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CARTAN-NOCHKA THEOREM WITH TRUNCATED COUNTING FUNCTIONS FOR MOVING TARGETS

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ABSTRACT. The purpose of this article is twofold. The first is to show the explicit truncations in the Cartan-Nochka theorem for nonconstant meromorphic mappings of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$ and moving targets. The second is to show the above type theorem over function fields.

CONTENTS

1. INTRODUCTION

Let $\{H_j\}_{j=1}^q$ be hyperplanes of $\mathbf{P}^n(\mathbf{C})$. Set the index set $Q = \{1, 2, \cdots, q\}$. Let $N \geq n$ and $q \geq N+1$. We say that the family $\{H_j\}_{j=1}^q$ are in N-*subgeneral position* if for every subset $R \subset Q$ with the cardinality $\sharp R = N+1$

$$\bigcap_{j \in R} H_j = \emptyset.$$

If they are in n-subgeneral position, we simply say that they are in *general position.*

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Let $f : \mathbf{C}^m \to \mathbf{P}^n(\mathbf{C})$ be a linearly nondegenerate meromorphic mapping and $\{H_j\}_{j=1}^q$ be hyperplanes in N-subgeneral position in $\mathbf{P}^n(\mathbf{C})$. Then the Cartan-Nochka theorem (see [4, 5, 6, 9]) stated that

$$|| \quad (q-2N+n-1)T(r,f) \leqslant \sum_{i=1}^{q} N^{(n)}(r, \operatorname{div}(f, H_i)) + o(T(r,f)).$$

The above Cartan-Nochka theorem plays an extremely important role in Nevanlinna theory, with many applications to Algebraic or Analytic geometry. Thus, much attention has been given to generalizing this theorem to abstract objects. For instance, motivated by the accomplishment of the second main theorem of meromorphic function for moving targets, Ru and Stoll [11], [12] gave a remarkable generalization of the Cartan-Nochka theorem to a finite set of moving hyperplanes (i.e, moving targets) in $\mathbf{P}^n(\mathbf{C})$.

Theorem A *Let $f : \mathbf{C}^m \longrightarrow \mathbf{P}^n(\mathbf{C})$ be a non-constant meromorphic mapping. Let $\{a_i\}_{i=1}^q$ be meromorphic mappings of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})^*$ in N-subgeneral position such that a_i are small with respect to f and f is linearly nondegenerate over $\mathcal{F}(\{a_i\}_{i=1}^q)$. Then for an arbitrary $0 < \epsilon < 1$,*

$$|| \quad (q-2N+n-1-\epsilon)T(r,f) \leqslant \sum_{i=1}^{q} N(r, \operatorname{div}(f, a_i)) + o(T(r,f)).$$

We see immediately a natural question from the Ru-Stoll theorem: Does there exist a truncated counting function which does not depend on ϵ? Precisely, we consider the following problem.

Problem *Let $f : \mathbf{C}^m \longrightarrow \mathbf{P}^n(\mathbf{C})$ be a nonconstant meromorphic mapping. Let $\{a_i\}_{i=1}^q$ be meromorphic mappings of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})^*$ in N-subgeneral position. Assume that a_i are small with respect to f and f is linearly nondegenerate over $\mathcal{F}(\{a_i\}_{i=1}^q)$. Is there a positive integer N_0 such that for an arbitrary $0 < \epsilon < 1$,*

$$|| \quad (q-2N+n-1-\epsilon)T(r,f) \leqslant \sum_{i=1}^{q} N^{(N_0)}(r, \operatorname{div}(f, a_i)) + o(T(r,f))?$$

Unfortunately, this problem is extremely difficult. We would like to explain the reason. It is well-known to the experts that the Ru-Stoll method which is the best available at present will lead truncations. However, the truncation level depends on the given ϵ and is big enough and, when ϵ goes to zero, the truncation level goes to infinite (so the truncation is totally lost).

Motivated by the establishment of the unicity theorems for moving targets, we would like to show the explicit truncations even though they still depend on the given ϵ. As far as we know, there has been no literature of such results. In the first part of this paper, the explicit truncations which depend on the given ϵ are given in Theorems 3.5 and 3.8. However, these are still weak and are far from sharp ones.

Nevanlinna theory is an approximation theory of complex numbers by meromorphic functions, as the Diophantine approximation is the approximation of algebraic numbers by rational or algebraic numbers of a fixed number field (the inverse of Vojta's observation). Over algebraic function fields, one may think the approximation of rational functions by rational functions. From this viewpoint, some Cartan-Nochka theorem with truncated counting function over function fields are showed in [15, 16, 7]. The second aim in this paper is to show two Cartan-Nochka theorems with truncated counting function for moving targets over function fields (see Theorems 4.3 and 4.5 below). In the proof we use the method of M. Shirosaki [13].

2. Basic notions and auxiliary results

(a) We set $||z|| = \big(|z_1|^2 + \cdots + |z_m|^2\big)^{1/2}$ for $z = (z_1, \ldots, z_m) \in \mathbf{C}^m$ and

$$B(r) := \{z \in \mathbf{C}^m : ||z|| < r\},$$
$$S(r) := \{z \in \mathbf{C}^m : ||z|| = r\} \quad (0 < r < \infty).$$

Define

$$d^c := \frac{i}{4\pi}(\bar{\partial} - \partial),$$
$$v_{m-1}(z) := \big(dd^c||z||^2\big)^{m-1}$$
$$\sigma_m(z) := d^c\mathrm{log}||z||^2 \wedge \big(dd^c\mathrm{log}||z||^2\big)^{m-1} \quad (z \neq 0).$$

(b) Let F be a nonzero holomorphic function on a domain Ω in $\mathbf{C}^m$. For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbf{Z}_+^m$, we set $|\alpha| = \alpha_1 + \ldots + \alpha_m$ and $\mathcal{D}^\alpha F = \partial^{|\alpha|}F/\partial^{\alpha_1}z_1 \cdots \partial^{\alpha_m}z_m$. We define a map $\mathrm{div}F : \Omega \to \mathbf{Z}$ by

$$\mathrm{div}F(z) := \max\ \{t : \mathcal{D}^\alpha F(z) = 0 \text{ for all } \alpha \text{ with } |\alpha| < t\}\ (z \in \Omega).$$

A divisor on a domain Ω in $\mathbf{C}^m$ is a map $\nu : \Omega \to \mathbf{Z}$ such that, for every $a \in \Omega$, there are nonzero holomorphic functions F and G on a connected neighbourhood $U \subset \Omega$ of a such that $\nu(z) = \mathrm{div}F(z) - \mathrm{div}G(z)$ for $z \in U$ outside an analytic subset of dimension $\leqslant m - 2$. Two divisors are equal to each other if they have the same value outside an analytic subset of dimension $\leqslant m - 2$. For a divisor ν on Ω we denote by $|\nu|$ the sum of $(m-1)$-dimensional irreducible components of $\overline{\{z : \nu(z) \neq 0\}}$.

Take a nonzero meromorphic function φ on a domain Ω in $\mathbf{C}^m$. For every $a \in \Omega$, we choose nonzero holomorphic functions F and G in a neighbourhood $U \subset \Omega$ such that $\varphi = F/G$ on U and $\dim(F^{-1}(0) \cap G^{-1}(0)) \leqslant m - 2$. We define the divisors

$$\mathrm{div}_0\varphi := \mathrm{div}F, \quad \mathrm{div}_\infty\varphi := \mathrm{div}G,$$

which are independent of choices of F and G and so globally well-defined on Ω.

