ACTA MATHEMATICA VIETNAMICA
Volume 35, Number 1, 2010, pp. 199–205

UNIFORMIZATION OF FAKE PROJECTIVE FOUR SPACES

SAI-KEE YEUNG

Abstract. We give criteria for a fake projective four space to be uniformized by the complex hyperbolic space of complex dimension four. This is achieved through a classification of the Chern numbers of a rational homology complex projective space of complex dimension four.

1. Statements of results

1.1. The problem of characterization of $P^n_{\mathbb{C}}$ in terms of topological conditions has a long history. In particular, Severi raised the question of whether a complex surface homeomorphic to $P^2_{\mathbb{C}}$ has to be biholomorphic to $P^2_{\mathbb{C}}$. The answer is affirmative and is achieved only after the solution of Calabi Conjecture in the case of negative scalar curvature by Aubin and Yau (cf. [16] or [1], page 5).

In higher dimensions we do not have the tools to approach the problem if the manifolds considered are not Kähler. From this point on, we assume that the complex manifold involved is equipped with a Kähler metric. In such a case, the result is proved by Hirzebruch and Kodaira [7] if all the Pontryagin classes are the same as $P^n_{\mathbb{C}}$. Fujita [3] deduced the same conclusion for a projective algebraic fourfold M if $c_1(M) > 0$ and the integral cohomology ring of M is the same as $P^4_{\mathbb{C}}$. More recently, Libgober and Wood in [10] showed that it was sufficient to assume that the cohomology ring of M was the same as $P^4_{\mathbb{C}}$.

The motivation of the present article comes from a complementary point of view. In [14], examples of Kähler fourfolds with the same Betti numbers as $P^4_{\mathbb{C}}$ but with infinite fundamental groups are constructed. These are called fake projective four spaces, a generalization of the notion of Mumford's fake projective plane in [11]. While fake projective planes are classified in Prasad-Yeung [13], see also Cartwright-Steger [2], the result is not known in higher dimensions. For such purpose, similar to complex dimension two as explained in the survey article [18], we may reduce the task into three steps, namely, uniformization of such complex manifolds as complex ball quotients, proof of arithmeticity of the lattices involved, and classification of such arithmetic quotients. The third step was achieved in [14] and the second step requires a generalization of the arguments in [9] and [17].

Received May 10, 2009; in revised form May 27, 2009.
2000 *Mathematics Subject Classification.* Primary 14F45, 32Q30, 32Q57.
Key words and phrases. Rational homology projective space, classification.
The author was partially supported by grants from the National Science Foundation and the National Security Agency.

Our main purpose here is to address the problem encountered in the first step. The main result is a uniformization statement about compact Kähler manifolds of complex dimension four which are rational homological projective spaces of the same dimension.

1.2. We begin with the following definition.

Definition 1.1.
(a) A connected compact Kähler manifold M of complex dimension n is said to be a (rational) homology complex projective space if M has the same (rational) homology group as the complex projective space of the same dimension.
(b) A connected compact Kähler manifold M of complex dimension n is said to be a fake projective space if M has the same Betti numbers as $P_{\mathbb{C}}^n$ but is different from $P_{\mathbb{C}}^n$.
(c) A fake projective space is said to be an arithmetic fake projective space if M is biholomorphic to the quotient of the complex hyperbolic space by an arithmetic lattice in $PU(n,1)$.

The definition in (b) follows the usual terminology of fake projective planes in dimension two. Fake projective spaces are rational homology complex projective spaces other than the projective spaces themselves. First examples of fake projective space were obtained in complex dimension 2 by Mumford [11]. More recently, two more examples of such fake projective planes were found in a related manner by Ishida and Kato, and an example with an automorphism of order 7 related to Mumford's example was constructed by Keum. A classification of fake projective planes was given in [13], [2]. We refer the readers to [13] and [18] for further details and references in complex dimension 2.

For $n > 2$, Gopal Prasad and the author classified all arithmetic fake projective spaces and constructed in complex dimension 4 four classes of fake projective four spaces [14]. Since only fake projective spaces of complex dimension four exist among all dimensions greater than 2, we concentrate on complex dimension four. To bridge the gap in the definition of fake projective four space and arithmetic fake projective four space, the first step is to have some uniformization result about manifolds having the same rational homology groups as the projective four space. This turns out to be rather intricate and is not completely solved. The following is a result in this direction, which needs one extra assumption apart from being a rational homology complex projective four space. It also limits the possibility of the Chern numbers of a homology complex projective four space.

