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SOME FIXED POINT THEOREMS FOR MAPPINGS OF CONTRACTIVE TYPE

ĐỖ HỒNG TÂN, *Institute of Mathematics.*

NGUYỄN ANH MINH, *Economical Institute.*

The purpose of this paper is to establish some new results on the existence of fixed points for some classes of mappings of contractive type. The paper consists of three sections. The first section extends the results of Banach, Rakotch, Boyd — Wong, Meir — Keeler, Edelstein and Sehgal to singlevalued mappings of contractive type. The second section extends an earlier result of Smithson to multivalued mappings of contractive type. In the last section we extend some results of Wong, Assad — Kirk and the others to multivalued generalized contractions.

1. Fixed points for singlevalued mappings of contractive type.

In this section we shall use the following notations: (X, d) denotes a metric space, T denotes a *continuous* mapping from X into X. For x, y in X let

$$r(x, y) = \max\left\{ d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}[d(x, Ty) + d(y, Tx)]\right\}.$$

A mapping T is said to belong to the class $\mathcal{A}$ iff $\exists\ \alpha \in [0, 1)$ such that

$$d(Tx, Ty) \leqslant \alpha\, r(x, y), \quad (x, y \in X).$$

The class $\mathcal{A}$ contains the class of contractions.

A mapping T is said to belong to the class $\mathcal{B}$ iff there exist monotone nonincreasing functions $\varphi_i : [0, \infty) \to [0, 1]$ such that $\varphi_i^{-1}(1) \cap (0, \infty) = \varnothing$ $(i = 1, 2, 3, 4)$ and

$$d(Tx, Ty) \leqslant \max\{\varphi_1(d(x, y))\, d(x, y),\ \varphi_2(d(x, Tx))\, d(x, Tx),$$

$$\varphi_3(d(y,Ty))\,d(y,Ty),\ \varphi_4\left[\frac{d(x,Ty)+d(y,Tx)}{2}\right]\frac{d(x,Ty)+d(y,Tx)}{2}\Big\}$$

for x, y in X. The class $\mathcal{B}$ contains the class of mappings studied by Rakotch [10].

A mapping T is said to belong to the class $\mathcal{C}$ iff there exist upper semi-continuous on the right functions $\Psi_i : [0, \infty) \to [0, \infty)$ such that $\Psi_i(0) = 0$, $\Psi_i(t) < t$ for all $t > 0$, $(i = 1, 2, 3, 4)$, and for x, y in X,

$$d(Tx, Ty) \leqslant \max\Big\{\Psi_1(d(x,y)),\ \Psi_2(d(x,Tx)),\ \Psi_3(d(y,Ty)),\ \Psi_4\left[\frac{d(x,Ty)+d(y,Tx)}{2}\right]\Big\}.$$

The class $\mathcal{C}$ contains the class of nonlinear contractions studied by Boyd — Wong [3].

A mapping T is said to belong to the class $\mathcal{D}$ iff

$$\forall\, \varepsilon > 0\ \exists\, \delta > 0 \text{ such that } r(x, y) < \varepsilon + \delta \Rightarrow d(Tx, Ty) < \varepsilon. \quad (1)$$

The class $\mathcal{D}$ contains the class of weakly uniformly contractive mappings studied by Meir — Keeler [8]. To seethis, it suffices to show that the condition of weak uniform contraction

$$\forall\, \varepsilon > 0\ \exists\, \delta > 0 \text{ such that } \varepsilon \leqslant d(x, y) < \varepsilon + \delta \Rightarrow d(Tx, Ty) < \varepsilon \quad (2)$$

is equivalent to the following condition

$$\forall\, \varepsilon > 0\ \exists\, \delta > 0 \text{ such that } d(x, y) < \varepsilon + \delta \Rightarrow d(Tx, Ty) < \varepsilon. \quad (3)$$

Indeed, if T satisfies (3) then it obviously also satisfies (2). Conversely, if T satisfies (2), we consider all x, y in X such that $d(x, y) < \varepsilon + \delta$. If $d(x, y) \geqslant \varepsilon$ then by (2) we obtain $d(Tx, Ty) < \varepsilon$. If $d(x, y) < \varepsilon$ then also by (2) we get $d(Tx, Ty) \leqslant d(x, y) < \varepsilon$ because T is contractive. Thus, in both cases we have $d(Tx, Ty) < \varepsilon$.

Further, a mapping T is said to belong to the class $\mathcal{E}$ iff

$$d(Tx, Ty) < r(x, y), \qquad (x \neq y).$$

The class $\mathcal{E}$ contains the class of contractive mappings studied by Edelstein [4] and the class of mappings studied by Sehgal [13].

For the above defined classes of mappings we have the following relation

$$\mathcal{A} \subset \mathcal{B} \subset \mathcal{C} \subset \mathcal{D} \subset \mathcal{E}.$$

It is easy to see that $\mathcal{A} \subset \mathcal{B} \subset \mathcal{C}$. To show that $\mathcal{D} \subset \mathcal{E}$ it suffices to note that the condition (1) is equivalent to the following condition

$$\forall \varepsilon > 0\ \exists\, \delta > 0 \text{ such that } \varepsilon \leqslant d(x, y) < \varepsilon + \delta \Rightarrow d(Tx, Ty) < \varepsilon.$$

To prove $\mathcal{C} \subset \mathcal{D}$, let $T \in \mathcal{C}$. From the condition on ψ_i in the definition of the class $\mathcal{C}$ we have : if $\varepsilon > 0$ then $\psi_i(\varepsilon) < \varepsilon$ and $\exists\, \delta > 0$ such that

$$\varepsilon \leqslant t < \varepsilon + \delta \Rightarrow \psi_i(t) < \varepsilon, \quad (i = 1, 2, 3, 4).$$

This implies $$t < \varepsilon + \delta \Rightarrow \psi_i(t) < \varepsilon, \quad (i = 1, 2, 3, 4),$$

for if $t < \varepsilon$ then we have also $\psi_i(t) \leqslant t < \varepsilon$. Now, if $r(x, y) < \varepsilon + \delta$ then $d(x, y) < \varepsilon + \delta$ and hence $\psi_1(d(x, y)) < \varepsilon$. Similarly, we have

$$\psi_2(d(x, Tx)) < \varepsilon,\ \psi_3(d(y, Ty)) < \varepsilon,\ \psi_4\left[\frac{d(x,Ty)+d(y,Tx)}{2}\right] < \varepsilon.$$

Then we get $d(Tx, Ty) < \varepsilon$ because $T \in \mathcal{C}$. This shows that $\mathcal{C} \subset \mathcal{D}$.

