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REMARKS ON THE CONNECTION BETWEEN MINIMAX THEORY AND DUALITY THEORY

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As is well known, the separation theorem for convex sets serves as a foundation of first-order necessary conditions and duality theory in extremal problems. On the other hand, optimality criteria and duality theorems in mathematical programming are often restated as minimax principles in the context of game theory (saddle-point characterization of optimal solutions), and several proofs are known for minimax propositions, which are based upon separation theorems.

This suggests that the connection between minimax theory and duality theory is far deeper than it seems. It is the aim of the present paper to display this connection in the converse direction by showing that various separation theorems can in turn be viewed as direct consequences of minimax propositions, and that, consequently, a meaningful duality theory could be built up, starting from minimax principles.

First, in section 1 we shall establish an elementary minimax proposition from which it is possible to derive in a simple unified manner some strong but little known separation theorems for polyhedral convex sets. Then, in section 2 we shall prove a general minimax proposition, using only topological assumptions. The proof is independent from separation theorems and fixed point principles and seems, therefore, to present some interest. Finally, in section 3 we shall show that from this minimax proposition one can easily deduce the Eidelheit-Mazur separation theorem (or, equivalently, the Hahn-Banach theorem). Thus, from a purely mathematical standpoint, minimax theory and duality theory are equivalent in most of their parts.

I. AN ELEMENTARY MINIMAX PROPOSITION AND ITS APPLICATION TO SEPARATION PROBLEMS FOR POLYHEDRAL CONVEX SETS.

In an earlier paper ([4]; see also [3]) we have proved an elementary property of convex sets, which turned out to be useful for the foundation of many existence propositions in Convex Analysis. Here we shall restate this property in a minimax form, more convenient for the application to special separation problems we are concerned with in this section.

Let C be a convex set in R^n. We shall say, as in [4], that a set E in R^n is a scheme for C if every element x of C can be represented in the form $x = \sum_{i \in J} \lambda_i e^i$ with $e^i \in E$, J finite, $\operatorname{supp} e^i \subset \operatorname{supp} x$, $\lambda_i \geqslant 0$ and $e^i_j x_j \geqslant 0$ for all $i \in J$ and all $j = 1,2, \dots, n$ (supp x denotes the support of x, i.e. the set of all i such that the i-component of x is non-zero).

THEOREM 1: *Let C be a non-empty convex set in R^n, E a finite scheme for C, Δ_i $(i = 1, \dots, n)$ a real interval (i.e. a non-empty connected subset of the real line, possibly degenerated to a single point) and let $\Delta = \Delta_1 \times \dots \times \Delta_n$. If we have for some real α:*

$$(\forall x \in E) \quad (\exists t \in \Delta) \qquad \langle t, x \rangle \geqslant \alpha \tag{1.1}$$

then

$$(\exists t \in \Delta) \quad (\forall x \in C) \qquad \langle t, x \rangle \geqslant \alpha \tag{1.2}$$

In other words:

$$\inf_{x \in E} \sup_{t \in \Delta} \langle t, x \rangle = \sup_{t \in \Delta} \inf_{x \in C} \langle t, x \rangle .$$

Proof. Since E is finite, one can choose for each i a closed bounded interval $[p_i, q_i] \subset \Delta_i$ such that $(\forall x \in E)$ $(\exists t \in [p, q])$ $\langle t, x \rangle \geqslant \alpha$, where $[p, q] = [p_1, q_1] \times \dots \times [p_n, q_n]$. Therefore, to simplify the notations we can assume $\Delta_i = [p_i, q_i]$.

If we define $\langle p, q, x \rangle = \sum_{x_j < 0} p_j x_j + \sum_{x_j > 0} q_j x_j$, then clearly $\langle p, q, x \rangle = \max_{t \in \Delta} \langle t, x \rangle$

and condition (1.1) can be rewritten as

$$(\forall x \in E) \qquad \langle p, q, x\rangle \geqslant \alpha .$$

Since E is a scheme for C, it is easy to see that the previous relation implies

$$(\forall x \in C) \qquad \langle p, q, x\rangle \geqslant \alpha, \text{ i.e.}$$

$$(\forall x \in C)\ (\exists\, t \in \Delta) \qquad \langle t, x\rangle \geqslant \alpha.$$

For each x let

$$\Delta\,(x) = \{\, t \in \Delta : \langle t, x\rangle \geqslant \alpha \}.$$

We have to prove that the family $\{\Delta\,(x),\ x \in C\}$ has a non-empty intersection. Since every $\Delta\,(x)$ is a non-empty closed subset of the compact set Δ, it will suffice to show that this family has the finite intersection property.