Theorem 1.2. (a). *The Chern numbers* $(c_1^4, c_1c_3, c_1^2c_2, c_2^2, c_4)$ *of a rational homology complex projective space of complex dimension* 4 *can only take one of the following two sets of values,*
(i) $(625, 50, 250, 100, 5)$, *or,*
(ii) $(225, 50, 150, 100, 5)$.
(b). *Let* M *be a fake projective four space. Suppose that the Chern numbers of* M *are not given by* (a)(ii). *Then* M *is biholomorphic to the quotient of* $B_{\mathbb{C}}^4$ *by*

a torsion free lattice in $PU(4,1)$. *In particular,* M *is a complex ball quotient of complex dimension* 4 *if any one of the following conditions is satisfied*
(i) $c_1^4(M) \neq 225$,
(ii) $H^4(M,\mathbb{Z})$ *modulo torsion is generated by* $\theta \cup \theta$*, where* θ *is a generator of* $H^2(M,\mathbb{Z})$ *modulo torsion,*
(iii) *The canonical line bundle* K_M *does not have length* 1, *in other words, the cycle corresponding to* K_M *is not the generator of the Neron-Severi group modulo torsion.*

1.3. The Chern numbers in Theorem 1.2 (a)(i) are satisfied by the complex projective space of complex dimension 4 and also by the four classes of arithmetic fake projective spaces constructed in Prasad-Yeung [14]. It is not clear to the author whether there are connected Kähler manifolds with Chern numbers given in Theorem 1.2 (a)(ii). To illustrate the intricate nature of the problem, let us mention two facts here.

The first fact is that there are actually examples of disconnected projective algebraic manifolds which have the same Chern numbers as given by Theorem 1.2 (a)(ii). This follows from a result of Milnor (cf. [5], see also [4]), since the data given above satisfy

$$\begin{aligned} -c_1^4+4c_1^2c_2+c_1c_3+3c_2^2-c_4 &\equiv 0 \mod 720,\\ 2c_1^4+c_1^2c_2 &\equiv 0 \mod 12,\\ c_1c_3-2c_4 &\equiv 0 \mod 4. \end{aligned}$$

The second fact is that in complex dimension 3, there are actually examples of fake projective three space which are not uniformized by the complex three balls. We refer the readers to §3.

1.4. Here is the organization of the paper. The proof of Theorem 1.2 is given in §2. This is achieved by generalizing an argument of Libgober and Wood [10] who assumed that the cohomology ring of M is the same as $P^4_{\mathbb{C}}$. We remark that for the cases that we are interested in, the cohomology ring may not be generated completely by the generator of Neron-Severi group. In §3, we give remarks about the situations in complex dimension 3.

2. Uniformization in complex dimension 4

2.1. We give a proof of the main result in this section.

Proof of Theorem 1.2

Libgober and Wood [10] proved that a compact Kähler manifold having the same cohomology ring as $P^4_{\mathbb{C}}$ is actually biholomorphic to $P^4_{\mathbb{C}}$. Here we modify their method and deal with the weaker assumption in terms of Betti numbers. To explain the difference, we let θ be an ample generator of $i^*H^2(M,\mathbb{Z})\cap H^{1,1}(M)$, where $i^*: H^2(M,\mathbb{Z}) \to H^2(M,\mathbb{C})$ is the map induced by the embedding $i:\mathbb{Z}\to\mathbb{C}$. The difficulty we need to deal with is that θ when regarded as class in $H^2(M,\mathbb{Z})$ may only generate a proper subring of cohomology ring $\oplus_j H^j(M,\mathbb{Z})$

modulo torsion instead of the whole $\oplus_j H^j(M,\mathbb{Z})$ modulo torsion. Hence the Chern classes are assumed to be only rational instead of integral multiples of θ.

In the following we denote by $c_i(X)$ the Chern number and $C_i(X)$ the Chern class of a manifold X. We simply denote $c_i(M)$ by c_i for our manifold M when there is no danger of confusion.

