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ON AN APPROXIMATION THEOREM FOR SET-VALUED MAPPINGS

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In this paper we shall consider the problem of approximating a section of a set-valued mapping U (from a real interval I to $\mathbb{R}^n$) by sections whose values at almost every $t \in I$ belong to exconv$U(t)$ (the set of extremal points of the convex hull of $U(t)$). Our result may be considered as an extension of the approximation lemma by Gamkrelidze and Kharatishvili in [1]. On the other hand it is known ([2], [3]) that under suitable assumptions the integral of a set-valued mapping is convex. As a consequence of our main theorem, we obtain that the set of all measurable sections of a set-valued mapping is quasiconvex in a sense.

Let $L_1^m(I)$ be the linear space of all Lebesgue integrable m-dimensional vector-valued functions defined on an interval I (finite or infinite). We shall introduce in this space the norm

$$\| g \|_{s_I} = \max_{1 \leqslant j \leqslant m} \sup_{t',t'' \in I} \left| \int_{t'}^{t''} g_j(t)dt \right|$$

where $$g = (g_1, g_2, \ldots, g_m).$$

Consider n vector-valued functions

$$f^{(i)} : I \to \mathbb{R}^m \qquad (i = 1,2, \ldots, n)$$

and a set-valued mapping

$$U : I \to 2^{\mathbb{R}^m}.$$

THEOREM 1. *Suppose that:*

1) *the functions* $f^{(i)}(t)$ $(i = 1,2, \ldots, n)$ *are measurable,*

2) *the graph of U is a Borel (mod 0) set ([4]) in $I \times \mathbb{R}^n$ and for almost all $t \in I$, $U(t)$ is a compact set in $\mathbb{R}^n$ contained in a ball around $0 \in \mathbb{R}^n$ of radius $\rho(t)$ such that $f^{(i)}(t)\rho(t) \in L_1^m(I)$.*

Then for every measurable section $\alpha(t) = (\alpha_1(t), \ldots, \alpha_n(t))$ of the mapping U and for every $\varepsilon > 0$ there exists a measurable section such that $\nu^{(\varepsilon)}(t) = \left(\nu_1^{(\varepsilon)}(t), \ldots, \nu_n^{(\varepsilon)}(t)\right) \in$ exconv$U(t)$ a. e. and

$$\left\| \sum_{i=1}^{n} \left(\alpha_i - \nu_i^{(\varepsilon)}\right) f^{(i)} \right\|_{s_I} < \varepsilon.$$

Proof. We first prove the theorem for the special case when I is a bounded interval and U is the constant mapping

$$U(t) \equiv S_n = \left\{\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{R}_+^n, \ \sum_{i=1}^{n} \lambda_i = 1\right\}.$$

From condition 2) it follows that each function $f^{(i)}(t)$ is integrable over I, so that there exists a piecewise constant function $\bar{f}^{(i)}(t)$ such that

$$\sum_{i=1}^{n} \int_I \left| f^{(i)}(t) - \bar{f}^{(i)}(t) \right| dt < \frac{\varepsilon}{4} \tag{1}$$

where $|.|$ denotes the euclidean norm in $\mathbb{R}^m$.

Let $\{I_p\}_{p=1}^{k}$ be a partition of I such that the functions $\bar{f}^{(i)}(t)$ take constant values on every I_p and

$$\sum_{i=1}^{n} \int_{I_p} |f^{(i)}(t)|\, dt < \frac{\varepsilon}{4}. \tag{2}$$

Since $\sum_{i=1}^{n} \int_{I_p} \alpha_i(t)\, dt = \int_{I_p} dt = \text{meas } I_p$ we can divide every interval I_p into n subintervals $\{I_{p,i}\}_{i=1}^{n}$ such that $(I_{p,i} \cap I_{p,j}) = \varnothing$ if $i \neq j$ and meas $I_{p,i} = \int_{I_p} \alpha_i(t)\, dt$. Put

$$\hat{f}(t) = f^{(i)}(t) \quad \text{if} \quad t \in I_{p,i}$$

$$\hat{\bar{f}}(t) = \bar{f}^{(i)}(t) \quad \text{if} \quad t \in I_{p,i}$$

i.e.
$$\hat{f}(t) = \sum_{i=1}^{n} \chi_{M_i^{(\varepsilon)}}(t) f^{(i)}(t)$$
$$\hat{\bar{f}}(t) = \sum_{i=1}^{n} \chi_{M_i^{(\varepsilon)}}(t) \bar{f}^{(i)}(t)$$

where $M_i^{(\varepsilon)} = \bigcup_{(p)} I_{p,i}$, $\chi_M(t)$ denotes the characteristic function of the set M.