Let us first show that for any two points a, $b \in C$ we have

$$\Delta\,(a) \cap \Delta\,(b) \neq \varnothing. \tag{1.3}$$

Indeed, assume the contrary, that $\Delta\,(a)$ and $\Delta\,(b)$ are disjoint for some $a, b \in C$. For every $x = \lambda a + (1-\lambda)\, b$ $(0 \leqslant \lambda \leqslant 1)$ we have obviously $\Delta\,(x) \subset \Delta\,(a) \cup \Delta\,(b)$, hence $\Delta\,(x) = G_a \cup G_b$ with $G_a = \Delta\,(x) \cap \Delta\,(a)$, $G_b = \Delta\,(x) \cap \Delta\,(b)$ two closed, disjoint sets. Since $\Delta\,(x)$ is convex, it follows that either G_a or G_b must be empty. That is, for every x in the segment $[a\,;b]$ one of the following alternatives holds, but not both:

$$(1) \quad \Delta\,(x) \subset \Delta\,(a)\,; \qquad (2) \quad \Delta\,(x) \subset \Delta\,(b).$$

Let M_a (M_b, resp.) denote the set of all $x \in [a\,;b]$ for which (1) occurs ((2) occurs, resp.). It is not hard to see that each set M_a, M_b is open in $[a\,;b]$. Indeed, if $x \in M_a$ (for example), then $(\forall\, t \in \Delta\,(b))$ $\langle t, x\rangle < \alpha$ and to every $t \in \Delta\,(b)$ there correspond a neighbourhood W_t of t in $\Delta\,(b)$ and a neighbourhood I_t of x in $[a\,;b]$, such that $\langle t', x'\rangle < \alpha$ for all $t' \in W_t$ and all $x' \in I_t$. Since $\Delta\,(b)$ is compact, a finite set Q can be found such that $\{W_t,\ t \in Q\}$ is a covering of $\Delta\,(b)$. Then for all $x' \in I = \bigcap_{t \in Q} I_t$ and all $t' \in \Delta\,(b)$ we have $\langle t', x'\rangle < \alpha$, which shows that $I \subset M_a$ and hence M_a is open.

Thus the segment $[a,b]$ is the union of two non-empty disjoint subsets M_a, M_b which are both open in it. This being impossible, we must have (1.3).

Let us now show that for any finite set $a^1, \dots, a^k \in C$:

$$\bigcap_{i=1}^{k} \Delta\,(a^i) \neq \varnothing. \tag{1.4}$$

Assume that the fact holds for $k = h-1$ and consider the case $k = h > 2$ (for $k = 2$ the fact has just been proved). Let $\Delta' = \Delta(a^h)$, $\Delta'(x) = \Delta(x) \cap \Delta'$. Then, by the above, $\Delta'(x)$ is non-empty for every $x \in C$, i.e. $(\forall x \in C)$ $(\exists t \in \Delta')$ $\langle t,x \rangle \geqslant \alpha$ and hence, all assumptions of the theorem are still fulfilled when we replace Δ by Δ'. Therefore, by the inductive assumption the sets $\Delta'(a^1)$, ..., $\Delta'(a^{h-1})$ have a non-empty intersection, i.e. (1.4) holds for $k = h$.

The proof is complete.

We now apply the previous proposition to derive some of the most strong theorems concerning separation properties of polyhedral convex sets. It is convenient to begin with the following

LEMMA 1: *Let C be a polyhedral convex set in R^n such that $\varnothing \neq \{x \in C : x \leqslant 0\} \subset \{x : (\forall i \in I)\ x_i = 0\}$. Then there exists a vector $t \geqslant 0$ such that $(\forall i \in I)\ t_i > 0$ and $(\forall x \in C)$ $\langle t, x\rangle \geqslant 0$.*

Proof. Let C be defined by the relation $Ax \leqslant b$, where $A : R^n \to R^m$ is a linear mapping and $b \in R^m$. Denote by $C^{\angle}$ the cone $Ax \leqslant 0$. From the resolution theorem for polyhedral convex sets we know that a scheme E of C may be chosen so that $E \subset C \cup C^{\angle}$. But it is easy to see that

$$\{x \in E : x \leqslant 0\} \subset \{x : (\forall i \in I)\ x_i = 0\}.$$

Indeed, in the contrary case there would exist an $x \in E$ with $x \leqslant 0$, and $x_i < 0$ for some $i \in I$; then $x \notin C$, so that $x \in C^{\angle}$ and taking an arbitrary $a \in C$ such that $a \leqslant 0$ we would have $a + x \in C$ and $a + x \leqslant 0$, $a_i + x_i < 0$ for some $i \in I$, which conflicts with the hypothesis.

