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ON A PROPERTY OF CONTRAVARIANT p-VECTOR-DENSITIES OF WEIGHT + 1 AND ITS APPLICATION

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INTRODUCTION

The present paper is devoted to the study of a property of the divergence of contravariant p-vector-densities of weight + 1. A new proof different from Schouten's one (see [3], § 3, IV,) is given.

The mentioned property will be used to describe Maxwell's equations and the equation of continuity in the field of electromagnetodynamics. These equations will be written in a system of curvilinear four-dimensional coordinates in space-time.

§ 1. SOME DEFINITIONS

Let $$x^i = (x^1, x^2, x^3, x^4) \; ; \; i = 1, 2, 3, 4.$$

$$x^{i'} = (x^{1'}, x^{2'}, x^{3'}, x^{4'}); \; i' = 1', 2', 3', 4'.$$

be the coordinates of a certain point M in a four-dimensional space, with respect to two systems of coordinates (i) and (i') respectively. The transformation of coordinates from system (i) (the basic system) to system (i') is given by :

$$x^{i'} = f^{i'}(x^i) \stackrel{\text{df}}{=} f^{i'}(x^1, x^2, x^3, x^4)^* \quad \begin{cases} i = 1, 2, 3, 4. \\ i' = 1', 2', 3', 4'. \end{cases} \tag{1-1}$$

with the conditions :

$$f^{i'}(x^i) \in c^r \; ; \; r \geqslant 2 \text{ and} \tag{1-2}$$

$$\Delta \stackrel{\text{df}}{=} \det A_i^{i'} \equiv \left| A_i^{i'} \right| \neq 0 \; ; \; A_i^{i'} \stackrel{\text{df}}{=} \frac{\partial x^{i'}}{\partial x^i} \tag{1-3}$$

It is known that if $\check{T} = \check{T}^{i_1 \dots i_p}$ is a contravariant p-vector-density of weight + 1, then from its definition we have two relations, namely :

$$\check{T}^{\dots i_k \, i_{k+1} \dots} = - \check{T}^{\dots i_{k+1} \, i_k \dots} \tag{1-4}$$

with every $k = 1, \dots, p-1$; $i_k = 1, 2, 3, 4$.

$$\check{T}^{i'_1 \dots i'_p} = \Delta^{-1} A^{i'_1 \dots i'_p}_{i_1 \dots i_p} \check{T}^{i_1 \dots i_p} \quad {}^{*)} \tag{1-5}$$

where

$$A^{i'_1 \dots i'_p}_{i_1 \dots i_p} \overset{\text{df}}{=} \frac{\partial x^{i'_1}}{\partial x^{i_1}} \dots \frac{\partial x^{i'_p}}{\partial x^{i_p}} \; ; \; i'_1, \dots, i'_p = 1', 2', 3', 4'.$$

The divergence of $\check{T}$ is defined by :

$$\text{Div} \; \check{T} \overset{\text{df}}{=} \partial_{i_p} \check{T}^{i_1 \dots i_{p-1} \, i_p} \tag{1-6}$$

where
$$\partial_{i_p} \overset{\text{df}}{=} \frac{\partial}{\partial x^{i_p}}$$

For the case $p = 1$, Div. of a contravariant vector-density of weight $+ 1$ is a scalar-density of the same weight (see [4], p. 280), then :

$$\partial_{i'} \check{T}^{i'} = \Delta^{-1} \partial_i \check{T}^i \tag{1-7}$$

§ 2. A PROPERTY OF DIV $\check{T}$

Div. of a contravariant $(p + 1)$-vector-density of weight $+ 1$ is a contravariant p-vector-density of the same weight.

We shall prove the above proposition by induction. Let us suppose that it is valid for the case of a p-vector-density.