Since the functions $f^{(i)}(t)$ take the constant vector values $f^{(i)}(t) = a_p^{(i)}$ on I_p we have :

$$\int_{I_p} \sum_{i=1}^{n} \alpha_i(t) \bar{f}^{(i)}(t)\, dt = \sum_{i=1}^{n} a_p^{(i)} \int_{I_p} \alpha_i(t)\, dt =$$
$$= \sum_{i=1}^{n} a_p^{(i)} \operatorname{meas} I_{p,i} = \int_{I_p} \hat{\bar{f}}(t)\, dt .$$

Consequently

$$\left| \int_{I_p} \left(\sum_{i=1}^{n} \alpha_i(t) f^{(i)}(t) - \hat{f}(t) \right) dt \right| \leqslant \left| \int_{I_p} \left(\sum_{i=1}^{n} \alpha_i(t) \bar{f}^{(i)}(t) - \hat{\bar{f}}(t) \right) dt \right|$$
$$+ \left| \int_{I_p} \sum_{i=1}^{n} \alpha_i(t) \left(f^{(i)}(t) - \bar{f}^{(i)}(t) \right) dt \right| + \left| \int_{I_p} \left(\hat{\bar{f}}(t) - \hat{f}(t) \right) dt \right| \leqslant$$
$$\leqslant \sum_{i=1}^{n} \int_{I_p} \left| f^{(i)}(t) - \bar{f}^{(i)}(t) \right| dt + \sum_{i=1}^{n} \int_{I_{p,i}} \left| \bar{f}^{(i)}(t) - f^{(i)}(t) \right| dt \leqslant$$
$$\leqslant 2 \sum_{i=1}^{n} \int_{I_p} \left| f^{(i)}(t) - \bar{f}^{(i)}(t) \right| dt. \tag{3}$$

Let us now estimate $\Phi(t', t'') = \left| \int_{t'}^{t''} \left(\sum_{i=1}^{n} \alpha_i(t) f^{(i)}(t) - \hat{f}(t) \right) dt \right|$. Denote by $I_{p'}$, $I_{p''}$ the subintervals containing t', t'' respectively. According to (1), (2), (3) we have

$$\Phi(t', t'') \leqslant \int_{I_{p'} \cup I_{p''}} \left| \sum_{i=1}^{n} \alpha_i(t) f^{(i)}(t) - \hat{f}(t) \right| dt + \sum_{p=p'+1}^{p''-1} \left| \int_{I_p} \left(\sum_{i=1}^{n} \alpha_i(t) f^{(i)}(t) - \hat{f}(t) \right) dt \right| \leqslant$$

$$\leqslant \frac{\varepsilon}{2} + \sum_{p=p'+1}^{p''-1} \sum_{i=1}^{n} \int_{I_p} \left| f^{(i)}(t) - \bar{f}^{(i)}(t) \right| dt \leqslant$$

$$\leqslant \frac{\varepsilon}{2} + 2 \sum_{i=1}^{n} \int_{\bigcup_{p=p'+1}^{p''-1} I_p} \left| f^{(i)}(t) - \bar{f}^{(i)}(t) \right| dt < \varepsilon.$$

Putting $\nu_i^{(\varepsilon)}(t) = \chi_{M_i^{(\varepsilon)}}(t)$ we see that the theorem holds in the case under consideration.

In order to prove the theorem in the general case, we shall need two lemmas.