Thus for every $x \in E$, either $x_i > 0$ for at least one $i = 1, \ldots, n$, or $x_i = 0$ for all $i \in I$. Taking $\Delta_i = (0, +\infty)$ for $i \in I$ and $\Delta_j = [0, +\infty)$, for $j \notin I$, $\Delta = \Delta_1 \times \cdots \times \Delta_n$, we then have

$$(\forall x \in E)\ (\exists t \in \Delta) \qquad \langle t, x\rangle \geqslant 0.$$

Indeed, if $x_{i_0} > 0$ we can take $t_i = +1$ $(i \in I)$, $t_i = 0$ $(i \notin I \cup \{i_0\})$ and $t_{i_0} > 0$ large enough; if $x_i = 0$ for all $i \in I$ we can take an arbitrary $t \in \Delta$ such that $t_i = 0$ $(i \notin I)$. The Lemma now follows readily from Theorem 1.

THEOREM 2: *If M_1 and M_2 are two polyhedral convex sets in R^n such that $M_1 \cap M_2 = \{0\}$, and if M_1 is a pointed cone (i.e. a cone M_1 such that x and $-x$ cannot belong both to M_1 unless $x = 0$), then there exists a hyperplane separating M_1 and M_2 and having no common point with M_1 other than 0.*

Proof. Let M_1 be defined by $Ax \leqslant 0$, where $A . R^n \to R^m$ is a linear mapping. Consider the set

$$C = A(M_2) = \left\{ y \in R^m : \; y = Ax \text{ for some } x \in M_2 \right\}.$$

Since M_2 is a polyhedral convex set, it can be easily verified, by making use for example of the resolution theorem, that C is also a polyhedral convex set. The hypothesis $M_1 \cap M_2 = \{0\}$ implies

$$\{ y \in C : y \leqslant 0 \} = \{0\},$$

so that Lemma 1 applies with $I = \{1, \ldots, n\}$. We then obtain a vector $t > 0$ such that $(\forall\, y \in C) \langle t, y \rangle \geqslant 0$. Setting $t' = A^*t$, so that $\langle t, Ax \rangle = \langle t', x \rangle$, we have $(\forall\, x \in M_2)$ $\langle t', x \rangle \geqslant 0$, and $(\forall\, x \in M_1)$ $\langle t', x \rangle = \langle t, Ax \rangle < 0$ provided $x \neq 0$ (which implies $Ax \neq 0$ since M_1 is a pointed cone).

As one sees, the proof is quite simple. Another proof of this theorem may be found in Rockafellar [8]. The next two propositions express deeper results and have been established in Rubinshtein [9] by more involved methods and starting from a different definition of polyhedral convex sets.

THEOREM 3: *If M_1 and M_2 are two polyhedral convex sets in R^n such that $M_1 \cap M_2 \subset F_1 \cap F_2$, where F_i is a facet of M_i with $\dim F_i < n$, then there exists a hyperplane P separating M_1 and M_2 such that $P \cap M_i \subset F_i$ $(i = 1,2)$.*

Proof. It suffices to show the existence of a separating hyperplane $P_1 : \langle t',x \rangle = \alpha_1$ such that $P_1 \cap M_1 \subset F_1$. Indeed, by a similar argument, one could prove the existence of a separating hyperplane $P_2 : \langle t^2,x \rangle = \alpha_2$ such that $P_2 \cap M_2 \subset F_2$; then the hyperplane $P : \langle t^1 + t^2, x \rangle = \alpha_1 + \alpha_2$ would be the required one, assuming that $\langle t^i, x \rangle \geqslant \alpha_i$ for all $x \in M_i$.

Let M_1 be defined by $Ax - b \leqslant 0$, where $A : R^n \to R^m$ is a linear mapping and $b \in R^m$. Consider the polyhedral convex set

$$C = A(M_2) - b = \left\{ y \in R^m : \; y = Ax - b \text{ for some } x \in M_2 \right\}.$$

Since F_1 is a facet of M_1, there is a set $I \subset \{1, \ldots, m\}$ such that F_1 is just the set of all $x \in M_1$ verifying

$$(Ax - b)_i = 0 \quad (i \in I)$$

The hypothesis $M_1 \cap M_2 \subset F_1$ means that

$$\{ y \in C : y \leqslant 0 \} \subset \{ y \cdot (\forall\, i \in I) \; y_i = 0 \}.$$