Since the Betti numbers of a rational homology four space M satisfy $b^{2i}(M) = b^{2i}(P^4_{\mathbb{C}}) = 1$ and $b^{2i+1}(M) = b^{2i+1}(P^4_{\mathbb{C}}) = 0$ for $0 \leqslant i \leqslant 3$, we conclude from Hodge decomposition of a compact Kähler manifold that the only non-vanishing Hodge numbers are given by $h^{p,p}(M) = 1$ for $0 \leqslant p \leqslant 4$.

It follows from Hirzebruch's Riemann-Roch Theorem that $\chi(M,\Omega^p) = \chi(P^4_{\mathbb{C}},\Omega^p)$ for $p = 0,1,2$, which give rise to three identities among the Chern numbers. The values $p = 3$ and 4 do not lead to new information. Equivalently we quote the formula on page 145 of [10],

$$c_4(M) = c_4(P^4_{\mathbb{C}}) = 5 \tag{2.1}$$

$$c_1c_3(M) = c_1c_3(P^4_{\mathbb{C}}) = 50 \tag{2.2}$$

$$(3c_2^2 + 4c_2c_1^2 - c_1^4)(M) = (3c_2^2 + 4c_2c_1^2 - c_1^4)(P^4_{\mathbb{C}}) = 675. \tag{2.3}$$

As $h^{2,0}$ is trivial, it follows that M is projective algebraic. We denote by θ_j for $j > 1$ the generator of $i^*H^{2j}(M,\mathbb{Z}) \cap H^{j,j}(M)$ normalized so that the pairing with the fundamental class of M of $\theta_j \wedge \theta^{n-j}$, $\int_M \theta_j \wedge \theta^{n-j}$, is positive. From definition, there is a positive integer k such that $\theta^2 = k\theta_2$. From Poincaré Duality, $\theta_2^2 = \theta_4$. Hence $\theta^4 = k^2\theta_4$.

As $h^{i,i} = 1$, we may write the i-th Chern classes as $C_i = a_i\theta^i \in H^{i,i}(M) \cap i^*H^{2i}(M,\mathbb{Z})$, where a_i are rational numbers for $1 < i \leqslant 4$, and a_1 is an integer. The third equation above can then be written as

$$3a_2^2k^2 + 4a_2a_1^2k^2 - a_1^4k^2 = 675.$$

We now narrow down the possibility of a_1 and k from the above equation. From the Chern number inequality which follows from the solution of Calabi Conjecture in the case of negative scalar curvature by Aubin and Yau (cf. [15]), we know that $(2c_2 - \frac{4}{5}c_1^2)\cdot\theta^2 \geqslant 0$. Since all the relevant cohomology groups $i^*H^{2j}(M,\mathbb{Z}) \cap H^{j,j}(M)$ are of dimension 1, it follows that $c_2c_1^2 \geqslant \frac{2}{5}c_1^4$ and $c_2^2 \geqslant \frac{4}{25}c_1^4$. Substituting to the equation above we conclude that

$$675 \geqslant 3\cdot(\frac{4}{25} + 4\cdot\frac{2}{5} - 1)c_1^4 = \frac{27}{25}a_1^4k^2.$$

Hence $|a_1^2|k \leqslant 25$. Since both a_1 and $k \geqslant$ are integers, $|a_1|$ can only take the values of 1 and 5 and k can only take the values of $1, 5$ or 25.

Solving a_2k from the same equation, we obtain

$$a_2k = \frac{-2a_1^2k \pm \sqrt{(7a_1^4k^2 + 2025)}}{3}.$$

As C_2 is an integral class, $a_2k \in \mathbb{Z}$. With the earlier constraint on $|a_1|$ and k, it is easy to check that the following are the only possibility for a_2k to be an integer,

Case (1) $a_1^2k = 25$, or
Case (2) $a_1^2k = 15$.