LEMMA 1. *Let T be a metric space and μ be a Radon measure on T, let $\{K_n\}_{n=1}^{\infty}$ be a sequence of compact subsets of T such that* $\mu(K_n \cap K_m) = 0$ *for $n \neq m$, $\mu(T \setminus \bigcup_{n=1}^{\infty} K_n) = 0$, let E be a separable complete metric space, let σ be an analytical (mod 0) set ([4]) in $T \times E$.*

Then

1) $\mathrm{pr}_T\, \sigma = T' \cup T''$, *where T' is analytical and T'' is μ-negligible.*

2) *There exists a μ-measurable function $u(t)$ defined on* $\mathrm{pr}_T\, \sigma$ *such that $(t, u(t)) \in \sigma$ a. e. on T.*

Proof. Put $H_1 = K_1, \dots, H_n = K_n \setminus \bigcup_{i=1}^{n-1} H_i$ for $n \geqslant 2$. It is obvious that H_n $(n = 1, 2, \dots)$ are Borel sets, $H_n \cap H_m = \varnothing$ for $n \neq m$, $H_n \subseteq K_n$, $\mu(K_n \setminus H_n) = 0$, $T = \bigcup_{n=1}^{\infty} (H_n \cup L_n) \cup L_0$ where $L_0 = T \setminus \bigcup_{n=1}^{\infty} K_n$, $L_n = K_n \setminus H_n$ for $n \geqslant 1$, the sets L_n $(n = 0, 1, 2, \dots)$ have measure zero by the assumptions and the construction of H_n. By the definition of an analytical (mod 0) set, there exists an analytical set σ' in $T \times E$ such that $\sigma' = \sigma \cap (\mathrm{pr}_T\, \sigma' \times E)$ and $\mu(\mathrm{pr}_T(\sigma \setminus \sigma')) = 0$. We have

$$\mathrm{pr}_T\sigma = \mathrm{pr}_T\sigma' \cup \mathrm{pr}_T(\sigma \setminus \sigma')$$

$$\mathrm{pr}_T\sigma' = \mathrm{pr}_T\,[\sigma' \cap (\bigcup_{n=1}^{\infty} K_n \cup L_0) \times E] =$$

$$= \bigcup_{n=1}^{\infty} \mathrm{pr}_{K_n}(\sigma' \cap (K_n \times E)) \cup \mathrm{pr}_{L_0}(\sigma' \cap (L_0 \times E))$$

Since σ' and $K_n \times E$ are analytical (K_n being compact), then so is $\sigma' \cap (K_n \times E)$. Hence, according to the Lusin — Yankoff's measurable choice theorem (see [5]) the set $\mathrm{pr}_{K_n}\sigma' \cap (K_n \times E)$ is analytical. Putting

$$T' = \bigcup_{n=1}^{\infty} \mathrm{pr}_{K_n}(\sigma' \cap (K_n \times E)) = \bigcup_{n=1}^{\infty} \mathrm{pr}_{K_n}(\sigma' \cap (H_n \times E))$$

$$T'' = (\mathrm{pr}_T\,\sigma \setminus \sigma') \cup \mathrm{pr}_T\,(\sigma' \cap (L_0 \times E))$$

we have that $\mu(T'') = 0$, T' is analytical and $\mathrm{pr}_T\sigma = T' \cup T''$. Further, by the Lusin — Yankoff's theorem, there exists a function $u_n(t)$ defined and measurable on $\mathrm{pr}_{K_n}(\sigma' \cap (K_n \times E))$ such that $(t, u_n(t)) \in \sigma'$. Putting $u(t) = u_n(t)$ for $t \in \mathrm{pr}_T\,\sigma' \cap (H_n \times E)$, we obtain the desired function $u(t)$.

LEMMA 2. *Let T be a metric compact space, let μ be a Radon measure on T, let Ω be a Borel (mod 0) subset of $T \times \mathbb{R}^n$ such that for almost all $t \in T$ the subset $U(t) = \{ x \in \mathbb{R}^n : (t, x) \in \Omega \}$ is non-empty and compact in $\mathbb{R}^n$.*

Then, for every measurable mapping $\alpha(t)$ of T into $\mathbb{R}^n$ such that $(t, \alpha(t)) \in U(t)$ for almost all $t \in T$, there exist $n+1$ numerical measurable functions $\lambda_j(t) \geqslant 0$, $j = 1, \dots, n+1$ such that $\sum_{j=1}^{n+1} \lambda_j(t) = 1$ for almost $t \in T$ and $(n+1)$ measurable mappings $\beta^{(j)}: T \to \mathbb{R}^n$ such that $\beta^{(j)}(t) \in exconvU(t)$ for almost all $t \in T$ $(j = 1, 2, \dots, n+1)$ and $\alpha(t) = \sum_{j=1}^{n} \lambda_j(t)\, \beta^{(j)}(t)$ for almost all $t \in T$.