(c) For a divisor ν on $\mathbf{C}^m$ and for positive integers k, M or $M = \infty$, we define the counting function of ν by

$$\nu^{(M)}(z) = \min\,\{M, \nu(z)\},$$

$$n^{(M)}(t,\nu) = \begin{cases} \int_{|\nu|\cap B(t)} \nu^{(M)}(z) v_{m-1} & \text{if } m \geq 2, \\ \sum\limits_{|z|\leqslant t} \nu^{(M)}(z) & \text{if } m = 1, \end{cases}$$

$$n(t,\nu) = n^{(\infty)}(t,\nu).$$

Define

$$N^{(M)}(r,\nu) = \int\limits_1^r \frac{n^{(M)}(t)}{t^{2n-1}} dt \quad (1 < r < \infty),$$

$$N(r,\nu) = N^{(\infty)}(r,\nu).$$

(d) Let $f : \mathbf{C}^m \longrightarrow \mathbf{P}^n(\mathbf{C})$ be a meromorphic mapping. For arbitrarily fixed homogeneous coordinates $(w_0 : \ldots : w_n)$ on $\mathbf{P}^n(\mathbf{C})$, we take a reduced representation $f = (f_0 : \ldots : f_n)$, which means that each f_i is a holomorphic function on $\mathbf{C}^m$ and $f(z) = \big(f_0(z) : \ldots : f_n(z)\big)$ outside the analytic set $\{f_0 = \ldots = f_n = 0\}$ of codimension ≥ 2. Set $\|f\| = \big(|f_0|^2 + \cdots + |f_n|^2\big)^{1/2}$.

The characteristic function of f is defined by

$$T(r,f) = \int\limits_{S(r)} \log\|f\|\sigma_m - \int\limits_{S(1)} \log\|f\|\sigma_m.$$

Let a be a meromorphic mapping of $\mathbf{C}^m$ into the dual projective space $\mathbf{P}^n(\mathbf{C})^*$ with reduced representation $a = (a_0 : \ldots : a_n)$. The duality is given by $(f,a) = \sum\limits_{i=0}^{n} a_i f_i$. We call a a *moving target* to deal with the intersection divisor $\operatorname{div}(f,a)$. Assume that $(f,a) \not\equiv 0$. We define the proximity function of f for a by

$$m(r,f;a) = \int\limits_{S(r)} \log\frac{||f||\cdot||a||}{|(f,a)|}\sigma_m - \int\limits_{S(1)} \log\frac{||f||\cdot||a||}{|(f,a)|}\sigma_m,$$

where $\|a\| = \big(|a_0|^2 + \cdots + |a_n|^2\big)^{1/2}$.

The first main theorem for moving targets in value distribution theory (see [11]) states that

$$T(r,f) + T(r,a) = m(r,f;a) + N(r, \operatorname{div}(f,a)), \quad (r > 1).$$

(e) As usual, by the notation $''||\ \mathcal{P}''$ we mean the assertion $\mathcal{P}$ holds for all $r \in [0,\infty)$ excluding a Borel subset E of the interval $[0,\infty)$ with $\int_E dr < \infty$.

Let f, a be as above. We say that a is "small" with respect to f if

$$||T(r,a) = o(T(r,f)), \qquad r \to \infty.$$

(f) Let φ be a nonzero meromorphic function on $\mathbf{C}^m$, which are occasionally regarded as a meromorphic map into $\mathbf{P}^1(\mathbf{C})$. The proximity function of φ is defined by

$$m(r,\varphi) := \int_{S(r)} \log\max\{|\varphi|,1\}\sigma_m.$$

(g) Let $\mathcal{M}(\mathbf{C}^m)$ be the field of all meromorphic functions on $\mathbf{C}^m$. Let $a_1,\dots,a_q$ $(q \geq n+1)$ be q meromorphic mappings of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})^*$ with reduced representations $a_j = (a_{j0}:\dots:a_{jn})$ $(1 \leqslant j \leqslant q)$. Denote by $\mathcal{F}\Big(\{a_j\}_{j=1}^q\Big) \subset \mathcal{M}(\mathbf{C}^m)$ the smallest subfield which contains $\mathbf{C}$ and all a_{jk}/a_{jl} with $a_{jl} \not\equiv 0$, where $1 \leqslant j \leqslant q, 0 \leqslant k,l \leqslant n$.

We say that that the family $\{a_j\}_{j=1}^q$ is in N-subgeneral position if and only if for every subset $R \subset Q$ with $|R| = N+1$ and for an arbitrary $(N+1,n+1)$-matrix $(a_{jk})_{j\in R, 0\leqslant k\leqslant n}$

$$\operatorname{rank}_{\mathcal{M}(\mathbf{C}^m)}\,(a_{jk})_{j\in R,0\leqslant k\leqslant n} = n+1.$$

We also denote the rank of the index subset R by

$$\operatorname{rank} R = \operatorname{rank}_{\mathcal{M}(\mathbf{C}^m)}\,(a_{jk})_{j\in R,0\leqslant k\leqslant n} \tag{1}$$

Let f be a meromorphic mapping of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$ with reduced representation $f = (f_0:\dots:f_n)$. Then $f := (f_0:\dots:f_n) : \mathbf{C}^m \to \mathbf{P}^n(\mathbf{C})$ is said to be k-nondegenerate over $\mathcal{F}\Big(\{a_j\}_{j=1}^q\Big)$ if there exist exactly $k+1$ linearly independent elements in $\{f_0,\dots,f_n\}$ over the field $\mathcal{F}\Big(\{a_j\}_{j=1}^q\Big)$.

3. Cartan-Nochka theorems over complex projective spaces

Put $Q = \{1,\dots,q\}$ $(q \geq 1)$. For a finite set R, $\sharp R$ denotes the cardinality of R. By Nochka (see [6, 1, 3]) we have the following.

Lemma 3.1. *Let $\{a_i\}_{i\in Q}$ be q moving targets in $\mathbf{P}^n(\mathbf{C})^*$ in N-subgeneral position, and assume that $q > 2N-n+1$. Then there are positive rational constants $\omega_j, j \in Q$ satisfying the following:*

(i) $0 < \omega_j \leqslant 1,\ \forall j \in Q$,

(ii) *Setting* $\tilde\omega = \max_{j\in Q}\omega_j$, *one gets*

$$\sum_{j=1}^q \omega_j = \tilde\omega(q-2N+n-1)+n+1.$$

(iii) $\dfrac{n+1}{2N-n+1} \leqslant \tilde\omega \leqslant \dfrac{n}{N}$.

(iv) *For* $R \subset Q$ *with* $0 < \sharp R \leqslant N+1$, $\sum_{j\in R}\omega_j \leqslant \operatorname{rank}_{\mathcal{F}\{a_i\}}\{a_i\}_{i\in R}$.

The above ω_j are called *Nochka weights*, and $\tilde\omega$ the *Nochka constant*.

