K$.

Proof. We introduce the space $R^{n(k+1)}$ of the variable $\widetilde{x} = (x(0)), x(1),\dots, x(K)$ and define the mappings T_i, $i = 1,2$, by the formulae

$$T_i(\widetilde{x}) = \prod_{k=0}^{K} [A_i^k(x(k)) - x(k+1)] \times \prod_{k=0}^{K} B_i^k(x(k)).$$

It is clear that Theorem 4.1 is an immediate consequence of Theorem 3.2.

Remark 4.1. Let functions $c^{A_2^k}(\psi_2(k), x)$ and $c^{B_2^k}(\varphi_2(k), x)$ be differentiable (with respect to x), where $\psi_2(k)$ and $\varphi_2(k)$ are the vectors mentioned in Theorem 4.1. Then the condition (d') can be replaced by

$$(d'')\ \frac{\partial H'(\psi_1(k), \psi_2(k), \varphi_1(k), \varphi_2(k), x)}{\partial x}\Bigg|_{x=\overset{\circ}{x}(k)} = \psi_1(k-1) + \psi_2(k-1),$$

$$k = 0,1,\dots, K,$$

where

$$H'(\psi_1(k), \psi_2(k), \varphi_1(k), \varphi_2(k), x) = \sum_{i=1}^{2} \{c^{A_i^k}(\psi_i(k), x) + c^{B_i^k}(\varphi_i(k), x)\} +$$

$$+ \langle\psi, Q(k, x)\rangle + \sum_{j=1}^{n+1} \sum_{i=1}^{m_2} \mu_i^j h_i(k, x, \alpha_i^j) + \sum_{i=1}^{m_3} \sum_{j=1}^{q_i} \nu_i^j l_i^j (k, x).$$

Définition 4.1. We shall say that the trajectory (4.4) satisfies a support principle if there exist vectors $\psi_i(k)$ ($i = 1,2$, $k = 0,1,\dots, K-1$), $\varphi_i(k)$ ($i = 1,2$, $k = 0,1,\dots, K$), $\psi \in M'$; numbers $\mu_i^j \leqslant 0$ ($j = 1,2, \dots, n+1$; $i = 1,2, \dots, m_2$), ν_i^j ($j = 1,2,\dots, q_i$; $i = 1,2,\dots, m_3$); and points $\alpha_i^j \in \Delta_j(\{\overset{\circ}{x}(k)\})$ ($j = 1,2,\dots, n+1$; $i = 1,2,\dots, m_2$) such that conditions (a')—(c'), (d'') hold.

From Theorem 4.1 and Remark 4.1 we have

Theorem 4.2. Assume, in addition to the already made assumptions, that the functions $c^{A_2^k}(\psi_2(k), x)$ and $c^{B_2^k}(\varphi_2(k), x)$ are differentiable with respect to x, where $\psi_2(k)$ and $\varphi_2(k)$ are the vectors mentioned in Theorem 4.1. Then the optimal trajectory (4.4) satisfies the support principle.

Remark 4.2. It follows from Theorem 4.2 that the support principle holds for a process without local sections [3] (i.e. for the case where mappings A_2^k and B_2^k are absent), as well as for a process with local sections [1,2] (i.e. for the case where mappings A_1^k and B_1^k are absent). It should be noted that the set $A_2^k(x)$ and B_2^k are not assumed to be compact.

Remark 4.3. From Theorem 3.2. we can also deduce the support principle for discrete time-lag processes, as well as for discrete distributed parameter systems [3].

Remark 4.4. Theorems 4.1 and 4.2 are also valid for the case where the mappings S'' and S''' are absent, provided that the cone M' is such that $M' \neq \{0\}$ and $M' \cap (-M') = \{0\}$.

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