Proof. Since Ω is Borel (mod 0), there exists a Borel set $T_1 \subset T$ such that $\mu(T \setminus T_1) = 0$ and $\Omega \cap T_1 \times \mathbb{R}^n$ is a Borel subset of $T \times \mathbb{R}^n$. From the hypothesis, we can assume that $U(t)$ is compact, for all $t \in T_1$. By the measurability of $\alpha(t)$, there exists a Borel subset T_2 of T such that $\mu(T \setminus T_2) = 0$ and the restriction of $\alpha(t)$ on T_2 is a Borel function. It is clear that $T' = T_1 \cap T_2$ is a Borel subset of T, $\Omega' = \Omega \cap (T' \times \mathbb{R}^n)$ is a Borel subset of $T \times \mathbb{R}^n$ and $T' = \mathrm{pr}_T \sigma'$.

But as has been proved in [4] (Lemma 1.2 and Theorem 1.7) the set $\mathcal{F} = \{ (t,\alpha) : t \in T', \alpha \in \operatorname{exconv} U(t) \}$ is a Borel set. Hence the set

$$\mathcal{L} \doteq \{ (t, \beta^1, \dots, \beta^{n+1}) : t \in T', \beta^{(j)} \in \operatorname{exconv} U(t), j = 1, \dots, n+1 \}$$

is a Borel set in $T \times \mathbb{R}^{n(n+1)}$ since it is the projection on $T \times \mathbb{R}^{n(n+1)}$ of the intersection of two Borel sets, namely, $\mathcal{F}^{n+1}$ and $(\operatorname{diag} T^{n+1}) \times \mathbb{R}^{n(n+1)}$ where $\operatorname{diag} T^{n+1} = \{ \tau = (t_1, \dots, t_{n+1}) \in T^{n+1} : t_1 = t_2 = \dots = t_{n+1} \}$

The set $\Phi_0 = \{ (t, \beta, \lambda) \in \mathcal{L} \times S_{n+1} : \sum_{j=1}^{n+1} \lambda_j \beta^{(j)} = \alpha(t) \}$ is a Borel set as the kernel of the Borel function $\varphi(t, \beta, \lambda) = \sum_{j=1}^{n} \lambda_j \beta^{(j)} - \alpha(t)$ defined on the Borel set $\mathcal{L} \times S_{n+1}$. Furthermore according to the Caratheodory theorem the projection of Φ_0 on T is T'. Then by the Lusin-Yankoff's theorem, there exist measurable mappings $\beta(t) = (\beta_1(t), \dots, \beta_{n+1}(t))$, $\lambda(t) = (\lambda_1(t), \dots, \lambda_{n+1}(t))$ such that $(t, \beta(t), \lambda(t)) \in \Phi_0$ for $t \in T'$. These functions may be considered to be defined on T and have obviously all desired properties.

We are now in a position to prove Theorem 1 in the general case. Let $J \subset I$ be a bounded interval such that

$$\int_{I \setminus J} \rho(t) \sum_{j=1}^{n} | f^{(j)}(t) | \, dt < \frac{\varepsilon}{2} . \tag{4}$$

According to Lemma 2, there exist measurable mappings

$$\lambda(t) = (\lambda_1(t), \dots, \lambda_{n+1}(t)) : J \to \mathbb{R}^{n+1},$$

$$\beta(t) = (\beta^{(1)}(t), \dots, \beta^{(n+1)}(t)) : T \to \mathbb{R}^{n(n+1)}$$

such that $\lambda(t) \in S_{n+1}$, $\beta^{(i)}(t) \in \operatorname{exconv} U(t)$ for almost all $t \in J$ and $\alpha(t) = \sum_{j=1}^{n} \lambda_j(t)\, \beta^{(j)}(t)$. We have a. e. on J

$$\sum_{i=1}^{n} \alpha_i(t)\, f^{(i)}(t) = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \lambda_j(t)\, \beta_i^{(j)}(t) \right) f^{(i)}(t) = \sum_{j=1}^{n} \lambda_j(t)\, g^{(j)}(t),$$

where $g^{(j)}(t) = \sum_{i=1}^{n} \beta_i^{(j)}(t) f^{(i)}(t)$.