For the class $\mathcal{D}$ (and hence, for classes $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$) using the same method as Meir — Keeler in [8], we have the following result.

Theorem 1. 1. *Let (X, d) be a complete metric space and $T \in \mathcal{D}$. Then T has a unique fixed point x^* and $T^n x \to x^*$ $(\forall x \in X)$.*

Proof. Let $x_0 \in X$, $x_{n+1} = Tx_n$ $(n = 0, 1, 2, \ldots)$. If $x_{m+1} = x_m$ for some m then x_m is a fixed point of T and $T^m x_0 = T^{m+1} x_0 = T^{m+2} x_0 = \ldots = x_m$. Thus we may suppose $x_{n+1} \neq x_n$ $(n = 0, 1, 2, \ldots)$.

Set $c_n = d(x_n, x_{n+1})$. Since $T \in \mathcal{E}$ we have

$$c_{n+1} = d(x_{n+1}, x_{n+2}) = d(Tx_n, Tx_{n+1}) < r(x_n, x_{n+1}) =$$

$$= \max \left\{ d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \frac{1}{2} d(x_n, x_{n+2}) \right\} \leqslant$$

$$\leqslant \max \left\{ d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \frac{d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})}{2} \right\}$$

$$= \max \left\{ d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}) \right\} =$$

$$= d(x_n, x_{n+1}) = c_n \ (n = 0, 1, 2, \ldots).$$

Thus $\{c_n\}$ is a decreasing sequence and hence

$$c_n \searrow \varepsilon \geqslant 0. \tag{4}$$

If $\varepsilon > 0$ then there exists $\delta > 0$ such that

$$r(x, y) < \varepsilon + \delta \Rightarrow d(Tx, Ty) < \varepsilon. \tag{5}$$

Due to (4) there is a natural number N such that $c_n < \varepsilon + \delta$ $(\forall n \geqslant N)$. Since $c_n = r(x_n, x_{n+1})$ we have

$$c_{n+1} = d(Tx_n, Tx_{n+1}) < \varepsilon,$$

contradicting (4). Thus $c_n \to 0$.

Now if $\{x_n\}$ is not a Cauchy sequence then an $\varepsilon > 0$ exists such that $\forall N \exists m, n \geqslant N$ with $d(x_n, x_m) > 2\varepsilon$. For this ε, select $\delta > 0$ such that (5) holds. Set $\delta' = \min\{\varepsilon, \delta\}$. Since $c_n \searrow 0$, there is N such that $c_n < \frac{\delta'}{4}$ $(\forall n \geqslant N)$. For this N, select $n, m \geqslant N$ with $d(x_m, x_n) > 2\varepsilon$.

For each $j \in \{m, \ldots, n\}$ we have

$$d(x_m, x_j) \leqslant d(x_m, x_{j+1}) + d(x_{j+1}, x_j),$$

$$d(x_m, x_{j+1}) \leqslant d(x_m, x_j) + d(x_j, x_{j+1})$$

and hence

$$| d(x_m, x_j) - d(x_m, x_{j+1}) | \leqslant c_j < \frac{\delta'}{4}.$$

In view of $c_m = d(x_m, x_{m+1}) < \varepsilon$, $d(x_m, x_n) > \varepsilon + \delta'$, from the above inequality it follows there is $k \in \{m, \ldots, n\}$ such that

$$\varepsilon + \frac{\delta'}{2} < d(x_m, x_k) < \varepsilon + \frac{3\delta'}{4}.$$

We shall verify the fololwing inequality

$$r(x_m, x_k) < \varepsilon + \delta'. \tag{6}$$

Indeed, we have

$$d(x_m, x_k) < \varepsilon + \frac{3\delta'}{4} < \varepsilon + \delta',$$

$$d(x_m, x_{m+1}) = c_m < \frac{\delta'}{4} < \varepsilon + \delta',$$

$$d(x_k, x_{k+1}) = c_k < \frac{\delta'}{4} < \varepsilon + \delta',$$

$$\frac{1}{2}[d(x_m, x_{k+1}) + d(x_{m+1}, x_k)] \leqslant d(x_m, x_k) + \frac{1}{2}(c_m + c_k) <$$

$$< \varepsilon + \frac{3\delta'}{4} + \frac{\delta'}{4} = \varepsilon + \delta'.$$

Thus, we get (6).

By (6) and (5) we obtain

$$d(x_{m+1}, x_{k+1}) = d(Tx_m, Tx_k) < \varepsilon. \tag{7}$$

On the other hand we have

$$d(x_{m+1}, x_{k+1}) \geqslant d(x_m, x_k) - d(x_m, x_{m+1}) - d(x_k, x_{k+1})$$

$$> \varepsilon + \frac{\delta'}{2} - \frac{\delta'}{4} - \frac{\delta'}{4} = \varepsilon,$$

contradicting (7). Thus $\{x_n\}$ is a Cauchy sequence. Since X is complete, $x_n \to x^* \in X$. By the continuity of T, $x_{n+1} = Tx_n \to Tx^*$. Hence $x^* = Tx^*$.

To prove the uniqueness of x^*, suppose there is $y^* = Ty^*$, $y^* \neq x^*$. Since $T \in \mathscr{E}$ we obtain the following contradiction

$$d(x^*, y^*) = d(Tx^*, Ty^*) < d(x^*, y^*).$$

Thus, the theorem is proved.

Remark 1. 1. The following example shows that without the continuity of T the above theorem does not hold.

Set $X = \left\{1, \frac{1}{2}, \ldots, \frac{1}{2^n}, \ldots, 0\right\}$ with the usual metric in R, $T(0) = 1$, $T\left(\frac{1}{2^n}\right) = \frac{1}{2^{n+1}}$

($n = 0, 1, 2,\ldots$). Then T satisfies all conditions in the above theorem except the continuity at 0, and T has no fixed point.

Remark 1. 2. The following example shows that the above theorem does not hold if in (1) $r(x, y)$ is replaced by

$$\rho(x, y) = \max \{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}.$$

Let X be the set of all integers with the usual metric, $T(n) = n + 1$ ($\forall n \in X$). Then T satisfies the condition (1) with $\rho(x, y)$ replacing $r(x, y)$ and T has no fixed point.

By the argument of Sehgal in [13] we get the following result.

Theorem 1. 2. *Let (X, d) be a metric space, $T \in \mathscr{C}$. If there exists a subsequence $\{T^{n_i}x_0\}$ of the iterate sequence $\{T^n x_0\}$ for some $x_0 \in X$, converging to $x^* \in X$ then x^* is a unique fixed point of T and $T^n x_0 \to x^*$.*

Proof. With the given x_0, we construct a sequence $\{x_n\}$ ($n = 0, 1, 2,\ldots$) by setting $x_{n+1} = Tx_n = T^{n+1}x_0$. Put $c_n = d(x_n, x_{n+1})$. As in the proof of Theorem 1. 1, we get $c_n \searrow \varepsilon \geqslant 0$. Especially, $c_{n_i} \searrow \varepsilon$.