Lemma 3.2. *Let the notation be as above. Let $E_j \geq 0, j \in Q$ be arbitrarily given numbers. Then for every subset $R \subset Q$ with $0 < \sharp R \leqslant N+1$, there is a subset $R^\circ \subset R$ such that $\sharp R^\circ = \operatorname{rank} R^\circ = \operatorname{rank} R$ and*

$$\sum_{i \in R} \omega_i E_i \leqslant \sum_{i \in R^\circ} E_i.$$

For a subset $\Phi \subset \mathcal{M}(\mathbf{C}^m)$ we denote by $\mathcal{L}(\Phi)$ the $\mathbf{C}$-vector space spanned by Φ. Assume that $q := \sharp\Phi < \infty$, and $1 \in \Phi$. Then for a positive integer p, we set $\Phi(p) = \{\varphi_1\varphi_2\cdots\varphi_k | \varphi_j \in \Phi; j = 1, \ldots, p\}$. Then

$$1 \in \Phi(p), \quad \Phi(p) \subset \Phi(p+1), \quad \sharp\Phi(p) = \binom{p+q-1}{p} = \binom{p+q-1}{q-1}.$$

Let $0 < \epsilon < 1$ be arbitrarily given. Then we denote by $p(\epsilon, q)$ the smallest positive integer p such that $\binom{p+q-1}{q-1} \leqslant (1+\epsilon)^p$. Set

$$P(\epsilon, q) = \sharp\Phi(p(\epsilon, q) + 1) = \binom{p(\epsilon, q) + q - 1}{q-1} \left(\leqslant (1+\epsilon)^{p(\epsilon,q)}\right).$$

Lemma 3.3. *Let the notations be as above. Then there exists an integer $p'(\epsilon, q) \leqslant p(\epsilon, q)$ such that*

$$\frac{\dim\mathcal{L}(\Phi(p'(\epsilon, q)+1))}{\dim\mathcal{L}(\Phi(p'(\epsilon, q)))} \leqslant (1+\epsilon), \quad \dim\mathcal{L}(\Phi(p'(\epsilon, q)+1)) \leqslant P(\epsilon, q).$$

Proof. Suppose that $\dim\mathcal{L}(\Phi(p+1))/\dim\mathcal{L}(\Phi(p)) > (1+\epsilon)$ for all $1 \leqslant p \leqslant p(\epsilon, q)$. Then

$$P(\epsilon, q) \geq \dim\mathcal{L}(\Phi(p(\epsilon, q)+1)) \geq \prod_{i=1}^{p(\epsilon,q)} \frac{\dim\mathcal{L}(\Phi(i+1))}{\dim\mathcal{L}(\Phi(i))} > (1+\epsilon)^{p(\epsilon,q)}.$$

Hence

$$P(\epsilon, q) > (1+\epsilon)^{p(\epsilon,q)}.$$

This is a contradiction. Thus, there exists a positive integer $p'(\epsilon, q) \leqslant p(\epsilon, q)$ such that

$$\frac{\dim\mathcal{L}(\Phi(p'(\epsilon, q)+1))}{\dim\mathcal{L}(\Phi(p'(\epsilon, q)))} \leqslant 1+\epsilon.$$

Moreover, we have

$$\dim\mathcal{L}(\Phi(p'(\epsilon, q)+1)) \leqslant \dim\mathcal{L}(\Phi(p(\epsilon, q)+1)) \leqslant P(\epsilon, q) \leqslant (1+\epsilon)^{p(\epsilon,q)}.$$

□

Remark 3.4. We give an evaluation of $p(\epsilon, q)$ for $0 < \epsilon < \sqrt{e} - 1$. For such an ϵ, we will show that $\binom{p+q-1}{q-1} \leqslant (1+\epsilon)^p$ for every integer $p \geq \frac{q}{\log^2(1+\epsilon)}$, which implies that

$$p(\epsilon, q) \leqslant \left[(1+\epsilon)^{\frac{q}{\log^2(1+\epsilon)}}\right] + 1 \quad (0 < \epsilon < \sqrt{e} - 1),$$

where $[\bullet]$ stands for Gauss' symbol.

Since $0 < \epsilon < \sqrt{e} - 1$, $p/q > 16$. Note that for $x > 16$

$$\sqrt{x} - 1 - \log(1 + x) > 0.$$

By assumption

$$p \log(1 + \epsilon) \geq q\sqrt{\frac{p}{q}},$$

and hence

$$\begin{aligned} p \log(1 + \epsilon) &\geq q\left(1 + \log(1 + \frac{p}{q})\right) = q + q \log(p + q) - q \log q \\ &> p \log(1 + \frac{q}{p}) + q \log(p + q) - q \log q \\ &= (p + q) \log(p + q) - (p + q) - p \log p + p - q \log q + q \\ &= \int_p^{p+q} (\log x - \log(x - p))dx > \sum_{i=1}^{p+q} (\log(p + i) - \log i) \\ &> \sum_{i=1}^{p+q-1} (\log(p + i) - \log i) = \log\tbinom{p+q-1}{q-1}. \end{aligned}$$

Thus, we have

$$\tbinom{p+q-1}{q-1} \leqslant (1 + \epsilon)^p.$$

By the definition of $p(\epsilon, q)$, we have $p(\epsilon, q) \leqslant [\frac{q}{\log^2(1+\epsilon)}] + 1$, and hence

$$P(\epsilon, q) = \binom{p(\epsilon, q) + q - 1}{q - 1} \leqslant \left[(1 + \epsilon)^{[\frac{q}{\log^2(1+\epsilon)}]+1}\right].$$

Theorem 3.5. *Let $f : \mathbf{C}^m \longrightarrow \mathbf{P}^n(\mathbf{C})$ be a non-constant meromorphic mapping. Let $\{a_i\}_{i=1}^q$ be meromorphic mappings of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})^*$ in N-subgeneral position such that a_i are small with respect to f and f is linearly nondegenerate over $\mathcal{F}(\{a_i\}_{i=1}^q)$. Then for an arbitrary $0 < \epsilon < 1$*

$$|| \quad (q - 2N + n - 1 - \epsilon)T(r, f) \leqslant \sum_{i=1}^q N^{((n+1)P(\epsilon,qN)-1)}(r, \mathrm{div}(f, a_i)) + o(T(r, f)).$$

Proof. Without loss of generality we may assume that $a_{i0} \not\equiv 0$ $(1 \leqslant i \leqslant q)$. Set $\tilde{a}_{ij} := a_{ij}/a_{i0}$, $||\tilde{a}_i|| := \sum_{j=0}^n |\tilde{a}_{ij}|$, $F_i := \sum_{j=0}^n f_j\tilde{a}_{ij}$. We put $\Phi = \{\tilde{a}_{ij}\}$. Then $\sharp\Phi = qN + 1$, and $\sharp\Phi(p) = \tbinom{p+qN}{qN}$. Take arbitrarily $0 < \epsilon < 1$. By Lemma 3.3 there exists a positive integer $p'(\epsilon, qN + 1) \leqslant p(\epsilon, qN + 1)$ such that

$$\frac{\dim\mathcal{L}(\Phi(p'(\epsilon, qN + 1) + 1))}{\dim\mathcal{L}(\Phi(p'(\epsilon, qN + 1)))} \leqslant (1 + \epsilon),$$

$$\dim\mathcal{L}(\Phi(p'(\epsilon, qN + 1) + 1)) \leqslant P(\epsilon, qN + 1).$$

Put

$$s = \dim\mathcal{L}(\Phi(p'(\epsilon, qN + 1))), \qquad t = \dim\mathcal{L}(\Phi(p'(\epsilon, qN + 1) + 1)).$$

Let $\{b_1, \ldots, b_s\}$ be a base of $\mathcal{L}(\Phi(p'(\epsilon, qN+1)))$ and $\{b_1, \ldots, b_t\}$ be a base of $\mathcal{L}(\Phi(p'(\epsilon, qN+1)+1))$. The followings are satisfied:

(i) $\dfrac{t}{s} \leqslant 1+\epsilon, \quad t \leqslant (1+\epsilon)^{P(\epsilon, qN+1)}$.

(ii) $\{b_j f_k (1 \leqslant i \leqslant t, 0 \leqslant k \leqslant n)\}$ are linearly independent over $\mathbf{C}$.