With the help of the formula :

$$\frac{\partial x^{i'}}{\partial x^i} \frac{\partial}{\partial x^{i'}} = \frac{\partial}{\partial x^i} \quad \Rightarrow \quad A^{i'}_i \partial_{i'} = \partial_i \tag{2-1}$$

and using (1-5), (1-6) we have :

$$\partial_{i'_p} \check{T}^{i'_1 \dots i'_{p-1} i'_p} = \Delta^{-1} A^{i'_1 \dots i'_{p-1} i'_p}_{i_1 \dots i_{p-1} i_p} \partial_{i'_p} \check{T}^{i_1 \dots i_{p-1} i_p} +$$

$$+ \check{T}^{i_1 \dots i_{p-1} i_p} \partial_{i'_p} \left(\Delta^{-1} A^{i'_1 \dots i'_{p-1} i'_p}_{i_1 \dots i_{p-1} i_p} \right) =$$

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*) The indices $i_1, \dots, i_p$ run over the range 1, 2, 3, 4 and the summation convection will be used with respect to these indices.

$$= \Delta^{-1} A^{i'_1 \dots i'_{p-1}}_{i_1 \dots i_{p-1}} \partial_{i_p} \check{T}^{i_1 \dots i_{p-1} i_p} +$$

$$+ \check{T}^{i_1 \dots i_{p-1} i_p} \partial_{i'_p} \left(\Delta^{-1} A^{i'_1 \dots i'_{p-1} i'_p}_{i_1 \dots i_{p-1} i_p} \right) \tag{2-2}$$

From the induction assumption it follows that for any contravariant p-vector-density of weight + 1 we have :

$$\check{T}^{i_1 \dots i_{p-1} i_p} \partial_{i'_p} \left(\Delta^{-1} A^{i'_1 \dots i'_{p-1} i'_p}_{i_1 \dots i_{p-1} i_p} \right) = 0 \tag{2-3}$$

In the same way for the case of a $(p + 1)$-vector-density we get :

$$\partial_{i_{p+1}} \check{T}^{i'_1 \dots i'_p i'_{p+1}} = \Delta^{-1} A^{i'_1 \dots i'_p}_{i_1 \dots i_p} \partial_{i_{p+1}} \check{T}^{i_1 \dots i_p i_{p+1}} +$$

$$+ R_1 + R_2 . \tag{2-4}$$

where :

$$R_1 \overset{\text{df}}{=} \check{T}^{i_1 \dots i_p i_{p+1}} A^{i'_p}_{i_p} \partial_{i'_{p+1}} (\Delta^{-1} A^{i'_1 \dots i'_{p-1} i'_{p+1}}_{i_1 \dots i_{p-1} i_{p+1}}) \tag{2-5}$$

and

$$R_2 \overset{\text{df}}{=} \check{T}^{i_1 \dots i_p i_{p+1}} \Delta^{-1} A^{i'_1 \dots i'_{p-1} i'_{p+1}}_{i_1 \dots i_{p-1} i_{p+1}} \partial_{i'_{p+1}} A^{i'_p}_{i_p} . \tag{2-6}$$

To complete the proof it remains to show

$$R_1 + R_2 = 0$$

We first pay attention to R_1. In virtue of the assumption that $\check{T}^{i_1 \dots i_{p+1}}$ is a contravariant $(p + 1)$-vector-density of weight + 1, the expression $\check{T}^{i_1 \dots i_p i_{p+1}} A^{i'_p}_{i_p}$ with any concrete index of $i'_p = (1', 2', 3', 4')$ is satisfied by (1-4) and (1-5). Therefore it is a contravariant p-vector-density of weight + 1, and hence, by the induction assumption, (formula (2-3)) we obtain:

$$R_1 = 0.$$

We next consider R_2. By making use of the relation (1-4) for $\check{T}^{i_1 \dots i_p i_{p+1}}$ with indices i_p and i_{p+1}, and of the formula (2-1) in virtue of transformation condition (1-2), we obtain from (2-6)