By hypothesis, $x_{n_i} = T^{n_i}x_0 \to x^*$. Since T is continuos, we obtain

$$x_{n_i+1} = T^{n_i+1}x_0 = Tx_{n_i} \to Tx^*.$$

Hence

$$c_{n_i} = d(x_{n_i}, x_{n_i+1}) \to d(x^*, Tx^*) = \varepsilon.$$

If $\varepsilon > 0$ then

$$d(Tx^*, T^2x^*) < \max\left\{d(x^*, Tx^*), d(Tx^*, T^2x^*), \frac{d(x^*, Tx^*) + d(Tx^*, T^2x^*)}{2}\right\}$$

It implies

$$d(Tx^*, T^2x^*) < d(x^*, Tx^*) = \varepsilon.$$

On the other hand, we have

$$d(Tx^*, T^2x^*) = \lim d(Tx_{n_i}, T^2x_{n_i}) = \lim d(x_{n_i+1}, x_{n_i+2}) =$$
$$= \lim c_{n_i+1} = \lim c_n = \varepsilon,$$

contracdicting the above inequality. Thus $\varepsilon = 0$ and $x^* = Tx^*$. The uniqueness of x^* is easy to prove as in the Theorem 1. 1. We shall prove that $T^n x_0 \to x^*$.

Since $x_{n_i} \to x^*$ and $c_{n_i} \to 0$, for every $\varepsilon > 0$ there is an integer j such that $\forall i \geqslant j$ we have

$$\max\{d(x_{n_i}, x^*), c_{n_i}\} < \varepsilon.$$

Then $\forall\, n \geqslant n_j$ we have

$$d(T^n x_0, x^*) = d(x_n, x^*) = d(Tx_{n-1}, Tx^*) <$$

$$< \max \left\{ d(x_{n-1}, x^*), d(x_{n-1}, x_n), \frac{d(x^*, x_n) + d(x^*, x_{n-1})}{2} \right\}$$

$$\leqslant \max \{ d(x_{n-1}, x^*), c_{n-1} \} \leqslant$$

$$\leqslant \max \{ d(x_{n-2}, x^*), c_{n-2}, c_{n-1} \} =$$

$$= \max \{ d(x_{n-2}, x^*), c_{n-2} \} < \dots \leqslant$$

$$\leqslant \max \{ d(x_{n_j}, x^*), c_{n_j} \} < \varepsilon.$$

The theorem is proved.

Remark 1.3. The following example shows that the theorem 1.2 does not hold if $r(x,y)$ is replaced by $\rho(x,y)$.

Let $X = A \cup B \cup C$, where $A = \{x_j,\ j = 1,2,\dots\}$, $B = \{x_j^i,\ i = 1,2,\dots;\ j = 1,\dots, i\}$, $C = \{y^i,\ i = 1,2,\dots\}$. We construct a metric in X as follows:

In A:
$$d(x_j, x_{j+k}) = 2 - \frac{1}{2^k},\ (k > 0;\ \forall\, j) \tag{8}$$

In B:
$$d(x_j^i, x_{j+k}^{i'}) = 2 - \frac{1}{2^k},\ (k > 0;\ \forall\, j,\ i\ i') \tag{9}$$

$$d(x_j^i, x_j^{i+k}) = \frac{1}{2^i} - \frac{1}{2^{i+k}},\ (k > 0;\ \forall\, j) \tag{10}$$

In C:
$$d(y^i, y^{i'}) = 2 \qquad (\forall\, i \neq i') \tag{11}$$

Between A and B:

$$d(x_j, x_j^i) = \frac{1}{2^i}, \qquad (\forall\, j) \tag{12}$$

$$d(x_j, x_{j+k}^i) = 2 - \frac{1}{2^k}, \qquad (k > 0;\ \forall\, j,\ i) \tag{13}$$

Between A and C:

$$d(x_j, y^i) = 2 + \frac{1}{2^{i+j}}. \tag{14}$$

Between B and C:

$$d(x_j^{i'}, y^i) = 2 + \frac{1}{2^{i+j}} + \frac{1}{2^{i'}} \tag{15}$$

d is a metric in X. Indeed, we must verify only the following inequality

$$d(x,z) \leqslant d(x,y) + d(y,z), \qquad (x,y,\ z \in X). \tag{16}$$

Consider the following cases:

a) $x, y, z \in A \cup B$.

α) $j(x) = j(y) = j(z)$. In this case the distances between $x_j, x_j^1, x_j^2, \ldots$ are similar to the distances between

$0, \frac{1}{2}, \frac{1}{2^2}, \ldots$ in R and (16) is obvious.

β) $j(x) \neq j(y) \neq j(z) \neq j(x)$. By (8), (9), (13) we have

$$1 < d(x,z),\ d(x,y),\ d(y,z) < 2,$$

from this we get (16).

γ) $j(x) = j(z) \neq j(y)$. By (10), (12), (8), (9), (13) we have

$$d(x,z) < 1;\ d(x,y),\ d(y,z) > 1.$$

from this we get (16).

δ) $j(x) = j(y) \neq j(z)$. By (8), (9), (13) we obtain $d(x,z) = d(y,z)$. Similarly, if $j(y) = j(z) \neq j(x)$ we get $d(x, z) = d(x, y)$. In both cases we obtain (16).

b) $\{x,y,z\} \cap C \neq \emptyset$. First, we observe that if $u \in C$ and $v \neq u$ then by (11), (14), (15) we have $2 \leqslant d(u,v) < 3$. If $x,z \in A \cup B$ then $y \in C$ and we have $d(x,z) < 2$. $d(x,y) > 2$. from this it follows (16). If $x, z \in C$, then $d(x,z) \leqslant d(x,y)$ and we have (16). Finally, if $x \in C$ and $y, z \in A \cup B$ then $x = y^i$, $y = x_j$, $z = z_{j'}^{i'}$ (we may suppose $y \in A$, $z \in B$, the other cases are similar to this). If $j \neq j'$, we have $d(y,z) > 1$, $d(x,y) > 2$ and $d(x,z) < 3$, it follows (16). If $j = j'$ then $d(x,z) = 2 + \frac{1}{2^{i+j}} + \frac{1}{2^{i'}}$, $d(x,y) = 2 + \frac{1}{2^{i+j}}$, $d(y, z) = \frac{1}{2^{i'}}$ by (15), (14), (12) and hence we also get (16). Thus d is a metric in X.