Claim 3.6. *If $\{a_{i_1}, \ldots, a_{i_l}\}$ are linearly independent over $\mathcal{M}(\mathbf{C}^m)$, then $\{b_j F_{i_k} (1 \leqslant j \leqslant s, 1 \leqslant k \leqslant l)\}$ are linearly independent over $\mathbf{C}$.*

Indeed, assume that $\sum_{1\leqslant j\leqslant s, 1\leqslant k\leqslant l} c_{jk} b_j F_{i_k} \equiv 0$, where $c_{jk} \in \mathbf{C}$. Then

$$\sum_{v=0}^{n}\left(\sum_{k=1}^{l}\left(\sum_{j=1}^{s} c_{jk} b_j\right)\tilde{a}_{i_k v}\right) f_v \equiv 0.$$

Since f is linearly nondegenerate over $\mathcal{R}\{a_i\}$, it implies that

$$\sum_{k=1}^{l}(\sum_{j=1}^{s} c_{jk} b_j)\tilde{a}_{i_k v} \equiv 0 \quad (0 \leqslant v \leqslant n).$$

Hence

$$\sum_{j=1}^{s} c_{jk} b_j \equiv 0 \quad (1 \leqslant k \leqslant l).$$

This yields

$$c_{jk} = 0 \quad (1 \leqslant\ k \leqslant l, 1 \leqslant j \leqslant s).$$

Claim 3.6 is proved.

Set $Q := \{1, \ldots, q\}$. Let $R \subset Q$ be such that $\sharp R = N+1$. Choose $R^\circ \subset R$ such that $\{a_i\}_{i\in R^\circ}$ are linearly independent over $\mathcal{M}(\mathbf{C}^m)$ and R° satisfies Lemma 3.2. Then $\{F_i\}_{i\in R^\circ}$ are linearly independent over $\mathcal{R}\{a_j\}$. Assume that $R := \{r_1, \ldots, r_{N+1}\}$ and $R^\circ := \{r_1^\circ, \ldots, r_{n+1}^\circ\}$.

Since $b_j F_{r_k^\circ} (1 \leqslant j \leqslant s, 1 \leqslant k \leqslant n+1)$ are linearly independent over $\mathbf{C}$, we can choose $\beta_{mj}^{kl} \in \mathbf{C}$ such that there is $C_{R^\circ} \in GL((n+1)t; \mathbf{C})$ satisfying

$$\begin{aligned}&\det(b_j F_{r_k^\circ} (1 \leqslant j \leqslant s, 1 \leqslant k \leqslant n+1), h_{ml}(s+1 \leqslant l \leqslant t, 0 \leqslant m \leqslant n))\\ &= C_{R^\circ} \det(b_j f_k (1 \leqslant j \leqslant t, 0 \leqslant k \leqslant n)),\end{aligned}$$

where $h_{mj} = \sum_{1\leqslant k\leqslant t, 0\leqslant l\leqslant n} \beta_{mj}^{kl} b_k f_l (s+1 \leqslant j \leqslant t, 0 \leqslant m \leqslant n)$, and C_{R° is a matrix of constants.

Let $\alpha := (\alpha_1, \ldots, \alpha_{(n+1)t}) \in (\mathbf{Z}_+^m)^{(n+1)t}$ be a minimal multi-index in the lexicographical order such that

$$W \equiv \det\big(\mathcal{D}^{\alpha_w} b_j f_k (1 \leqslant j \leqslant t, 0 \leqslant k \leqslant n)\big)_{1\leqslant w\leqslant (n+1)t} \not\equiv 0.$$

By [2], Proposition 4.5, we have $|\alpha_i| \leqslant (n+1)t - 1, \forall 1 \leqslant i \leqslant (n+1)t$. Set

$$W_{R^\circ} \equiv \det\big(\mathcal{D}^{\alpha_w} b_j F_{r_k^\circ}, \mathcal{D}^{\alpha_w} h_{vl}\big),$$

where $1 \leqslant j \leqslant t, 1 \leqslant k \leqslant n+1$, $s+1 \leqslant l \leqslant t, 0 \leqslant v \leqslant n$, and $1 \leqslant w \leqslant (n+1)t$. It is easy to see that $W_{R^\circ} = W \cdot \det C_{R^\circ}$.

Let z be a fixed point. Then there exists $R \subset Q$ with $\sharp R = N+1$ such that $|F_i(z)| \leqslant |F_j(z)|$, $\forall i \in R, j \notin R$. On the other hand, we have

$$F_{r_k^\circ} := \sum_{j=0}^{n} \tilde{a}_{r_k^\circ j} f_j.$$

This implies that

$$f_k := \sum_{j=1}^{n+1} A_{kj} F_{r_j^\circ},$$

where $A_{kj} \in \mathcal{F}(\{a_i\})$. We put $A_R := \sum_{j=1}^{n+1} \sum_{k=0}^{n} |A_{kj}|$. Then

$$||f(z)|| \leqslant A_R(z)|F_j(z)|, \quad \forall j \notin R.$$

Set $A := \sum_{R \subset Q} A_R$. Then

$$|| \int_{S(r)} \log^+ A(z)\sigma_n = o(T(r,f)).$$

We also have

$$\begin{aligned}
(2) \qquad & \frac{||f(z)||^{\tilde{\omega}\cdot s(q-2N+n-1)}|W(z)|}{|F_1(z)|^{\omega_1 s}\cdots|F_q(z)|^{\omega_q s}\cdot||f(z)||^{(n+1)(t-s)}} \\
&= \frac{||f(z)||^{s(\sum_{i=1}^{q}\omega_i - n-1)}|W(z)|}{|F_1(z)|^{\omega_1 s}\cdots|F_q(z)|^{\omega_q s}\cdot||f(z)||^{(n+1)(t-s)}} \\
&\leqslant \frac{A^{s\sum_{i\notin R}\omega_i}(z)||f(z)||^{s\sum_{i\in R}\omega_i}|W(z)|}{\prod_{i\in R}|F_i(z)|^{\omega_i s}||f(z)||^{(n+1)t}} \\
&= \left(\prod_{i\in R}\left(\frac{||f(z)||\cdot||\tilde{a}_i(z)||}{|F_i(z)|}\right)^{\omega_i}\right)^{s} \frac{A^{s\sum_{i\notin R}\omega_i}(z)|W(z)|}{\prod_{i\in R}||\tilde{a}_i(z)||^{s\omega_i}\cdot||f(z)||^{(n+1)t}} \\
&\leqslant \left(\prod_{i\in R^\circ}\frac{||f(z)||\cdot||\tilde{a}_i(z)||}{|F_i(z)|}\right)^{s} \frac{A^{s\sum_{i\notin R}\omega_i}(z)|W_{R^\circ}(z)\det C_{R^\circ}|}{\prod_{i\in R}||\tilde{a}_i(z)||^{s\omega_i}\cdot||f(z)||^{(n+1)t}} \\
&= \frac{\prod_{i\in R^\circ}||\tilde{a}_i(z)||\cdot|\det C_{R^\circ}|}{\prod_{i\in R}||\tilde{a}_i(z)||^{s\omega_i}} \cdot \frac{A^{s\sum_{i\notin R}\omega_i}(z)|W_{R^\circ}(z)|}{\prod_{i\in R^\circ}|F_i(z)|^{s}\cdot||f(z)||^{(n+1)(t-s)}}.
\end{aligned}$$

Put

$$B_R := \frac{\prod_{i\in R^\circ}||\tilde{a}_i||\cdot\det C_{R^\circ}}{\prod_{i\in R}||\tilde{a}_i||^{s\omega_i}} \cdot \frac{A^{s\sum_{i\notin R}\omega_i}(z)|W_{R^\circ}|}{\prod_{i\in R^\circ}|F_i|^{s}\cdot||f||^{(n+1)(t-s)}}.$$

It follows easily that

$$|| \int_{S(r)} \log^+ B_R(z)\sigma_m = o(T(r,f)).$$

By (2), we have

$$\log\left(\frac{||f(z)||^{\tilde{\omega}\cdot s(q-2N+n-1)}|W(z)|}{|F_1(z)|^{\omega_1 s}\cdots|F_q(z)|^{\omega_q s}\cdot||f(z)||^{(n+1)(t-s)}}\right)\leqslant\sum_{R\subset Q}\log^+ B_R$$

for $z\in\mathbf{C}^m$. Integrating both sides of the above inequality over $S(r)$, we have

$$\begin{aligned}(3)\qquad || \ (q-2N+n-1)T(r,f) &\leqslant \sum_{i=1}^{q}\frac{\omega_i}{\tilde{\omega}}N(r,\mathrm{div}(f,a_i))(r)\\ &+\frac{n+1}{\tilde{\omega}}\left(\frac{t}{s}-1\right)T(r,f)-\frac{1}{\tilde{\omega}s}N(r,\mathrm{div}_0W)\\ &+\frac{1}{\tilde{\omega}s}N(r,\mathrm{div}_\infty W)+o(T(r,f)).\end{aligned}$$

Claim 3.7. $||\ N(r,\mathrm{div}_\infty W)=o(T(r,f))$.