In case (1), $a_2k = 10$ or $-80/3$. The latter possibility is ruled out again by the Chern number inequality. Hence $a_2k = 10$, and therefore $(2c_2 - \frac{4}{5}c_1^2) \cdot \theta^2 = 0$. From the result of Yau mentioned earlier, the Chern number equality implies that the universal covering of M is the complex hyperbolic space $B_{\mathbb{C}}^4$ equipped with the Bergman metric unless M is just $P_{\mathbb{C}}^4$ equipped with the Fubini-Study metric. Hence the Chern numbers are given by those written in Theorem 1.2 (a)(i).

In case (2), it follows from $a_1^2k = 15$ that $a_1 = 1$ and $k = 15$. In such a case, the Chern numbers of M are precisely the ones written in Theorem 1.2 a(ii).

Hence for a fake projective four space M with Chern numbers satisfying Theorem 1.2 a(i), M is a quotient of $B_{\mathbb{C}}^4$ by a torsion free lattice in $PU(4,1)$. On the other hand, if the Chern numbers are given by Theorem 1.2 (a)(ii), which corresponds to Case (2) above, K_M is the generator θ of the Neron-Severi group modulo torsion as $a_1 = 1$, and $\theta \wedge \theta = 15\theta_2$ does not generate $H^4(M, \mathbb{Z})$ modulo torsion. Theorem 1.2 (b) follows.

This concludes the proof of Theorem 1.2.

3. Remarks on complex dimension 3

3.1. For completeness, we collect some results about fake projective spaces in complex dimension three.

3.2. It is observed in Prasad-Yeung [14] from Hirzebruch's Proportionality Principle that there is no arithmetic fake projective space of odd complex dimension. We observe that there is a slightly stronger result in complex dimension 3 in the sense that there is no fake projective three-fold of general type. We give a brief argument as follows.

From the same proof as in Theorem 1.2, we know that a generator θ of $H^{1,1}$ as a d-closed differential form is cohomologous to the Kähler form ω. From Hodge decomposition $b^2 = h^{2,0} + h^{1,1} + h^{0,2}$, we conclude that $H^2(M, \mathbb{Q})$ has rank one and is represented by θ. The vanishing of $h^{2,0}$ once again shows that M is projective algebraic.

With the assumption on Betti numbers and hence on Hodge numbers, it follows from Hirzebruch's Riemann-Roch Theorem for $\chi(\Omega^p), p = 1, 2$ that $c_3(M) = c_3(P_{\mathbb{C}}^3) = 4$ and $c_1c_2(M) = c_1c_2(P_{\mathbb{C}}^3) = 24$. As the second Betti number is 1, the first Chern class satisfies $C_1(M) = a\theta$. The coefficient a cannot be 0, for otherwise $c_1c_2(M) = 0$ instead of 24 mentioned earlier. The Chern number inequality following the work of Aubin and Yau (cf. [15]) for negative c_1 implies that

$$(2c_2(M) - \frac{3}{4}c_1^2)\theta \geqslant 0.$$

Since M is of general type, we know that $a < 0$. This implies that $2c_2c_1(M) \leqslant \frac{3}{4}c_1^3 < 0$, contradicting to $c_2c_1(M) = 24$.

3.3. In the opposite direction, there does exist an algebraic threefold which has the same rational homology group as $P^3_{\mathbb{C}}$ if one is not confined to threefolds of general type. In fact, it is known that the hyperquadric $Q_2 \subset P^4_{\mathbb{C}}$ has the same rational homology group as $P^3_{\mathbb{C}}$. This follows for example from Theorem 22.1.1 on page 160 of Hirzebruch [6]. There are other examples as mentioned in [8].

In general, the discussions of 3.2 implies that for a fake projective three space M, the first Chern class satisfies $C_1(M) = a\theta$ for some $a > 0$. It follows that a fake projective three space has to be a Fano manifold.

3.4. If one imposes the stronger condition that the integral (co)homology ring of M of dimension 3 is isomorphic as a ring to the corresponding one of $P^3_{\mathbb{C}}$, we can conclude that M is biholomorphic to $P^3_{\mathbb{C}}$. This was proved in [8]. The problem was reduced to results on classification of Fano threefolds. Here the ring structure of the cohomology ring is used to exclude the hyperquadrics.