Now we construct a mapping T in X as follows:

$$Tx_j = x_{j+1}\ (\forall j),\ Tx_j^i = x_{j+1}^i\ (j < i),\ Tx_i^i = y^i\ (\forall i),\ Ty^i = x_1^{i+1}\ (\forall i) \quad (17)$$

It is clear, that T is continuous, $T^{n_i} x_1^1 \to x_1$ and T has no fixed point. Thus we must only show that T satisfies the condition

$$d(Tx, Ty) < \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\} \text{ for } x \neq y. \quad (18)$$

Let us consider the following cases. Set

$$\Delta = \{x_n^n,\ n = 1, 2, \ldots\}$$

a) $x, y \in A \cup B$.

α) $x, y \notin \Delta$. We may suppose $x \in A$, $y \in B$, hence $x = x_j, y = x^i_{j+k}$. If $k=0$, by (17), (12), (13) we have $d(Tx_j, Tx^i_j) = d(x_{j+1}, x^i_{j+1}) = \frac{1}{2^i} < 2 - \frac{1}{2} = d(x_j, Tx^i_j)$. If $k > 0$ then by (17), (13) we get

$$d(Tx_j, Tx^i_{j+k}) = d(x_{j+1}, x^i_{j+k+1}) = 2 - \frac{1}{2^k} < 2 - \frac{1}{2^{k+1}} = d(x_j, Tx^i_j)$$

Thus we always obtain $d(Tx, Ty) < d(x, Ty)$ and hence (18).

β) $x \notin \Delta$, $y \in \Delta$. If $x \in A$ we have $x = x_j$, $y = x^i_i$. Then

$$d(Tx_j, Tx^i_i) = d(x_{j+1}, y^i) = 2 + \frac{1}{2^{i+j+1}} < 2 + \frac{1}{2^{i+j}} = $$
$$= d(x_j, Tx^i_i).$$

We obtain the same inequality when $x \in B$. Thus in this case we get also (18).

γ) $x, y \in \Delta$. Since $x \in B$, $Ty \in C$ we get

$d(Tx, Ty) = d(y^i, y^{i'}) = 2 < d(x, Ty)$, hence (18) also holds.

b) $x \in C$.

α) $y \notin \Delta$. Since. $Tx \in A$, $Ty \in A \cup B$, $x \in C$, we have

$$d(Tx, Ty) < 2 \leqslant d(x, x),$$

hence (18) also holds.

β) $y \in \Delta$. Then $x = y^i$, $y = x^{i'}_{i'}$.

If $i' > i$ then

$$d(Tx, Ty) = d(x^{i+1}_1, y^{i'}) = 2 + \frac{1}{2^{i'+1}} + \frac{1}{2^{i+1}} < 2 + \frac{1}{2^{i+1}} + \frac{1}{2^{i+1}}$$
$$= d(y^i, x^{i+1}_1) = d(x, Tx).$$

If $i' \leqslant i$ then $i = i' + k\,(k \geqslant 0)$. It is easy to verify that $\frac{1}{2^{i'+1}} + \frac{1}{2^{i'+k+1}} < \frac{1}{2^{2i'}} + \frac{1}{2^{i'}}$. Then

$$d(Tx, Ty) = d(x^{i'+k+1}_1, y^{i'}) = 2 + \frac{1}{2^{i'+1}} + \frac{1}{2^{i'+k+1}} < 2 + \frac{1}{2^{i'}} + \frac{1}{2^{i'}} =$$

$(dx^{i'}_{i'}, y^{i'}) = d(y, Ty)$.

Thus in both cases we obtain (18). The proof is complete.

2. Fixed points for multivalued mappings of contractive type.

In the sequel we shall use the following notations: (X, d) denotes a metric space, $CB(X)$ is the class of all nonempty closed bounded subsets of X, $K(X)$ is the class of all nonempty compact subsets of X, D is the Hausdorff metric generated by d in $CB(X)$, and finally,

$$d(x, A) = \inf \{d(x, y) \mid y \in A\},$$
$$(x \in X, \ A \subset X).$$

Let T be a multivalued mapping of X into $CB(X)$. $O(x)$ denotes the set $\{x_n \mid n = 0, 1, 2, \ldots; x_0 = x, x_{n+1} \in Tx_n \ (\forall n)\}$ called the orbit of T at x. $O(x)$ is said to be *normal* if

$$\sum_{n=0}^{\infty} [d(x_n, x_{n+1}) - d(x_n, Tx_n)] < \infty,$$

and *quasinormal* if

$$\Sigma^+ [d(x_n, x_{n+1}) - D(Tx_{n-1}, Tx_n)] < \infty,$$

where Σ^+ means that the sum consists only of positive terms.

It is clear that every normal orbit is quasinormal and that every regular (in the sense of Smithson [14]) orbit is quasinormal.

Theorem 2. 1. *Let (X, d) be a metric space, T be a mapping of X into $CB(X)$, continuous in the Hausdorff metric D and satisfying*

$$D(Tx, Ty) < \max \{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}[d(x, Ty) + d(y, Tx)]\}, \ (x \neq y) \qquad (1)$$

Suppose there is a quasinormal orbit $O(x_0)$ satisfying the following condition (S): $O(x_0)$ contains two successive convergent subsequences:

$$x_{n_i} \to x^*, \ x_{n_i+1} \to y^*. \ \textit{Then } x^* = y^* \in Tx^*.$$

Proof. Let $O(x_0) = \{x_n \mid n = 0, 1, 2, \ldots\}$ be the quasinormal orbit given in the hypothesis, set $a_n = d(x_n, x_{n+1}) - D(Tx_{n-1}, Tx_n)$. For every n, we have

$$D(Tx_{n-1}, Tx_n) < \max \Big\{ d(x_{n-1}, x_n), d(x_{n-1}, Tx_{n-1}),$$
$$d(x_n, Tx_n), \frac{1}{2} d(x_{n-1}, Tx_n) \Big\} \leqslant \max \Big\{ d(x_{n-1}, x_n),$$
$$D(Tx_{n-1}, Tx_n), \frac{1}{2}\Big[d(x_{n-1}, x_n) + d(x_n, Tx_n)\Big]\Big\}.$$