Therefore, by Lemma 1 a vector $t \geqslant 0$ can be found such that $(\forall\, i \in I)\, t_i > 0$ and $(\forall\, y \in C) \langle t, y \rangle \geqslant 0$. Setting $t' = A^*t$, $\langle t, b \rangle = \alpha$, we have

$$(\forall x \in M_1) \qquad \langle t', x\rangle = \langle t, Ax\rangle \leqslant \langle t, b\rangle = \alpha$$

$$(\forall x \in M_2) \qquad \langle t, Ax - b\rangle \geqslant 0, \text{ hence } \langle t', x\rangle \geqslant \alpha,$$

which shows that the hyperplane $P: \langle t', x\rangle = \alpha$ separates M_1 from M_2. On the other hand, since $(\forall i \in I)$ $t_i > 0$, since every $x \in M_1 \smallsetminus F_1$ must verify $(Ax - b)_i < 0$ for at least one $i \in I$, we can write

$$(\forall x \in M_1 \smallsetminus F_1) \qquad \langle t', x\rangle = \langle t, Ax\rangle < \langle t, b\rangle = \alpha.$$

Thus $P \cap M_1 \subset F_1$, completing the proof.

COROLLARY : *If M_1 and M_2 are two polyhedral convex sets in R^n such that $M_1 \cap M_2 \neq \varnothing$ and $\dim(M_1 \cap M_2) < n$, if F_i is the smallest facet of M_i containing $M_1 \cap M_2$, then there exists a hyperplane P separating M_1 and M_2 such that $P \cap M_i = F_i$ $(i = 1,2)$.*

Proof. By the preceding theorem, there exists a separating hyperplane $P = \{x : \langle t, x\rangle = \alpha\}$ such that $P \cap M_i \subset F_i$ $(i = 1,2)$. Furthermore, under the conditions of the corollary, P is a supporting plane for each set M_1, M_2 since for every $x \in M_1 \cap M_2$ one must have $\langle t, x\rangle = \alpha$. Therefore, $P \cap M_i$ is a facet of M_i and hence $P \cap M_i = F_i$.

THEOREM 4 : *If M_1 and M_2 are two polyhedral convex sets in R^n such that $M_1 \cap M_2 = \varnothing$, then there exists a hyperplane P separating them strictly, i.e. such that the normal vector t to P satisfies the relation :*

$$\sup \{\langle t, x\rangle : x \in M_1\} < \inf \{\langle t, x\rangle : x \in M_2\}. \tag{1.5}$$

Proof. By an argument quite similar to that used for the proof of Theorem 3, one can prove the existence of a vector t and a number γ such that $(\forall x \in M_1)$ $(\forall x' \in M_2)$ $\langle t, x\rangle < \gamma < \langle t, x'\rangle$. The relation (1.5) then follows, since a linear function which is bounded from above (from below) on a polyhedral convex set attains its maximum (minimum) on this set.

COROLLARY : *If M_1 and M_2 are two partial polyhedral convex sets in R^n such that $M_1 \cap M_2 = \varnothing$, then there exists a hyperplane P separating them such that $P \cap M_i = \varnothing$ $(i = 1,2)$.*

(A partial polyhedral convex set is a polyhedral convex set from which some of the facets have been removed).

This proposition can easily be deduced from Theorems 3 and 4, but can also be established directly by the same method as that used for the proof of the previous results.

Note that the above theorems and their proofs are still valid for polyhedral convex sets in an arbitrary linear (not necessarily finite-dimensional) space, X, i.e. for sets defined by inequalities of the form $Ax \leqslant b$, where A is a linear mapping from X into a finite-dimensional space R^m and $b \in R^m$. Indeed, in the preceding arguments Theorem 1 was needed only to show the existence of some vector t in the range space of the mapping A, the required separating hyperplane being given by the functional $t' = A^*t$.

Thus, Theorem 1 permits a straightforward derivation of most of the special separation theorems for polyhedral convex sets. It could also be used to deduce many known results on linear inequalities (Farkas' lemma, Stiemke's theorem, Tucker's theorem, strong complementary slackness theorem, and so on), as well as a number of theorems of a combinatorial character. As an example, let us prove the following

THEOREM 5: (Gallai [1]). *Any (directed) graph containing no directed elementary path of length $\geqslant k$ ($k \geqslant 1$) is k-colourable.*

Proof. Let C be the set of all circulations in the given graph (a circulation is a vector $x \in R^n$ such that $Sx = 0$, where n is the number of arcs of the graph, S is the incidence-matrix). Then, as can be easily verified, the set E of all elementary cycles is a scheme for C. Let us set for every arc j: $p_j = 1$, $q_j = k-1$. From the assumptions of the Theorem, it follows that for every elementary cycle x we have:

$$\sum_{x_j=1} q_j - \sum_{x_j=-1} p_j \geqslant 0.$$

That is, $(\forall x \in E)\ (\exists\, t \in \Delta)$. $\langle t, x\rangle \geqslant 0$. Therefore, by Theorem 1 there exists a $t \in \Delta$ such that $(\forall x \in C)$ $\langle t, x\rangle \geqslant 0$; hence such that $(\forall x \in C)$ $\langle t, x\rangle = 0$, since C is a subspace. Let $v_1 = 0$ and for every vertex $i \neq 1$ let