First of all we see that if f and g are non-zero meromorphic functions on $\mathbf{C}^m$, then the followings are satisfied for $\alpha\in\mathbf{Z}^m_+$ and $z\in\mathbf{C}^m$ outside an analytic subset of dimension $\leqslant n-2$:

(i) $\mathrm{div}(fg)(z)=\mathrm{div}f(z)+\mathrm{div}g(z)$.
(ii) $\mathrm{div}\mathcal{D}^\alpha(fg)(z)\geq\mathrm{div}\mathcal{D}^\alpha f(z)-\mathrm{div}_\infty\mathcal{D}^\alpha g(z)$.
(iii) $\mathrm{div}_\infty\mathcal{D}^\alpha f(z)\leqslant(|\alpha|+1)\mathrm{div}_\infty f(z)$.
(iv) $\mathrm{div}_0 f(z)\leqslant\mathrm{div}_0\mathcal{D}^\alpha f(z)+|\alpha|$.

Put $\mathcal{I}=\bigcup_{i=1}^q I(a_i)\cup I(f)$, and

$$\lambda=\sum_{\substack{1\leqslant j\leqslant t\\ 1\leqslant\omega\leqslant(n+1)t}}(n+1)\mathrm{div}_\infty(\mathcal{D}^{\alpha_\omega}b_j)+\sum_{\substack{s+1\leqslant j\leqslant t,0\leqslant v\leqslant n\\ \sharp R=N+1,1\leqslant\omega\leqslant(n+1)t}}\mathrm{div}_\infty(\mathcal{D}^{\alpha_\omega}h^R_{vj}).$$

By the above properties (i)–(iii) we have that $||\ N(r,\lambda)=o(T(r,f))$. Since $N(r,\mathrm{div}_\infty W)\leqslant N(r,\lambda)$, Claim 3.7 follows.

We are going to show

$$\begin{aligned}&\sum_{i=1}^{q}\omega_i N(r,\mathrm{div}(f,a_i))-\frac{1}{s}N(r,\mathrm{div}_0 W)\\ (4)\qquad &\leqslant\sum_{i=1}^{q}\omega_i N^{((n+1)t-1)}(r,\mathrm{div}(f,a_i))+o(T(r,f)).\end{aligned}$$

Assume that z is a zero of some (f,a_i). We consider two cases.

Case 1. z is a common zero of at least $N+1$ functions in the family $\{(f,a_i)\}$.

Suppose that $(f,a_i)(z)=0$ for $1\leqslant i\leqslant p$ with $p\geq N+1$, and that $(f,a_i)(z)\neq0$ for $i>p$. Without loss of generality one may assume that

$$\mathrm{div}(f,a_1)(z)\geq\mathrm{div}(f,a_2)(z)\geq\cdots\geq\mathrm{div}(f,a_p)(z).$$

Put $R:=\{1,2,\dots,N+1\}$, Choose $R^\circ:=\{r_1^\circ,\dots,r_{n+1}^\circ\}\subset R$ such that $\{a_i\}_{i\in R^\circ}$ are linearly independent over $\mathcal{M}(\mathbf{C}^m)$ and R° satisfies Lemma 3.2. Then

z either is a zero with multiplicity at least $\operatorname{div}(f, a_{r^\circ_{n+1}})$ of $\det(a_{r^\circ_i,j})_{1\leqslant i\leqslant n+1, 0\leqslant j\leqslant n}$, or z is in $I(f)$.

In fact, suppose that $z \notin I(f)$. Without loss of generality we may assume that $f_0(z) \neq 0$. Then, there exists a neighbourhood U of z such that f_0 is non-vanishing on U. We have

$$\begin{aligned}
&\det(a_{r^\circ_i,j})_{1\leqslant i\leqslant n+1, 0\leqslant j\leqslant n}\\
&= \det\begin{vmatrix} a_{r^\circ_1,0} + \frac{f_1}{f_0}a_{r^\circ_1,1} + \cdots + \frac{f_n}{f_0}a_{r^\circ_1,n} & a_{r^\circ_1,1} & \cdots & a_{r^\circ_1,n}\\ \vdots & \vdots & \vdots & \\ a_{r^\circ_{n+1},0} + \frac{f_1}{f_0}a_{r^\circ_{n+1},1} + \cdots + \frac{f_n}{f_0}a_{r^\circ_1,n} & a_{r^\circ_{n+1},1} & \cdots & a_{r^\circ_{n+1},n}\end{vmatrix}\\
&= \frac{1}{f_0^{n+1}}\det\begin{vmatrix} (f, a_{r^\circ_1}) & a_{r^\circ_1,1} & \cdots & a_{r^\circ_1,n}\\ \vdots & \vdots & \vdots & \\ (f, a_{r^\circ_{n+1}}) & a_{r^\circ_{n+1},1} & \cdots & a_{r^\circ_{n+1},n}\end{vmatrix}
\end{aligned}$$

on U. Hence

$$\operatorname{div}_0(\det(a_{r^\circ_i,j})_{1\leqslant i\leqslant n+1, 0\leqslant j\leqslant n})(z) \geqslant \operatorname{div}(f, a_{r^\circ_{n+1}})(z).$$

This implies that

$$\begin{aligned}
\sum_{i=N+2}^{p}\operatorname{div}(f, a_i)(z) &\leqslant (p-N-1)\operatorname{div}(f, a_{r^\circ_{n+1}})(z)\\
&\leqslant (q-N-1)\operatorname{div}_0\det(a_{r^\circ_i,j})_{1\leqslant i\leqslant n+1, 0\leqslant j\leqslant n}(z).
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
&\sum_{i=1}^{N+1}\omega_i(\operatorname{div}(f, a_i)(z) - \min\{\operatorname{div}(f, a_i)(z), (n+1)t-1\})\\
&\leqslant \sum_{i=1}^{n+1}(\operatorname{div}(f, a_{r^\circ_i})(z) - \min\{\operatorname{div}(f, a_{r^\circ_i})(z), (n+1)t-1\})
\end{aligned}$$

and

$$\begin{aligned}
&\operatorname{div}_0 W(z) = \operatorname{div}_0 W_{R^\circ}(z)\\
&\geq \min_{\sigma\in S_{(n+1)t}}\operatorname{div}\Bigg(\prod_{\substack{1\leqslant j\leqslant s\\ 1\leqslant k\leqslant n+1}}\mathcal{D}^{\alpha_{\sigma((j-1)(n+1)+k)}}b_jF_{r^\circ_k}\times\\
&\qquad\qquad \times\prod_{\substack{s+1\leqslant j\leqslant t\\ 0\leqslant v\leqslant n}}\mathcal{D}^{\alpha_{\sigma((j-1)(n+1)+v+1)}}h^R_{vj}\Bigg)(z)\\
&\geq \min_{\sigma\in S_{(n+1)t}}\Bigg(\sum_{\substack{1\leqslant j\leqslant s\\ 1\leqslant k\leqslant n+1}}\operatorname{div}\mathcal{D}^{\alpha_{\sigma((j-1)(n+1)+k)}}F_{r^\circ_k}(z)
\end{aligned}$$