Hence

$$D(Tx_{n-1}, Tx_n) < d(x_{n-1}, x_n).$$

From this

$$d(x_n, x_{n+1}) = a_n + D(Tx_{n-1}, Tx_n) < a_n + d(x_{n-1}, x_n).$$

Setting $b_n = d(x_n, x_{n+1})$, we have

$$0 \leqslant b_n < a_n + b_{n-1},$$
$$\Sigma^+ a_n < \infty.$$

This follows $b_n \to b \geqslant 0$. Indeed, set $b = \lim b_n$. Then for every $\varepsilon > 0$ there is an integer N such that $\Sigma^+_{n \geqslant N} a_n < \frac{\varepsilon}{2}$ and $\sup_{n \geqslant N} b_n < b + \varepsilon$. On the other hand, for every $n \geqslant N$ there exists $n' > n$ such that $b_{n'} > b - \frac{\varepsilon}{2}$. We have

$$b_{n'} < b_{n'-1} + a_{n'} < \ldots < b_n + \Sigma^+_{m \geqslant N} a_m < b_n + \frac{\varepsilon}{2}.$$

Thus, for every $n \geqslant N$ we get

$$b + \varepsilon > b_n > b_{n'} - \frac{\varepsilon}{2} > b - \varepsilon,$$

i.e. $b_n \to b$. Observe that

$$d(y^*, Tx^*) \leqslant d(y^*, x_{n_i+1}) + d(x_{n_i+1}, Tx^*)$$
$$\leqslant d(y^*, x_{n_i+1}) + D(Tx_{n_i}, Tx^*).$$

By the continuity of T we get $y^* \in Tx^*$, hence

$$D(Tx^*, Ty^*) \geqslant d(y^*, Ty^*). \qquad (2)$$

Since $O(x_0)$ is quasinormal we have

$$d(x^*, y^*) - D(Tx^*, Ty^*) = \lim d(x_{n_i}, x_{n_i+1}) - \lim D(Tx_{n_i}, Tx_{n_i+1})$$
$$= \lim [d(x_{n_i+1}, x_{n_i+2}) - D(Tx_{n_i}, Tx_{n_i+1})] \leqslant 0.$$

Thus

$$D(Tx^*, Ty^*) \geqslant d(x^*, y^*) \qquad (3)$$

On the other hand, if $x^* \neq y^*$ then we get

$$D(Tx^*, Ty^*) < \max \left\{d(x^*, y^*), d(x^*, Tx^*), d(y^*, Ty^*), \frac{1}{2} d(x^*, Ty)\right\} =$$
$$\max \{d(x^*, y^*), d(y^*, Ty^*)\},$$

contradicting (2) and (3). Consequently, $x^* = y^*$ and the proof is complete.

Remark 2. 1. The above theorem implies a result of Smithson in [14].

Remark 2. 2. *If in addition to the hypotheses of Theorem 2.1 we assume T is of X into K (X) then the condition (S) can be replaced by a weaker one: there is a convergent subsequence* $x_n \to x^*$.

Indeed, by the continuity of T we have

$$d\,(x_{n_{i+1}},\, Tx^*) \leqslant D\,(Tx_{n_i},\, Tx^*) \to 0.$$

Since Tx^* is compact, there is a subsequence $\{x_{n_{i_k}+1}\}$ of $\{x_{n_{i+1}}\}$ such that $x_{n_{i_k}+1} \to y^* \in Tx^*$. Applying the theorem. 2.1. we get the required result.

Proposition 2. 1. *Let* (X, d) *be a metric space,* T *be a mapping of* X *into* $CB\,(X)$. *Set* $\varphi\,(x) = d\,(x, Tx)$ $(\forall x \in X)$. *Then*

(i) *If* T *is upper semicontinuous then* φ *is lower semicontinuous.*
(ii) *If* T *is lower semicontinuous then* φ *is upper semicontinuous.*

Proof. Since (i) has been proved in [7] we must only prove (ii). Set $a \in R$ and put $A = \{x \in X \mid \varphi\,(x) < a\}$. We shall show that A is open. Obviously we may assume $a > 0$. Set $x_0 \in A$, then $\varphi\,(x_0) = d\,(x_0, Tx_0) < a$. Denote $r = \varphi\,(x_0)$, $\delta = a - r$. Then there exists $x_1 \in Tx_0$ such that $d\,(x_0, x_1) < r + \frac{\delta}{3}$. Let $B\left(x_1; \frac{\delta}{3}\right)$ be the open ball with radius $\frac{\delta}{3}$ and centre x_1. Then $B\left(x_1; \frac{\delta}{3}\right) \cap Tx_0 \neq \varnothing$. By the lower semicontinuity of T, there is a ball $B\,(x_0; \rho)$ such that $B\left(x_1; \frac{\delta}{3}\right) \cap Tx \neq \varnothing$ $(\forall x \in B\,(x_0; \rho))$. Set $\delta_1 = \min\left\{\frac{\delta}{3}, \rho\right\}$, then $\forall x \in B\,(x_0; \delta_1)$ $\exists\, y \in B\left(x_1; \frac{\delta}{3}\right) \cap Tx$ and hence

$$\varphi\,(x) = d\,(x, Tx) \leqslant d\,(x, y) \leqslant d\,(x, x_0) + d\,(x_0, x_1) + d\,(x_1, y) <$$
$$< \frac{\delta}{3} + r + \frac{\delta}{3} + \frac{\delta}{3} = r + \delta = a.$$

Thus $B\,(x_0; \delta_1) \subset A$ and A is open. The proposition is proved.

Remark 2. 3. *If* (X, d) *is a compact metric space,* T *is a upper semicontinuous mapping of* X *into* $CB\,(X)$ *satisfying (1) then* T *has a fixed point.*

Indeed, by Proposition 2. 1 (i) $\varphi(x) = d(x, Tx)$ is lower semicontinuous. Since X is compact, there is $x^* \in X$ such that

$$d(x^*, Tx^*) = \varphi(x^*) = \min_{x \in X} \varphi(x) = \min_{x \in X} d(x, Tx) = \alpha \geqslant 0.$$

If $\alpha = 0$ then $x^* \in Tx^*$. If $\alpha > 0$, by the compact—ness of Tx^* there is $y^* \in Tx^*$ such that

$$d(x^*, y^*) = d(x^*, Tx^*) = \alpha.$$

Since $x^* \neq y^*$, by (1) we obtain

$$D(Tx^*, Ty^*) < \max \{ d(x^*, y^*), d(x^*, Tx^*), d(y^*, Ty^*), \frac{1}{2} d(x^*, Ty^*) \}.$$

Since $d(x^*, Ty^*) \leqslant d(x^*, y^*) + d(y^*, Ty^*)$ and

we get
$$d(x^*, y^*) = d(x^*, Tx^*) = \alpha \leqslant d(y^*, Ty^*)$$
$$D(Tx^*, Ty^*) < d(y^*, Ty^*). \tag{4}$$

On the other hand, since $y^* \in Tx^*$ then $d(y^*, Ty^*) \leqslant D(Tx^*, Ty^*)$, contradicting (4). Thus $\alpha = 0$ and hence $x^* \in Tx^*$.