$v_i = \min\left\{\sum_{j\in\mu^+} t_j - \sum_{j\in\mu^-} t_j\right\}$, where the minimum is taken over the set M_i of all paths μ from vertex 1 to vertex i and μ^+ (μ^-) denotes the set of all arcs of μ oriented in the running direction (in the converse direction) of μ. The condition $\sum t_j x_j = 0$ for every cycle x ensures that $t_j = v_{i_1} - v_{i_2}$ for every arc j going from vertex i_1 to vertex i_2. Therefore, if u_i is the residue of $[v_i]$ modulo k, then $0 \leqslant u_i \leqslant k-1$ and $u_{i_1} \neq u_{i_2}$ for any two adjacent vertices i_1, i_2 (here $[v]$ denotes the greatest integer not exceeding v). This shows that we may assign to every vertex i the colour « u_i », completing the proof.

2. THE GENERAL MINIMAX THEOREM

A common feature to all minimax theorems known up to the present is that they are based upon a fixed point principle, or a separation theorem, or some other equivalent proposition. Moreover they all require convexity or quasiconvexity assumptions, which in some applications appear to be too stringent.

Recently, the author has obtained in [5] a new minimax theorem using topological assumptions weaker than traditional algebraic ones and including as special cases most of the results known in this field. In this section we shall present an improved version of that result, the essential improvements concerning the assumptions and some points in the proof (which appeals neither to the fixed point principle nor to the separation theorem).

Let us consider a real function $F: C \times D \to R^1$, where C, D are subsets of two topological Hausdorff spaces X, Y respectively. For any finite system $x^1, \dots, x^k \in C$ and for any real α we define

$$D_\alpha(x^1, \dots, x^k) = \{ y \in D : F(x^i, y) \geq \alpha,\ i = 1, \dots, k \}.$$

(in games-theoretical terminology this is the set of all strategies which guarantee to the second player a pay-off not less than α, if the first player chooses one of the strategies $x^1, \dots, x^k$).

We shall say that the function F has *property* (P_α) on $C \times D$ if for any finite system $x^1, \dots, x^k \in C$ and for any pair $a, b \in C$ such that $\overline{D'_\alpha(a)} \cap \overline{D'_\alpha(b)} = \emptyset$ — where $D'_\alpha(x)$ stands for $D_\alpha(x^1, \dots, x^k, x)$ and the bar denotes the topological closure operation — there exists a continuous mapping $u: [0,1] \to C$ verifying $u(0) = a$, $u(1) = b$ such that for every interval $[s_0, s_1] \subset [0,1]$ and every $s \in [s_0, s_1]$ we have

1) either $$D'_\alpha(u(s)) \subset D'_\alpha(u(s_0)) \qquad (2.1)$$

2) or $$D'_\alpha(u(s)) \subset D'_\alpha(u(s_1)) \qquad (2.2)$$

Example. If X, Y are linear topological spaces, if C, D are convex subsets of X, Y resp., if the sets $D_\alpha(x) = \{ y \in D : F(x, y) \geq \alpha \}$ and $C_\alpha(y) = \{ x \in C : F(x, y) < \alpha \}$ are convex for all $x \in C$ and all $y \in D$, then F has property (P_α) on $C \times D$.

It should be noticed, however, that the class of functions having property (P_α) is larger than the class of functions quasiconvex in $x \in C$ and quasiconcave in $y \in D$.

THEOREM 6 : *Assume that : 1) the set D is compact ; 2) there exists a non-decreasing sequence α_n such that $\alpha_n \to \gamma$, where*

$$\gamma = \inf_{x \in C} \sup_{y \in D} F(x, y) \tag{2.3}$$

and the function F has property (P_{α_n}) on $C \times D$ for every n ; 3) $F(x, y)$ is upper semi-continuous in y for every fixed $x \in C$; either of the following conditions holds : a) $\alpha_n < \gamma$ for every n and $F(x, y)$ is lower semi-continuous in x for every fixed $y \in D$; b) $F(x, y)$ is upper semi-continuous in x for every fixed $y \in D$. Then

$$\inf_{x \in C} \sup_{y \in D} F(x, y) = \sup_{y \in D} \inf_{x \in C} F(x, y). \tag{2.4}$$

Proof. The proof is similar to that of Theorem 1, although is more involved.