$$
\begin{aligned}
&\qquad - \sum_{\substack{1\leqslant j\leqslant s\\ 1\leqslant k\leqslant n+1}} \mathrm{div}_\infty \mathcal{D}^{\alpha_{\sigma((j-1)(n+1)+k)}} b_j(z)\\
&\qquad - \sum_{s+1\leqslant j\leqslant t, 0\leqslant v\leqslant n} \mathrm{div}_\infty \mathcal{D}^{\alpha_{\sigma((j-1)(n+1)+v+1)}} h^R_{vj}(z)\Bigg)\\
&\geq \min_{\sigma\in S_{(n+1)t}} \Bigg(\sum_{\substack{1\leqslant j\leqslant s\\ 1\leqslant k\leqslant n+1}} \Big(\mathrm{div}\mathcal{D}^{\alpha_{\sigma((j-1)(n+1)+k)}}(f, a_{r^\circ_k})(z)\\
&\qquad - n\mathrm{div}_\infty \mathcal{D}^{\alpha_{\sigma((j-1)(n+1)+k)}} a_{r^\circ_k 0}(z)\Big) - \lambda(z)\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&\geq \min_{\sigma\in S_{(n+1)t}} \Bigg(\sum_{\substack{1\leqslant j\leqslant s\\ 1\leqslant k\leqslant n+1}} \Big(\mathrm{div}(f, a_{r^\circ_k})(z) - \min\{\mathrm{div}(f, a_{r^\circ_k})(z), |\alpha_{\sigma((j-1)(n+1)+k)}|\}\\
&\qquad - n(|\alpha_{\sigma((j-1)(n+1)+k)}| + 1)\mathrm{div}_\infty a_{r^\circ_k 0}(z)\Big) - \lambda(z)\Bigg)\\
&\geq s\Big(\sum_{1\leqslant k\leqslant n+1} \mathrm{div}(f, a_{r^\circ_k})(z) - \min\{\mathrm{div}(f, a_{r^\circ_k})(z), (n+1)t-1\}\Big)\\
&\qquad - \sum_{1\leqslant k\leqslant n+1} n(n+1)ts\,\mathrm{div}_\infty a_{r^\circ_k 0}(z) - \lambda(z),
\end{aligned}
$$

where $S_{(n+1)t}$ is the $(n+1)t$-th symmetric group.

This implies that

$$
\begin{aligned}
&\sum_{i=1}^{N+1} \omega_i \Big(\mathrm{div}(f, a_i)(z) - \min\{\mathrm{div}(f, a_i)(z), (n+1)t-1\}\Big)\\
&\leqslant \sum_{i=1}^{n+1} \Big(\mathrm{div}(f, a_{r^\circ_i})(z) - \min\{\mathrm{div}(f, a_{r^\circ_i})(z), (n+1)t-1\}\Big)\\
&\leqslant \mathrm{div}_0 W(z) + \sum_{1\leqslant k\leqslant n+1} n(n+1)ts\,\mathrm{div}_\infty a_{r^\circ_k 0}(z) + \lambda(z).
\end{aligned}
$$

Thus, we have either

$$
\begin{aligned}
&\sum_{i=1}^{q} \omega_i \Big(\mathrm{div}(f, a_i)(z) - \min\{\mathrm{div}(f, a_i)(z), (n+1)t-1\}\Big)\\
&\leqslant \frac{1}{s}\mathrm{div}_0 W(z) + \sum_{1\leqslant k\leqslant n+1} n(n+1)t\,\mathrm{div}_\infty a_{r^\circ_k 0}(z)
\end{aligned}
$$

$$+\frac{1}{s}\lambda(z)+\frac{1}{s}(q-N-1)\mathrm{div}_0\det(a_{r_i^\circ j})_{1\leqslant i\leqslant n+1,0\leqslant j\leqslant n}(z)$$

or $z\in I(f)$.

Case 2. z is a common zero of at most N functions in the family $\{(f,a_i)\}$.

Suppose that $(f,a_i)(z)=0$ for $1\leqslant i\leqslant p$ with $p\leqslant N$, and that $(f,a_i)(z)\neq 0$ for $i>p$. Consider the set $R=\{1,\dots,N+1\}$. Repeating the argument of Case 1, we have

$$\begin{aligned}&\sum_{i=1}^q\omega_i\Big(\mathrm{div}(f,a_i)(z)-\min\{\mathrm{div}(f,a_i)(z),(n+1)t-1\}\Big)\\&\leqslant\frac{1}{s}\mathrm{div}_0W(z)+\sum_{1\leqslant k\leqslant n+1}n(n+1)t\mathrm{div}_\infty a_{r_k^\circ 0}(z)+\frac{1}{s}\lambda(z).\end{aligned}$$

From the consequence of the above two cases we infer that for $z\in\mathbf{C}^m$ outside an analytic subset of dimension $\leqslant n-2$

$$\begin{aligned}&\sum_{i=1}^q\omega_i\Big(\mathrm{div}(f,a_i)(z)-\min\{\mathrm{div}(f,a_i)(z),(n+1)t-1\}\Big)\\&\leqslant\frac{1}{s}\mathrm{div}_0W(z)+\sum_{1\leqslant k\leqslant q}n(n+1)t\mathrm{div}_\infty a_{k0}(z)+\frac{1}{s}\lambda(z)\\&\quad+\frac{1}{s}(q-N-1)\sum_{\sharp R=N+1}\mathrm{div}_0\det(a_{r_i^\circ j})_{1\leqslant i\leqslant n+1,0\leqslant j\leqslant n}(z).\end{aligned}$$

Integrating both sides, we have

$$\begin{aligned}&\sum_{i=1}^q\omega_i\left(N(r,\mathrm{div}(f,a_i))-N^{((n+1)t-1)}(r,\mathrm{div}(f,a_i))\right)\\&\leqslant\frac{1}{s}N(r,\mathrm{div}_0W)+\sum_{1\leqslant k\leqslant q}n(n+1)tN(r,\mathrm{div}_\infty a_{k0})+\frac{1}{s}N(r,\lambda)\\&\quad+\frac{1}{s}(q-N-1)\sum_{\sharp R=N+1}N(r,\mathrm{div}_0\det(a_{r_i^\circ,j})_{1\leqslant i\leqslant n+1,0\leqslant j\leqslant n})\\&=\frac{1}{s}N(r,\mathrm{div}_0W)+o(T(r,f)).\end{aligned}$$

Hence (4) is proved. Thus, we have

$$||\ (q-2N+n-1-\epsilon)T(r,f)\leqslant\sum_{i=1}^qN^{((n+1)p(\epsilon)-1)}(r,\mathrm{div}(f,a_i))+o(T(r,f)).$$

□

The following is a reformulation of Theorem 3.5:

Theorem 3.8. *Let $f : \mathbf{C}^m \longrightarrow \mathbf{P}^n(\mathbf{C})$ be a nonconstant meromorphic mapping, and let $\{a_i\}_{i=1}^q$ be small (with respect to f) meromorphic mappings of $\mathbf{C}^m$ into $\mathbf{P}^n(\mathbf{C})$ in general position such that f is k-nondegenerate over $\mathcal{F}(\{a_i\}_{i=1}^q)$. Let $0 < \epsilon < 1$ be arbitrary. Then the following holds*

$$|| \quad (q - 2n + k - 1 - \epsilon)T(r, f) \leqslant \sum_{i=1}^{q} N^{((k+1)P(\epsilon,kq+3)-1)}(r, \operatorname{div}(f, a_i)) + o(T(r, f)).$$

4. Cartan-Nochka theorem over function fields

(a) (First Main Theorem over function fields (see [7])) Let $\mathbf{k}$ be an algebraically closed field of characteristic 0 (for simplicity we assume that $\mathbf{k} = \mathbf{C}$), let R be a smooth projective algebraic variety of dimension N over $\mathbf{k}$, and let K denote the rational function field of R.