$$R_2 = \check{T}^{i_1 \dots i_p i_{p+1}} \Delta^{-1} A^{i'_1 \dots i'_{p-1}}_{i_1 \dots i_{p-1}} \partial_{i_{p+1}} A^{i'_p}_{i_p} =$$

$$= \check{T}^{i_1 \dots i_p i_{p+1}} \Delta^{-1} A^{i'_1 \dots i'_{p-1}} \partial_{i_p} A^{i'_p}_{i_{p+1}} =$$

$$= - \check{T}^{i_1 \dots i_{p+1} i_p} \Delta^{-1} A^{i'_1 \dots i'_{p-1}}_{i_1 \dots i_{p-1}} \partial_{i_p} A^{i'_p}_{i_{p+1}} =$$

$$= - R_2$$

Thus $$R_2 = 0$$

which completes our proof.

Note : The property established above in four-dimensional space without a metric is also valid in n-dimensional space, whether a metrical tensor is given in it or not.

§ 3. INVARIANT FORM OF MAXWELL'S EQUATIONS IN ELECTROMAGNETOGASDYNAMICS

Maxwell's equations of electromagnetic field used in plasma dynamics are : (see [5], IV)

$$\nabla \times \vec{E} = - \frac{\partial \vec{B}}{\partial t} , \quad \nabla . \vec{B} = 0 \tag{3-1}$$

$$\nabla \times \vec{H} = - \frac{\partial \vec{D}}{\partial t} + \vec{J}, \quad \nabla . \vec{D} = \rho_e \tag{3-2}$$

where ∇ is the gradient operator, $\vec{E}$ is the vector electric field strength with components E_1 , E_2 , E_3 ; $\vec{H}$ is the vector magnetic field strength with components H_1 , H_2 , H_3 ; $\vec{J}$ is the electric current density with components J^1 , J^2 , J^3 ; t is the time ; and

$\vec{D} = \varepsilon \vec{E}$: the dielectric displacement

ε : the inductive capacity

$\vec{B} = \mu_e \vec{H}$: the magnetic flux density

μ_e : the magnetic permeability.

Usually we may assume that both ε and μ_e are constant for a given isotropic material. ρ_e is the excess electric charge of the plasma.

Let us consider equations (3-1), (3-2) in the inertial system of coordinates (i) with the space coordinates (x^1 , x^2 , x^3) and the time coordinate $x^4 = t$.

For the study of the internal structure of the electromagnetic field (see [2]) it is convenient to introduce the covariant bivector F_{ij} instead of vectors $\vec{B}$ and $\vec{D}$ and the contravariant bivector density of weight + 1 instead of vectors $\vec{H}$ and $\vec{D}$. We have, with the coordinates transformation (i) → (i'):

$$F_{ij} \overset{df}{=} \left\| \begin{matrix} 0 & B^3 & -B^2 & E_1 \\ -B^3 & 0 & B^1 & E_2 \\ B^2 & -B^1 & 0 & E_3 \\ -E_1 & -E_2 & -E_3 & 0 \end{matrix} \right\| \quad ; \quad F_{i'j'} = A^{ij}_{i'j'} F_{ij} \tag{3-3}$$

$$\check{G}^{ij} \overset{df}{=} \left\| \begin{matrix} 0 & H_3 & -H_2 & -D^1 \\ -H_3 & 0 & H_1 & -D^2 \\ H_2 & -H_1 & 0 & -D^3 \\ D^1 & D^2 & D^3 & 0 \end{matrix} \right\| \quad ; \quad \check{G}^{i'j'} = \Delta^{-1} A^{i'j'}_{ij} \check{G}^{ij} \tag{3-4}$$

The extension connected with the electric current density $\vec{J}$ and the excess electric charge of the plasma ρ_e is given by (see [2]) :

$$\check{J}^i \overset{df}{=} (J^1, J^2, J^3, \rho_e) \tag{3-5}$$

$$\check{J}^{i'} = \Delta^{-1} A^{i'}_i \check{J}^i \tag{3-6}$$

In the system of coordinates (i), using the mentioned extended formulae we obtain the equations equivalent to (3-1) and (3-2) respectively

$$\varepsilon^{ijkl} \partial_i F_{jk} \equiv \partial_i \varepsilon^{ijkl} F_{jk} = 0 \tag{3-7}$$

$$\partial_j \check{G}^{ij} = \check{J}^i \tag{3-8}$$

where $\partial_i \overset{df}{=} \frac{\partial}{\partial x^i}$; i, j, k and l = 1,2,3,4.