I. We shall first show that for every pair $a, b \in C$ and every n :

$$D_{\alpha_n}(a) \cap D_{\alpha_n}(b) \neq \emptyset. \tag{2.5}$$

Assume the contrary, that this does not hold for some $a, b \in C$ and some n. For the sake of simplicity let $\alpha_n = \alpha$, so that we have

$$D_\alpha(a) \cap D_\alpha(b) = \emptyset. \tag{2.6}$$

Observe that, D being compact, $\gamma = \inf_{x \in C} \max_{y \in D} F(x, y)$, and so for every $x \in C$ the set $D_\alpha(x)$ is non-empty. Further, $D_\alpha(x)$ is closed, because $F(x, y)$ is upper semi-continuous in y by assumption 3). Now, let u be the continuous mapping that corresponds to the pair a, b according to property (P_α). Then for every $s \in [0,1]$ one of the following alternatives holds but not both :

$$\text{(i) } D_\alpha(u(s)) \subset D_\alpha(a) \quad ; \quad \text{(ii) } D_\alpha(u(s)) \subset D_\alpha(b) .$$

Denote by M_a (M_b, resp.) the set of all $s \in [0,1]$ for which (i) ((ii), resp.) holds. Obviously, $0 \in M_a$, $1 \in M_b$, $M_a \cup M_b = [0,1]$ and, according to (2.1) and (2.2), for every $s \in [0,1]$: if $s \in M_a$ then $[0,s] \subset M_a$, if $s \in M_b$ then $[s, 1] \subset M_b$.

Let $\bar{s} = \sup M_a = \inf M_b$ and suppose, for example, $\bar{s} \in M_a$. We shall show that (2.6) leads to a contradiction.

Assume first that condition 3a) holds. Since $\bar{s} \in M_a$, we have $D_\alpha(u(\bar{s})) \subset D_\alpha(a)$, hence $F(u(\bar{s}), y) < \alpha$ for every $y \notin D_\alpha(a)$. From this and the relation $\alpha < \gamma \leqslant \sup_{y \in D} F(u(\bar{s}), y)$ it follows that a point $\bar{y} \in D_\alpha(a)$ must exist such that

$F(u(\bar{s}), \bar{y}) > \alpha$. Then, $F(x, \bar{y})$ being lower semi-continuous in x, there exists a neighbourhood V of $u(\bar{s})$ such that

$$(\forall x \in V) \qquad F(x, \bar{y}) > \alpha .$$

Because of the continuity of u the set $I = \overset{-1}{u}(V)$ is a neighbourhood of $\bar{s}$ in $[0,1]$, and for all $s \in I$ we have $F(u(s), \bar{y}) > \alpha$, i.e. $\bar{y} \in D_\alpha(u(s))$ and hence $D_\alpha(u(s)) \subset D_\alpha(a)$ since $\bar{y} \in D_\alpha(a)$. Thus $s \in M_a$ for all s in a neighbourhood I of $\bar{s}$, which conflicts with $\bar{s} = \inf M_b$.

It remains to consider the case where 3b) holds. Since $\bar{s} \in M_a$, we have $D_\alpha(u(\bar{s})) \cap D_\alpha(b) = \varnothing$, i.e.

$$(\forall y \in D_\alpha(b)) \qquad F(u(\bar{s}), y) < \alpha .$$

Using the upper semi-continuity of $F(x, y)$ in x, we can find for every fixed $y \in D_\alpha(b)$ a neighbourhood V_y of $u(\bar{s})$ such that

$$(\forall x \in V_y) \qquad F(x, y) < \alpha .$$

Since $\overset{-1}{u}(V_y)$ is a neighbourhood of $\bar{s}$, two numbers $s_i = s_i(y) \in \overset{-1}{u}(V_y)$ $(i = 0,1)$ exist such that $I_y = [s_0, s_1]$ is still a neighbourhood of $\bar{s}$ in $[0,1]$. We have $F(u(s_i), y) < \alpha$ $(i = 0,1)$ and hence, using the upper semi-continuity of $F(x, y')$ in y', we can find for each $i = 0,1$ a neighbourhood $W_i(y)$ of y satisfying $(\forall y' \in W_i(y))$ $F(u(s_i), y') < \alpha$. Then $W_y = W_0(y) \cap W_1(y)$ will be a neighbourhood of y such that $(\forall y' \in W_y)$ $F(u(s_i), y') < \alpha$, i.e. $y' \notin D_\alpha(u(s_i))$ for $i = 0,1$. Therefore, according to (2.1) and (2.2), $y' \notin D(u(s))$ for every $s \in I_y$ and we have thus associated to every $y \in D_\alpha(b)$ a neighbourhood W_y and an interval I_y such that

$$(\forall s \in I_y)(\forall y' \in W_y) \qquad F(u(s), y') < \alpha .$$

Since $D_\alpha(b)$ is a closed subset of the compact set D, it is itself compact and so there exists a finite subset Q of $D_\alpha(b)$ such that the family $\{ W_y, y \in Q \}$ covers $D_\alpha(b)$. If $s \in I = \bigcap \{ I_y : y \in Q \}$ and if $y \in D_\alpha(b)$, then $y \in W_y$, for some $y' \in Q$ and hence $F(u(s), y) < \alpha$. Therefore, $D_\alpha(u(s)) \subset D_\alpha(a)$ for every $s \in I$, so that $I \subset M_a$, which again conflicts with $\bar{s} = \inf M_b$. This proves (2.5).