Theorem 2. 2. *Let (X,d) be a metric space, T be a closed lower semicontinuous mapping of X into $CB(X)$ satisfying (1). Suppose there is a normal orbit $O(x_0)$ satisfying (S). Then the conclusion of Theorem 2.1 still holds.*

Proof. Let $O(x_0) = \{x_n | n = 0,1,2,...\}$, set $c_n = d(x_{n+1}, x_{n+2}) - d(x_{n+1}, Tx_{n+1})$,

Then, similarly to the proof of Theorem 2. 1, we have

$$D(Tx_n, Tx_{n+1}) < d(x_n, x_{n+1})$$

and hence

$$d(x_{n+1}, x_{n+2}) = C_n + d(x_{n+1}, Tx_{n+1}) \leqslant c_n + D(Tx_n, Tx_{n+1})$$
$$< c_n + d(x_n, x_{n+1}).$$

Setting $b_n = d(x_n, x_{n+1})$ we get

$$0 \leqslant b_{n+1} < c_n + b_n, \quad c_n \geqslant 0, \quad \Sigma c_n < \infty,$$

from this $b_n \to b \geqslant 0$.

Since T is closed, $y^* \in Tx^*$. If $x^* \neq y^*$ we have

$$d(x^*, Ty^*) \leqslant D(Tx^*, Ty^*) < \max \left\{ d(x^*, y^*), \ d(x^*, Tx^*), \ d(y^*, Ty^*), \frac{1}{2} d(x^*, Ty^*) \right\}.$$

Hence

$$d(x^*, Ty^*) < d(x^*, y^*) = \lim d(x_{n_i}, x_{n_i+1}) = b. \tag{5}$$

On the other hand,

$$d(x_{n_i+1}, Tx_{n_i+1}) = d(x_{n_i+1}, x_{n_i+2}) - c_n,$$

i.e. $d(x_{n_i+1}, Tx_{n_i+1}) \to b$. By Proposition 2. 1 (ii), φ is upper semicontinuous, we obtain

$$\varphi(y^*) = d(y^*, Ty^*) \geqslant b,$$

contradicting (5). Thus $x^* = y^*$ and the proof is complete.

3. Fixed points for multivalued generalized contractions.

Combining the methods of Wong in [16] and of Nadler in [9] we can prove the following result.

Theorem 3.1. *Let (X,d) be a complete metric space, S, T be two mappings of X into $CB(X)$, Suppose there exist nonnegative numbers $a_1, \ldots, a_5$ with $\Sigma a_i < 1$ and*

$$(a_2 - a_1)(a_4 - a_3) \geqslant 0 \tag{1}$$

such that

$$D(Sx, Ty) \leqslant a_1 d(x,Sx) + a_2 d(y, Ty) + a_3 d(x,Ty) + a_4 d(y,Sx) + a_5 d(x,y), \qquad (\forall x, y \in X) \tag{2}$$

Then the fixed point set of each S, T is nonempty and these two sets coincide.

Proof. Without loss of generality, we may assume $a_5 > 0$. Set $\alpha = a_1 + a_3 + a_5$, $\beta = a_2 + a_4 + a_5$,

$$r = \frac{\alpha}{1-a_2-a_3}, \ s = \frac{\beta}{1-a_1-a_4}.$$

Let $x_0 \in X$, $x_1 \in Sx_0$, there exists $x_2 \in Tx_1$ such that

$$d(x_1, x_2) \leqslant D(Sx_0, Tx_1) + \alpha.$$

By (2) we have

$$\begin{aligned} d(x_1, x_2) &\leqslant a_1 d(x_0, Sx_0) + a_2 d(x_1, Tx_1) + a_3 d(x_0, Tx_1) + \\ &+ a_5 d(x_0, x_1) + \alpha \leqslant a_1 d(x_0, x_1) + a_2 d(x_1, x_2) + \\ &+ a_3 d(x_0, x_2) + a_5 d(x_0, x_1) + \alpha \leqslant a_1 d(x_0, x_1) + \\ &+ a_2 d(x_1, x_2) + a_3 [d(x_0, x_1) + d(x_1, x_2)] + a_5 d(x_0, x_1) \\ &+ \alpha = \alpha\, d(x_0, x_1) + (a_2 + a_3) d(x_1, x_2) + \alpha. \end{aligned}$$

From this

$$d(x_1, x_2) \leqslant r d(x_0, x_1) + r.$$

Select $x_3 \in Sx_2$ such that

$$d(x_2, x_3) \leqslant D(Tx_1, Sx_2) + r\beta.$$

Similarly, we have

$$d(x_2, x_3) \leqslant \beta\, d(x_1, x_2) + (a_1+a_4)\, d(x_2, x_3) + r\beta.$$

So

$$d(x_2, x_3) \leqslant sd(x_1, x_2) + rs.$$

Generally, for $x_{2n+1} \in Sx_{2n}$ select $x_{2n+2} \in Tx_{2n+1}$ such that

$$d(x_{2n+2}, x_{2n+1}) \leqslant D(Sx_{2n}, Tx_{2n+1}) + (sr)^n r, (sr)^n r$$

then select $x_{2n+3} \in Sx_{2n+2}$ such that

$$d(x_{2n+2}, x_{2n+3}) \leqslant D(Tx_{2n+1}, Sx_{2n+2}) + (sr)^n r\beta.$$

Repeating the above argument, we obtain, for each $n = 0, 1, 2, \ldots,$

$$d(x_{2n+1}, x_{2n+2}) \leqslant rd(x_{2n}, x_{2n+1}) + r(sr)^n;$$

$$d(x_{2n+2}, x_{2n+3}) \leqslant sd(x_{2n+1}, x_{2n+2}) + (sr)^{n+1}.$$

From this,

$$d(x_{2n+2}, x_{2n+3}) \leqslant s[rd(x_{2n}, x_{2n+1}) + r(sr)^n] + (sr)^{n+1} =$$

$$= srd(x_{2n}, x_{2n+1}) + 2(sr)^{n+1} \leqslant \ldots$$

$$\leqslant (sr)^{n+1} d(x_0, x_1) + 2(n+1)(sr)^{n+1}.$$

Similarly,

$$d(x_{2n+1}, x_{2n+2}) \leqslant r(sr)^n d(x_0, x_1) + (2n+1) r(sr)^n.$$

Consequently,

$$\sum_{m=1}^{\infty} d(x_m, x_{m+1}) \leqslant d(x_0, x_1) \sum_{n=0}^{\infty} (sr)^{n+1} + 2 \sum_{n=0}^{\infty} (n+1)(sr)^{n+1}$$

$$+ rd(x_0, x_1) \sum_{n=0}^{\infty} (sr)^n + r \sum_{n=0}^{\infty} (2n+1)(sr)^n. \tag{3}$$