We fix a Hodge metric form ω on R. For a divisor D on R, we define the counting function of D with respect to ω by

$$N(D; \omega) = \int_D \omega^{N-1}.$$

Let $a_j \in K, j = 0, \ldots, m$, be not all zero; so say, $a_0 \neq 0$. We define a divisor on R by

$$((a_j))_\infty = -\min\left\{\operatorname{div}\frac{a_j}{a_0}; 0 \leqslant j \leqslant m\right\}.$$

Then the (projective) height $\operatorname{ht}((a_j); \omega)$ of $(a_0, \ldots, a_m)$ with respect to ω is defined by

$$\operatorname{ht}((a_j); \omega) = N(((a_j))_\infty; \omega).$$

By [7], Section 2.1, we have

$$\operatorname{ht}((a_j); \omega) = \int_R dd^c \log\Big(\sum_{j=0}^{m} |a_j|^2\Big) \wedge \omega^{N-1}.$$

There is another interpretation of $\operatorname{ht}((a_j); \omega)$. Let $L \to R$ be a line bundle determined by the divisor $((a_j))_\infty$, and $\sigma_0 \in \Gamma(R, L)$ be a global holomorphic section determining the divisor $\operatorname{div}\sigma_0 = ((a_j))_\infty$. Then $N(\operatorname{div}\sigma_0; \omega)$ is considered as a *counting function.*

Setting $\sigma_j = (a_j/a_0)\sigma_0 \in \Gamma(R, L), 0 \leqslant j \leqslant m$, one gets the following reduced representation of a rational mapping f from R into $\mathbf{P}^m(\mathbf{C})$:

$$f = (\sigma_0 : \ldots : \sigma_m) : R \to \mathbf{P}^m(\mathbf{C}).$$

Let Ω denote the Fubini-Study form on $\mathbf{P}^m(\mathbf{C})$. We define the *characteristic* or *order function* of f by

$$T(f; \omega) = \int_R f^*\Omega \wedge \omega^{N-1}.$$

Then we have the following *First Main Theorem* over function fields:

$$T(f;\omega) = \mathrm{ht}((a_j);\omega) = N(\mathrm{div}\sigma_j;\omega).$$

Thus we write $\mathrm{ht}(f;\omega) = T(f;\omega)$.

(b) (Wronskian) We use the same notations as in Subsection (a). Let $z_1, \ldots, z_N$ be a transcendental base of K. Then there exists a *Zariski* open subset U of R such that the holomorphic vector fields $\partial/\partial_j, 1 \leqslant j \leqslant N$ are defined on U and

$$\frac{\partial}{\partial z_1} \wedge \cdots \wedge \frac{\partial}{\partial z_N} \neq 0$$

at every point of U. Without loss of generality we may assume that the bundle $L \to R$ is trivial on U. Then the restriction to U of every holomorphic section in $\Gamma(R, L)$ can be considered as a holomorphic function on U.

Let $\{a_1, \ldots, a_t\}$ be a subset of $\Gamma(R, L)$ such that the family $\{a_1, \ldots, a_t\}$ is linearly independent over $\mathbf{C}$. We set $g = (a_1 : \ldots : a_t) : R \to \mathbf{P}^{t-1}(\mathbf{C})$. Then g is a linearly non-degenerate rational mapping. Let r be the rank of the differential dg at a general point. Then, by [2] and the construction of the Wronskian in [7], Section 2, we have a generalized Wronskian $W((a_i)) = W(a_1, \ldots, a_t)$ as described below.

For $x \in U$ we consider the vectors

$$(\mathcal{D}^\alpha a_1(x), \ldots, \mathcal{D}^\alpha a_t(x)) \in \mathbf{C}^t,$$

where $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbf{Z}_+^N$ are non-negative multi-indices and $\mathcal{D}^\alpha = \partial^{|\alpha|}/\partial z_1^{\alpha_1} \cdots z_N^{\alpha_N}$ with $|\alpha| = \alpha_1 + \cdots + \alpha_N$. We write $\mathrm{ord}\, \mathcal{D}^\alpha = |\alpha|$.

We denote $V_l(x)$ by the linear subspace of $\mathbf{C}^t$ spanned by

$$(\mathcal{D}^\alpha a_1(x), \ldots, \mathcal{D}^\alpha a_t(x))$$

with $|\alpha| \leqslant l$, and set

$$\lambda_l = \max_{x \in U} \dim V_l(x).$$

Starting from $(a_1(x), \ldots, a_t(x))$, we can take $(\mathcal{D}_i^\alpha a_1(x), \ldots, \mathcal{D}_i^\alpha a_t(x))$, $1 \leqslant i \leqslant \lambda_1 - 1$, with $|\alpha_i| = l$ such that for all $x \in U$ outside a thin analytic subset of U, the vectors $(a_1(x), \ldots, a_t(x))$ and $(\mathcal{D}_i^\alpha a_1(x), \ldots, \mathcal{D}_i^\alpha a_t(x))$, $1 \leqslant i \leqslant \lambda_1 - 1$ form a maximal linearly independent subset of

$$\{(\mathcal{D}^\alpha a_1(x), \ldots, \mathcal{D}^\alpha a_t(x)), |\alpha| \leqslant 1\}.$$

Here one notes that $\lambda_1 - 1 \geq r$. Similarly, we take $(\mathcal{D}_i^\alpha a_1(x), \ldots, \mathcal{D}_i^\alpha a_t(x))$, $\lambda_1 \leqslant i \leqslant \lambda_2 - 1$ with $|\alpha_i| = 2$. Thus, we inductively find the family $\{\alpha_i\}, 1 \leqslant i \leqslant t-1$, and obtain the generalized Wronskian

$$W((a_j))(x) = W(a_1, \ldots, a_t)(x) = \begin{vmatrix} a_1(x) & \cdots & a_t(x) \\ \mathcal{D}^{\alpha_1} a_1(x) & \cdots & \mathcal{D}^{\alpha_1} a_t(x) \\ \vdots & \vdots & \vdots \\ \mathcal{D}^{\alpha_{t-1}} a_1(x) & \cdots & \mathcal{D}^{\alpha_{t-1}} a_t(x) \end{vmatrix} \not\equiv 0$$

such that

$$|\alpha_i| = 1, \qquad 1 \leqslant i \leqslant r,$$
$$|\alpha_i| \leqslant i - r + 1, \; r+1 \leqslant i \leqslant t-1.$$

By the contruction we have $W((\zeta a_j)) = \zeta^t W((a_j))$ for all $\zeta \in K$.

There exists an effective divisor J on R given by the family of local trivializations $\{(R_\nu, \xi_\nu)\}$ such that $R = \cup R_\nu$ is a Zariski open covering and ξ_ν are holomorphic functions on R_ν with $(\xi_\nu) = J|R_\nu, L|R_\nu \equiv R_\nu \times \mathbf{C}$, and $\xi_\nu \partial/\partial z_i, 1 \leqslant i \leqslant N$, are holomophic on R_ν. By [7] §2 we have

$$W((a_j)) \in \Gamma(R, [pJ] \otimes L^t),$$
$$\triangle((a_j)) = \frac{W((a_j))}{a_1 \cdots a_t} \in \Gamma_{\mathrm{rat}}(R, [pJ]),$$

where $\Gamma_{\mathrm{rat}}(\cdot)$ denotes the space of rational sections and $p = \sum_{i=1}^t |\alpha_i| \leqslant \dfrac{m(m+1)}{2}$.