II. Fixing an arbitrary natural number n, we now show that for every finite system $a^1, \ldots, a^k \in C$ we have

$$\bigcap_{i=1}^{k} D_{\alpha_n}(a^i) \neq \varnothing. \tag{2.7}$$

For $k = 2$ this has just been proved. Assuming it to hold for $k = h - 1$, let us consider the case $k = h$. Let $D' = D_{\alpha_n}(a^h)$, $D'_{\alpha_n}(x) = D' \cap D_{\alpha_n}(x)$.

From the above argument it follows that for every $m > n$ we have $D_{\alpha_m}(a^h) \cap D_{\alpha_m}(x) \neq \varnothing$, whatever may be $x \in C$. But $D_{\alpha_m}(a^h) \subset D_{\alpha_n}(a^h)$ because $m > n$ implies $\alpha_m \geqslant \alpha_n$. Therefore, for every $x \in C$ we have $D_{\alpha_n}(a^h) \cap D_{\alpha_m}(x) \neq \varnothing$, i.e. $(\exists y \in D')\ F(x, y) \geqslant \alpha_m$. This means that $\inf_{x \in C} \sup_{y \in D'} F(x, y) \geqslant \alpha_m$ and hence, by letting $m \to \infty$, we get $\inf_{x \in C} \sup_{y \in D'} F(x, y) = \gamma$. It is then easily verified that all assumptions of Theorem 6 still hold when D is replaced by D'. Consequently, by the inductive assumption, $\bigcap_{i=1}^{h-1} D'_{\alpha_n}(a^i) \neq \varnothing$, that is, $\bigcap_{i=1}^{h} D_{\alpha_n}(a^i) \neq \varnothing$.

Thus we have shown that the family $\{ D_{\alpha_n}(x), x \in C \}$ has the finite intersection property. Since every $D_{\alpha_n}(x)$ is a non-empty closed subset of the compact set D, the family must have a non-empty intersection. Let y^n be a common element to all $D_{\alpha_n}(x)$, $x \in C$, and let $\bar{y}$ be a cluster point of the sequence $\{ y^n \} \subset D$. Then $\bar{y} \in D$, $F(x, y^n) \geqslant \alpha_n$, and using the upper semi-continuity of $F(x, y)$ in y, we conclude that $F(x, \bar{y}) \geqslant \gamma$ for every fixed $x \in C$. Thus $\sup_{y \in D} \inf_{x \in C} F(x, y) \geqslant \gamma$. Since the converse inequality is obvious, the proof is complete.

Remark. One of the first «topological» minimax theorems have been proved by Wu Wen-tzun [11]. However the result of Wu Wen-tzun does not include entirely those of Nikaido [7] and Sion [10]. It is not hard to see that the class of functions $F(x, y)$ having property (P_α) on $C \times D$ contains as a proper subclass the class of functions α-connected in the sense of our paper [5] and hence, as was shown in [5], it also contains as a proper subclass the class of functions strongly connected in the sense of Wu Wen-tzun in [11]. Therefore Theorem 6 includes as special cases all the main results of Nikaido [7], Sion [10], Wu Wen-tzun [11], as well as Theorems 1 and 2 in [5].

3. THE EIDELHEIT-MAZUR SEPARATION THEOREM AS A COROLLARY OF THE MINIMAX THEOREM

In this section we shall show that, starting from minimax theorems, it is possible to obtain in a simple way even the most general result on the separation of convex sets. More specifically, we shall provide a new proof of the Eidelheit-Mazur theorem, based upon the following minimax proposition :

Let C, D be two convex sets in two real linear topological spaces X, Y resp. If D is compact and $F(x, y)$ is a continuous bilinear function on $C \times D$, then the minimax equality (2.4) holds.

(This is only a special case of Theorem 6 and could be established directly by an argument as simple as that used for Theorem 1).