Since by (1), $sr < 1$ hence the right side of (3) converges. So $\{x_n\}$ is Cauchy. By completeness of X, $x_n \to x^* \in X$. We shall prove that x^* is a fixed point of S. Indeed, let n be given. Then

$$d(x^*, Sx^*) \leqslant d(x^*, x_{2n+2}) + d(x_{2n+2}, Sx^*) \leqslant d(x^*, x_{2n+2}) +$$

$$+ D(Sx^*, Tx_{2n+1}) \leqslant d(x^*, x_{2n+2}) + a_1\, d(x^*, Sx^*) +$$

$$a_2 d(x_{2n+1}, Tx_{2n+1}) + a_3\, d(x^*, Tx_{2n+1}) + a_4\, d(x_{2n+1}, Sx^*) +$$

$$+ a_5\, d(x^*, x_{2n+1}) \leqslant d(x^*, x_{2n+2}) + a_1\, d(x^*, Sx^*) +$$

$+ a_2 d(x_{2n+1}, x_{2n+2}) + a_3 d(x^*, x_{2n+2}) + a_4 d(x_{2n+1}, Sx^*) + a_5 d(x^*, x_{2n+1})$.

Letting $n \to \infty$ we obtain

$$d(x^*, Sx^*) \leqslant (a_1 + a_4)\, d(x^*, Sx^*).$$

Hence $d(x^*, Sx^*) = 0$, i. e. $x^* \in Sx^*$.

We shall prove that if y^* is a fixed point of S then it is also a fixed point of T. Indeed, by (2) for $x = y = y^*$ we get

$$d(y^*, Ty^*) \leqslant D\,(Sy^*, Ty^*) \leqslant (a_2 + a_3)\, d(y^*, Ty^*).$$

Hence $y^* \in Ty^*$. By the symmetry of S and T we conclude that two fixed point sets of S and T coincide and the proof is complete.

Remark 3. 1. The above theorem generalizes the following results: When $S = T$ we obtain a proposition of Alesina — Massa — Roux [1]. If S and T are singlevalued, we have a theorem of Wong [16]. When $S = T$ is singlevalued, this is a theorem of Hardy — Rogers [5]. If $S = T$, $a_3 = a_4 = 0$ we get a theorem of Reich [12]. When $a_3 = a_4 = a_5 = 0$ this is a theorem of Ray [11]. If S and T are singlevalued, $a_3 = a_4 = a_5 = 0$ we have a theorem of Srivastava — Gupta [15]. When $S = T$ is singlevalued, $a_3 = a_4 = a_5 = 0$ we obtain a theorem of Kannan [6]. Finally, if $a_1 = a_2 = a_3 = a_4 = 0$ we have a theorem of Nadler [3], the singlevalued form of which is the well-known Banach contraction principle.

Definition 3. 1. A metric space (X,d) is said to be (metrically) *convex* (in the sense of K. Menger) if $\forall\, x, y \in X$, $x \neq y$ there exists $z \in X$, $z \neq x$, $z \neq y$ such that

$$d\,(x, z) + d\,(z, y) = d\,(x, y). \tag{4}$$

For each subset K of X we denote ∂K the boundary of K. We shall use the following fact [2]: If (X, d) is convex and complete, $K \subset X$, $x \in K$, $y \notin K$ then there exists $z \in \partial K$ satisfying (4).

Following the argument of Assad — Kirk in [2], we obtain

Theorem 3. 2. *Let (X,d) be a complete convex metric space, K be a closed subset of X, S, T be two mappings of K into $CB(X)$ satisfying the boundary condition: $Sx \subset K$, $Tx \subset K$ $(\forall x \in \partial K)$. Suppose there exist nonnegative numbers $a_1, \ldots, a_5$ with*

$$a_i + a_j < \frac{1 - a_5}{3 + a_5}, (i = 1,2\,;\ j = 3,4) \tag{5}$$

such that (2) holds. Then the conclusion of the theorem 3.1 still holds.

Proof. First, by (5), $\Sigma a_i < 1$ and hence, similarly to the proof of Theorem 3. 1, the fixed point sets of S and T coincide. Thus, we have only to show that S has at least one fixed point. Without loss of generality we may assume $a_5 > 0$. Let α; β, r, s as in the proof of Theorem 3. 1. It is easy to verify that by (5) we have $r < 1$, $s < 1$. We construct a sequence of mappings

$\{T_n\}$ $(n = 1,2,3,\dots)$ with $T_n = S$ if n is odd, and $T_n = T$ if n is even.

Let $x_0 \in X$, $x_1' \in T_1 x_0$. If $x_1' \in K$ put $x_1 = x_1'$, if $x_1' \notin K$ we denote by x_1 a point of ∂K satisfying

$$d(x_0, x_1) + d(x_1, x_1') = d(x_0, x_1').$$

Then select $x_2' \in T_2 x_1$ such that

$$d(x_1', x_2') \leqslant D(T_1 x_0, T_2 x_1) + \alpha.$$

If $x_2' \in K$ put $x_2 = x_2'$; otherwise let x_2 be a point of ∂K such that $d(x_1, x_2) + d(x_2, x_2') = d(x_1, x')$. Then select $x_3' \in T_3 x_2$ such that

$$d(x_2', x_3') \leqslant D(T_2 x_1, T_3 x_2) + \beta.$$

Generally, for a given x_n we select $x'_{n+1} \in T_{n+1} x_n$ such that

$$d(x'_n, x'_{n+1}) \leqslant \begin{cases} D(T_n x_{n-1}, T_{n+1} x_n) + r^{n-1} & \text{if } n \text{ is odd,} \\ D(T_n x_{n-1}, T_{n+1} x_n) + s^{n-1} & \text{if } n \text{ is even.} \end{cases}$$

Then put $x_{n+1} = x'_{n+1}$ if $x'_{n+1} \in K$; otherwise let x_{n+1} be a point of ∂K such that

$$d(x_n, x_{n+1}) + d(x_{n+1}, x'_{n+1}) = d(x_n, x'_{n+1}). \tag{6}$$

In the result we obtain two sequences $\{x_n\}$, $\{x'_n\}$. Denote

$$P = \{x_n : x_n = x'_n\},$$

$$Q = \{x_n : x_n \neq x'_n\}.$$

Observe that if $x_n \in Q$ for some n, then $x_{n+1} \in P$.