3.5. We summarize the above discussion as follows.

Theorem 3.1.
(a) *There is no fake projective three space M of general type.*
(b). *There is no fake projective three space M with the same rational cohomology ring as $P^3_{\mathbb{C}}$.*
(c). *Any fake projective threefold has to be Fano.*
(d) *A complete list of Fano fake projective threefolds is known and can be found in* [8]. *Examples are given by hyperquadrics in $P^4_{\mathbb{C}}$.*

3.6. As mentioned in the introduction, the classification of fake projective planes and thus rational homology complex projective two spaces is complete following the work of [13] and [2], see also [18].

Acknowledgments

Parts of this work were done while the author was visiting the Institute of Mathematics of the University of Hong Kong, the Department of Mathematics of Osaka University and the Korea Institute for Advanced Study. The author would like to thank Jun-Muk Hwang, Toshiki Mabuchi and Ngaiming Mok for their hospitality during the visit. Furthermore, the author is grateful to Martin Deraux, Akira Fujiki, Hansjörg Geiges, Anatoly Libgober, and Gopal Prasad for their interests and helpful discussions, and to the referee for helpful comments.

References

[1] W. P. Barth, K. Hulek, Ch. Peters and A. Van de Ven, *Compact Complex Surfaces*, Second edition. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics 4. Springer-Verlag, Berlin, 2004.
[2] D. Cartwright, T. Steger, private communication, in preparation.
[3] T. Fujita, On topological characterizations of complex projective spaces and affine linear spaces, *Proc. Japan Acad. Ser. A Math. Sci.* **56** (1980), no. 5, 231-234

[4] H. Geiges, Chern numbers of almost complex manifolds, *Proc. Amer. Math. Soc.* **129** (2001), 3749–3752

[5] F. Hirzebruch, Komplexe Mannigfaltigkeiten, Proc. Internat. Congress Math. 1958, Cambridge University Press, 1960, pp. 119-136

[6] F. Hirzebruch, *Topological Methods in Algebraic Geometry*, third edition, Springer-Verlag 1995.

[7] F. Hirzebruch, K. Kodaira, On the complex projective spaces, *J. Math. Pures Appl.* **36**(1957) 201-216.

[8] V. A. Iskowskih, V. V. Shokurov, Biregular theory of Fano 3-folds, in *Algebraic Geometry* (L. Lonsted, ed.), Lect. Notes. Math. 1016, Springer 1983.

[9] B. Klingler, Sur la rigidité de certains groupes fondamentaux, l'arithméticité des résearx hyperboliques complexes, et les "faux plans projectifs", *Invent. Math.* **153** (2003), no. 1, 105–143.

[10] A. S. Libgober, J. W. Wood, Uniqueness of the complex structure on Kähler manifolds of certain homotopy types, *J. Differential Geom.* **32** (1990), no. 1, 139–154.

[11] D. Mumford, An algebraic surface with K ample, $K^2 = 9$, $p_g = q = 0$, *Amer. J. Math.* **101**(1979), 233–244.

[12] G. Prasad, Volumes of S-arithmetic quotients of semi-simple groups, *Publ. Math. IHES*, **69**(1989), 91–117.

[13] G. Prasad, S. -K. Yeung, Fake projective planes, *Invent. Math.* **168** (2007), no. 2, 321–370.

[14] G. Prasad, S. -K. Yeung, Arithmetic fake projective spaces and arithmetic fake Grassmannians, *Amer. J. Math.* **131**(2009) 179-207.

[15] Y. -T. Siu, Lectures on Hermitian-Einstein metrics for stable bundles and Kähler-Einstein metrics. DMV Seminar, 8. Birkhäuser Verlag, Basel, 1987.

[16] S. -T. Yau, Calabi's conjecture and some new results in algebraic geometry, *Proc. Nat. Acad. Sci. U.S.A.* **74** (1977), no. 5, 1798–1799.

[17] S. -K. Yeung, Integrality and arithmeticity of co-compact lattices corresponding to certain complex two-ball quotients of Picard number one, *Asian Jour. of Math.* **8**(2004), 104-130, Erratum, to appear in Asian Jour. of Math.

[18] S. -K. Yeung, Classification and construction of fake projective planes, to appear in Handbook in Geometric Analysis II.

Mathematics Department, Purdue University
West Lafayette, IN 47907 USA
E-mail address: yeung@math.purdue.edu