We shall first prove the following

LEMMA 2 : *Let A be a closed convex set in a real locally convex space X. If A does not contain the origin O, then there exists a continuous linear functional t such that* $(\forall x \in A) \quad \langle t, x \rangle \geqslant 1.$

Proof. Since $0 \notin A$ and A is closed, one can find a convex balanced neighbourhood V of 0 such that V and $C = A + V$ are disjoint. Let $\{ e^i, i \in I \}$ be an algebraic basis of the space X, such that $\{ e^i, i \in I \} \subset V$ and let W be the absolutely convex hull of the set $\{ e^i, i \in I \}$, i.e. W is the set of all finite linear combinations $\sum \lambda_i e^i$ with $\sum |\lambda_i| \leqslant 1$. Then W is a convex, balanced, absorbing subset of V, and if X^* denotes the algebraic dual of X, then the set D of all $t \in X^*$ for which $\sup \{ |\langle t, x \rangle| : x \in W \} \leqslant 1$ is convex and compact in the topology $\sigma(X^*, X)$. For every $x \in C$ let us define in the following way an element $t \in D$ such that $\langle t, x \rangle \geqslant 1$: if $x = \sum \lambda_i e^i$ (with completely defined λ_i), then t is the linear function such that $\langle t, e^i \rangle = \operatorname{sgn} \lambda_i$ for every $i \in I$. Clearly for every $y = \sum \mu_i e^i$ with $\sum |\mu_i| \leqslant 1$ we have $|\langle t, y\rangle| \leqslant \sum |\mu_i| \leqslant 1$, so that, actually, $t \in D$; on the other hand, we must have $\langle t, x\rangle = \sum |\lambda_i| \geqslant 1$, otherwise $x \in W$, conflicting with the hypothesis $x \in C$. Thus $(\forall x \in C)\ (\exists t \in D)$ $\langle t, x \rangle \geqslant 1$. Hence, by the minimax theorem, $(\exists t \in D)\ (\forall x \in C)$ $\langle t, x \rangle \geqslant 1$. Since int $C \neq \varnothing$, the linear functional t must be continuous and so the lemma is proved.

COROLLARY. *Let A be a convex set in a real locally convex space X. If $0 \notin A$ and int $A \neq \varnothing$, then there exists a continuous linear functional $t \neq 0$ such that* $(\forall x \in A)$ $\langle t, x \rangle \geqslant 0$.

Proof. Let a be an interior point of A. For every $\varepsilon > 0$ we must have $-\varepsilon a \notin \overline{A}$ (the closure of A), for otherwise from the convexity of A it would follow that $0 \in$ int A. Therefore, $0 \notin \varepsilon a + \overline{A}$ and by the previous lemma there is a linear functional t_ε such that $(\forall x \in \varepsilon a + \overline{A})$ $\langle t_\varepsilon, x \rangle \geqslant 1$. Now, using the fact that $a \in$ int A we can choose a convex balanced neighbourhood V of 0 such that $a + V \subset \varepsilon a + A$ for all $\varepsilon > 0$ small enough. Then $\langle t_\varepsilon, a \rangle \geqslant 1$ and setting $u_\varepsilon(x) = \dfrac{1}{\langle t_\varepsilon, a \rangle} \cdot \langle t_\varepsilon, x \rangle$ we have $u_\varepsilon(a) = 1$, $(\forall x \in \varepsilon a + \overline{A})$ $u_\varepsilon(x) \geqslant 0$. For every $x \in V$, since $x + a \in \varepsilon a + A$, we have $u_\varepsilon(x+a) \geqslant 0$, hence $u_\varepsilon(x) \geqslant -u_\varepsilon(a) = -1$; on the other hand, since $-x \in V$, $u_\varepsilon(-x) \geqslant -1$, hence $u_\varepsilon(x) \leqslant 1$. Thus $|u_\varepsilon(x)| \leqslant 1$, so that u_ε belongs to the set $V^o = \{u \in X^* : |u(x)| \leqslant 1$ for all $x \in V\}$. Since V^o is compact in the topology $\sigma(X^*, X)$, the sequence $\{u_\varepsilon, \varepsilon \to 0\}$ has a cluster point t and we have clearly $\langle t, a \rangle = 1$, $(\forall x \in A)$ $\langle t, x \rangle \geqslant 0$. As previously, the continuity of the functional t is ensured by the fact that int $A \neq \varnothing$.

The Eidelheit-Mazur separation theorem now follows readily :

If two convex sets A, B in a real locally convex space X are disjoint and such that int $A \neq \varnothing$, then there exists a continuous linear functional $t \neq 0$ such that $(\forall x \in A)\ (\forall y \in B)$ $\langle t, x \rangle \geqslant \langle t, y \rangle$.

Indeed, since $0 \notin A - B$ and int $(A - B) \neq \varnothing$, there exists a continuous linear functional $t \neq 0$ such that $(\forall x \in A)\ (\forall y \in B)$ $\langle t, x - y \rangle \geqslant 0$.

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