We shall consider three following cases:

1. $x_n \in P$, $x_{n+1} \in P$. In this case, if n is odd we have

$$d(x_n, x_{n+1}) \leqslant D(Sx_{n-1}, Tx_n) + r^{n-1}\alpha \leqslant a_1 d(x_{n-1}, Sx_{n-1}) + a_2 d(x_n, Tx_n) + a_3 d(x_{n-1}, x_{n+1}) + a_5 d(x_{n-1}, x_n) + r^{n-1}\alpha \leqslant a_1 d(x_{n-1}, x_n) + a_2 d(x_n, x_{n+1}) + a_3 [d(x_{n-1}, x_n) + d(x_n, x_{n+1})] + a_5 d(x_{n-1}, x_n) + r^{n-1}\alpha.$$

From this

$$d(x_n, x_{n+1}) \leqslant rd(x_{n-1}, x_n) + r^n.$$

Similarly, for even n we have

$$d\,(x_n,\, x_{n+1}) \leqslant sd\,(x_{n-1},\, x_n) + s^n.$$

2. $x_n \in P$, $x_{n+1} \in Q$. If n is odd, we have

$$d(x_n,\, x_{n+1}) < d(x_n,\, x'_{n+1}) \leqslant rd(x_{n-1},\, x_n) + r^n. \tag{7}$$

Similarly, for even n we obtain

$$d(x_n,\, x_{n+1}) < d(x_n,\, x'_{n+1}) \leqslant sd(x_{n-1},\, x_n) + s^n. \tag{8}$$

3. $x_n \in Q$, $x_{n+1} \in P$. Then $x_{n-1} \in P$ and if n is odd we have

$$d(x_n, x_{n+1}) \leqslant d(x_n,\, x'_n) + d(x'_n,\, x'_{n+1}) \leqslant d(x_n,\, x'_n) + a_1\, [d(x_{n-1},\, x_n) +$$

$$+ d(x_n,\, x'_n)] + a_2 d(x_n,\, x_{n+1}) + a_3\, [d(x_{n-1},\, x_n) + d(x_n,\, x_{n+1})] +$$

$$+ a_4 d(x_n,\, x'_n) + a_5 d(x_{n-1},\, x_n) + r^{n-1}\alpha.$$

It follows

$$d\,(x'_n,\, x_{n+1}) \leqslant td(x_n,\, x'_n) + rd(x_{n-1},\, x_n) + r^n, \text{ where } t = \frac{1 + a_1 + a_4}{1 - a_2 - a_3}.$$

Since $t > 1$, $r > 1$, by we obtain (6)

$$d(x_n,\, x_{n+1}) \leqslant td\,(x_{n-1},\, x'_n) + r^n.$$

Observe that $n - 1$ is even, by (8) we get

$$d(x_n,\, x_{n+1}) \leqslant tsd\,(x_{n-2},\, x_{n-1}) + ts^{n-1} + r^n.$$

Similarly, if n is even we have

$$d(x_n,\, x_{n+1}) \leqslant urd\,(x_{n-2} x_{1-n}) + ur^{n-1} + s^n,$$

with $$u = \frac{1 + a_2 + a_3}{1 - a_1 - a_4}.$$

Put $\gamma = \max\,\{ ts,\, ur \}$, we have

$$d\,(x_n,\, x_{n+1}) \leqslant \begin{cases} \gamma d\,(x_{n-1},\, x_n) + \gamma^n & \text{or} \\ \gamma d\,(x_{n-2},\, x_{n-1}) + \gamma^{n-1} + \gamma^n. \end{cases}$$

From (5) it is easy to show that $\gamma < 1$.

Setting $\delta = \frac{1}{\sqrt{\gamma}} \max \{ d(x_0, x_1), d(x_1, x_2) \}$, by induction we can easily prove the inequality

$$d(x_n, x_{n+1}) \leqslant \sqrt{\gamma^n} (\delta + n). \qquad (n = 1, 2, 3, \ldots).$$

From this $\{x_n\}$ is Cauchy, and hence $x_n \to x^* \in K$. We shall prove $x^* \in Sx^*$. Indeed, fix an even n, if $x_n \in P$ we have

$$d(x^*, Sx^*) \leqslant d(x^*, x_n) + d(x_n, Sx^*) \leqslant d(x^*, x_n) + D(Tx_{n-1}, Sx^*) \leqslant$$
$$\leqslant d(x^*, x_n) + a_1 d(x^*, Sx^*) + a_2 d(x_{n-1}, x_n) + a_3 d(x^*, x_n) +$$
$$+ a_4 d(x_{n-1}, Sx^*) + a_5 d(x_{n-1}, x^*).$$

If there is an infinite sequence of x_n in P with even n then letting n tend to infinity, we obtain

$$d(x^*, Sx^*) \leqslant (a_1 + a_4)\, d(x^*, Sx^*),$$

and hence $x^* \in Sx^*$.

Otherwise, there exists an infinite sequence of x_n in Q with even n. Then, for every n large enough we have $x_{n-1} \in P$ and

$$d(x^*, Sx^*) \leqslant d(x^*, x_{n-1}) + d(x_{n-1}, x'_n) + d(x'_n, Sx^*) \leqslant$$
$$\leqslant d(x^*, x_{n-1}) + d(x_{n-1}, x'_n) + D(Tx_{n-1}, Sx^*) \leqslant d(x^*, x_{n-1}) +$$
$$+ d(x_{n-1}, x'_n) + a_1 d(x^*, Sx^*) + a_2 d(x_{n-1}, x_n) + a_3 [\, d(x^*, x_{n-1}) +$$
$$+ d(x_{n-1}, x'_n)] + a_4 d(x_{n-1}, Sx^*) + a_5 d(x_{n-1}, x^*) \leqslant$$
$$\leqslant (1 + a_3 + a_5)\, d(x^*, x_{n-1}) + a_1 d(x^*, Sx^*) + a_4 d(x_{n-1}, Sx^*) +$$
$$+ (1 + a_2 + a_3)\, d(x_{n-1}, x'_n).$$

From this and (7) it follows

$$d(x^*, Sx^*) \leqslant (1 + a_3 + a_5)\, d(x^*, x_{n-1}) + a_1 d(x^*, Sx) +$$
$$+ a_4 d(x_{n-1}, Sx^*) + (1+a_2+a_3)\, [\, r d(x_{n-2}, x_{n-1}) + r^{n-1}].$$

Letting n tend to infinity, we obtain

$$d(x^*, Sx^*) \leqslant (a_1 + a_4)\, d(x^*, Sx^*),$$

and hence $x^* \in Sx^*$. The proof is complete.

Remark. 3. 2. When $a_1 = a_2 = a_3 = a_4 = 0$ and $S = T$ we obtain a theorem of Assad — Kirk [2].

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