ε^{ijkl} denotes the usual permutation indicator.

We see that $\varepsilon^{ijkl} F_{kl}$ and $\check{G}^{ij}$ are contravariant bivector – densities of weight + 1, therefore, by virtue of the property of Div. proved in § 2, equations (3-7) and (3-8) are tensor equations ; they are invariant under coordinates transformation given by (1-1), (1-2) and (1-3). Finally in any system of coordinates (i') Maxwell's equations are found to be :

$$\varepsilon^{i'j'k'l'} \partial_{i'} F_{k'l'} = 0 \tag{3-9}$$

$$\partial_{j'} \check{G}^{i'j'} = \check{J}^{i'} \tag{3-10}$$

i', j', k' and $l' = 1', 2', 3', 4'$.

These equations have been built from the macroscopic point of view (see [5] and [2]).

§ 4 INVARIANT FORM OF THE EQUATION OF CONTINUITY

The law of conservation of mass gives the equation of continuity. Let us consider this equation in a certain inertial system of coordinates (i). As in § 3, it may be written :

$$\frac{\partial}{\partial x^1} \rho v^1 + \frac{\partial}{\partial x^2} \rho v^2 + \frac{\partial}{\partial x^3} \rho v^3 + \frac{\partial}{\partial x^4} \rho = 0 \tag{4-1}$$

where v^1, v^2 and v^3 are the velocity components along the x^1, x^2 and x^3 axes respectively, ρ is the density.

For the study of equation (4-1) in the class of systems of coordinates (i') which can be obtained from system (i) by (1-1), (1-2) and (1-3) we may extend the notion of the velocity vector and of the density in the space — time by the relations :

$$u^i \overset{\text{df}}{=} (v^1, v^2, v^3, 1) ; \; i = 1,2,3,4. \tag{4-2}$$

and

$$\underset{(i')}{\check{\rho}} \overset{\text{df}}{=} \Delta^{-1} \underset{(i)}{\check{\rho}} \; ; \; \underset{(i)}{\check{\rho}} \overset{\text{df}}{=} \rho. \tag{4-3}$$

According to (1-1) the vector derivative dx^i as well as the vector velocity $u^i \overset{\text{df}}{=} \frac{dx^i}{dt}$ is a contravariant vector in four-dimensional space. Therefore we have

$$u^{i'} = A^{i'}_{i} u^{i} \tag{4-4}$$

where $$u^{i'} \overset{\text{df}}{=} \frac{dx^{i'}}{dt}.$$

Using the extension relations (4-2) and (4-3) we can write the equation of continuity in the form :

$$\partial_j \left(\underset{(i)}{\check{\rho}} \, u^j \right) = 0 \tag{4-5}$$

In view of (4-3) and (4-4) vector $\underset{(i)}{\check{\rho}} \, u^j$ is a covariant vector density of weight $+1$, therefore from (1-7) we see that the left-hand side of equation (4-5) is a scalar density. This shows the invariance of the equation and

$$\partial_{j'} \left(\underset{(i')}{\check{\rho}} \, u^{j'} \right) = 0 \tag{4-6}$$

Note that it is quite simple to write equation (4-6) in spherical or cylindrical coordinates. The form obtained in this manner for that equation completely coincides with the form given in standard text-books on hydrodynamics. (See for example, Theoretical hydrodynamics, T.I, p. 26 by N.E. Kochin, J.A. Kibel and N.V. Roze, Moscow 1963, in Russian.).

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