Since the theorem holds for the case $U(t) \equiv S_{n+1}$ there exist Borel sets $M_j^{(\varepsilon)} \subset J\ (j = 1, \ldots, n+1)$ such that $M_j^{(\varepsilon)} \cap M_k^{(\varepsilon)} = \varnothing$ for $j \neq k$, $\bigcup_{j=1}^{n+1} M_j^{(\varepsilon)} = J$ and

$$\left\| \sum_{j=1}^{n+1} \left(\lambda_j(t) - \chi_{M_j^{(\varepsilon)}}(t) \right) g^{(j)}(t) \right\|_{s_J} < \frac{\varepsilon}{2}. \tag{5}$$

Putting $$\tilde{\nu}^{(\varepsilon)}(t) = \sum_{j=1}^{n} \chi_{M_j^{(\varepsilon)}}(t)\, \beta^{(j)}(t)$$

we have a. e. on J :

$$\sum_{i=1}^{n} \tilde{\nu}_i^{(\varepsilon)}(t)\, f^{(i)}(t) = \sum_{j=1}^{n} \chi_{M_j^{(\varepsilon)}}(t)\, g^{(j)}(t).$$

It follows from (5) that

$$\left\| \sum_{i=1}^{n} \left(\alpha_i(t) - \tilde{\nu}_i^{(\varepsilon)}(t) \right) f^{(i)}(t) \right\|_{s_J} < \frac{\varepsilon}{2}. \tag{6}$$

Since $\beta^{(j)}(t) \in \operatorname{exconv} U(t)$ for almost all $t \in J$, we also have $\tilde{\nu}^{(\varepsilon)}(t) \in \operatorname{exconv} U(t)$ for almost all $t \in J$. On the other hand, since graph$U$ is a Borel (mod 0) set in $I \times \mathbb{R}^n$, the set

$$\sigma = \{ (t, \alpha) : t \in I \setminus J,\ \alpha \in \operatorname{exconv} U(t) \}$$

is a Borel (mod 0) set in $(I \setminus J) \times \mathbb{R}^n$ (see Lemma 1.2 and Theorem 1.7, [4]). By virtue of Lemma 1 there exists a measurable vector function $\tilde{\tilde{\nu}}^{(\varepsilon)}(t)$ defined on $I \setminus J$ such that $\tilde{\tilde{\nu}}^{(\varepsilon)}(t) \in \operatorname{exconv} U(t)$ for almost all $t \in I \setminus J$.

From (4) and (6) it follows that the function

$$\nu^{(\varepsilon)}(t) = \begin{cases} \tilde{\nu}^{(\varepsilon)}(t) & \text{if } t \in J \\ \tilde{\tilde{\nu}}^{(\varepsilon)}(t) & \text{if } t \in I \setminus J \end{cases}$$

has all desired properties. The proof of Theorem 1 is complete.

As a corollary of Theorem 1 we obtain the following result which provides an answer to a question put by Hoang Tuy:

THEOREM 2. *Let* $U: I \to 2^{\mathbb{R}^n}$ *be such that:*

1) *graphU is a Borel (mod 0) set in* $I \times \mathbb{R}^n$

2) *for almost all* $t \in I$, $U(t)$ *is a compact set in* $\mathbb{R}^n$, *contained in a ball around* $0 \in \mathbb{R}^n$ *of radius* $\rho(t) \in L_1(I)$.

Then, every measurable section $\alpha(t)$ *of* U *may be approximated in the norm* $\|\cdot\|_{s_I}$ *by sections taking values only in the sets of extremal points of conv*$U(t)$.

Indeed, it suffices to apply Theorem 1 to the functions

$$f^{(i)}(t) = (0,\ldots,0,\overset{(i)}{1},0,\ldots,0) \quad (i = 1, 2, \ldots, n).$$

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