(c) (Definition) Being given a family $\mathcal{B} = \{b_0, \ldots, b_{q-1}\} \subset (K^*)^{m+1}$, where $K^* = K \setminus \{0\}$ and $b_i = (b_{i0} : \cdots : b_{im})$ $(0 \leqslant i \leqslant q-1)$, we say that the family $\mathcal{B}$ is *nondegenerate over* K if $\dim(\mathcal{B})_K = m+1$ and for every nonempty proper subset $\mathcal{B}_1$ of $\mathcal{B}$

$$(\mathcal{B}_1)_K \cap (\mathcal{B} \setminus \mathcal{B}_1)_K \cap \mathcal{A} \neq \emptyset,$$

where $(\mathcal{B})_K$ is the linear span of a subset $\mathcal{B}$ of K^{m+1} over the field K.

The set $\mathcal{L} \subset K^{m+1}$ is said to be *minimal (over K)* if it is linearly dependent over K and every proper subset of $\mathcal{L}$ is linearly independent over K.

Lemma 4.1. *Assume that the family $\mathcal{B} = \{b_0, \ldots, b_{q-1}\} \subset (K^*)^{m+1}$ is nondegenerate over K. Then there exist subsets $I_1, \ldots, I_k$ of $\mathcal{B}$ such that*

(i) *I_1 is minimal and I_i is linearly independent over K $(2 \leqslant i \leqslant k)$,*

(ii) *for each $2 \leqslant i \leqslant k$, there exist a meromorphic function $c_\alpha \in (K^*)$ satisfying*

$$\sum_{\alpha \in I_i} c_\alpha b_\alpha \in \left(\bigcup_{j=1}^{i-1} I_j \right)_K \quad \textit{and} \quad \left(\bigcup_{j=1}^{k} I_j \right)_K = (\mathcal{B})_K.$$

Proof. Since $b_0 \in (\mathcal{B} \setminus \{b_0\})_K$, we can choose a set I_1 such that I_1 is the minimal subset of $\mathcal{B}$ containing $\{b_0\}$. Assume that $I_1 = \{b_0, \ldots, b_{t_1}\}$. Then there exist meromorphic functions c_i, $1 \leqslant i \leqslant t_1$, and $c_0 = 1$ such that $\sum_{i=0}^{t_1} c_i b_i = 0$.

If $I_1 = \mathcal{B}$, then the proof is finished.

Otherwise, one of the following two cases holds:

i) There exists $b \in \mathcal{B} \setminus \{I_1\}$. We may assume that $b = b_{t_1+1}$ and $b \in (I_1)_K$. Put $I_2 = \{b_{t_1+1}\}$ and $c_{t_1+1} = 1$. Then there exist $c_{2j}, b_j \in I_1$ and $c_{2t_1+1} = c_{t_1+1}$ such that $\sum_{j=0}^{t_1+1} c_{2j} b_j = 0$. Moreover, we also may assume that $\{b_j \mid b_j \in I_1, \; c_{2j} \neq 0\}$ is independent over K.

ii) There exists $b \in \{I_1\}$. We may assume that $b = b_{t_1}$ and $b \in (\mathcal{B} \setminus I_1)_K$. Then there exists a subset of $\mathcal{B} \setminus I_1$ which is independent over K. We may assume that this subset is $\{b_{t_1+1}, \ldots, b_{t_2}\}$. On the other hand, there are c_i $(i \leqslant t_1 + 1 \leqslant t_2)$ such that $b_{t_1} + \sum_{j=t_1+1}^{t_2} c_j b_j = 0$. Set $I_2 = \{b_{t_1+1}, \ldots, b_{t_2}\}$.

If $I_1 \cup I_2 = \mathcal{B}$, then the proof is finished; otherwise, by repeating the above argument, we have the subset I_3.

Continuiting this process, there exist the subsets $I_1, \ldots, I_k$ satisfying the assertions of Lemma 4.1. □

Remark 4.2. Set

$$\nu_1 = \max\{\operatorname{div}\det\big(a_{i_j t_j}\big)_{1\leqslant j,h\leqslant m+1} \mid \det\big(a_{i_j t_j}\big)_{1\leqslant j,h\leqslant m+1} \not\equiv 0\},$$
$$\nu_2 = \min\{\operatorname{div}\det\big(a_{i_j t_j}\big)_{1\leqslant j,h\leqslant m+1} \mid \det\big(a_{i_j t_j}\big)_{1\leqslant j,h\leqslant m+1} \not\equiv 0\}.$$

By solving linear equations, for c_i and c_{ij} as in the above we see that $\operatorname{div} c_i \leqslant \nu_1 - \nu_2$ and $\nu_{c_{ij}} \leqslant \nu_1 - \nu_2$.

(d) (Second Main Theorem over function fields) Denote by $\mathcal{R}(\mathcal{B})$ the smallest subfield of K containing $\mathbf{C}$ and all $\{\frac{b_{il}}{b_{ik}}\}$ with $b_{ik} \neq 0$, $0 \leqslant i \leqslant q-1$, $0 \leqslant k, l \leqslant m$. By solving linear equations $\sum_{j=0}^{t_i+1} c_{(i+1)j} b_j = 0$ or $b_{t_i} + \sum_{j=t_i+1}^{t_{i+1}} c_j b_j = 0$ over the field $\mathcal{R}(\mathcal{B})$, it is easy to see that all elements c_i and c_{ij} belong to $\mathcal{R}(\mathcal{B})$.

Theorem 4.3. *Let $f = (\sigma_0 : \ldots : \sigma_m) : R \to \mathbf{P}^m(\mathbf{C})$ be a rational mapping with $\sigma_j \in \Gamma(R, L)$. Let $\mathcal{B} = \{b_0, \ldots, b_{q-1}\} \subset (K^*)^{m+1}$ be a finite family which is nondegenerate. Assume that f is linearly nondegenerate over $\mathcal{R}(\mathcal{B})$, i.e. $(f, c) = \sum_{i=0}^m c_i \sigma_i \neq 0$ for all $c = (c_0, \ldots, c_m) \in (\mathcal{R}(\mathcal{B}))^{m+1} \setminus \{0\}$. Then*

$$\begin{aligned}\operatorname{ht}(f; \omega) \leqslant \sum_{i=0}^{q-1} N^{(m)}(\operatorname{div}(f, b_i); \omega) + \frac{m(m+1)}{2} N(J; \omega)\\ + qN(\nu_1; \omega) + 2(q-1)N(\nu_2; \omega).\end{aligned}$$

Proof. By Lemma 4.1, we may assume that there exist the subsets $I_i = \{b_{t_{i-1}+1}, \ldots, b_{t_i}\}$ $(1 \leqslant i \leqslant k)$, where $t_0 = -1$, which satisfy the assertions of Lemma 4.1. Since I_1 is minimal, there exists a linear relation among I_1. That is, there exist $c_{1j} \in \mathcal{R}(\mathcal{B})$ such that

$$\sum_{j=0}^{t_1} c_{1j} \cdot b_j = 0.$$

Define $c_{1j} = 0$ for all $j > t_1$. Then $\sum_{j=0}^{t_k} c_{1j} \cdot b_j = 0$. Since f is linearly nondegenerate over $\mathcal{R}(\mathcal{B})$, it implies that $\{c_{1j}(f, b_j)\}_{j=1}^{t_1}$ is linearly independent over $\mathbf{C}$. Hence there exists $\{\alpha_{11}, \ldots, \alpha_{1t_1}\} \subset \mathbf{Z}_+^{N+1}$ $(|\alpha_{1j}| \leqslant t_1 - 1 \leqslant N)$ such that

$$A_1 \equiv \begin{vmatrix} \mathcal{D}^{\alpha_{11}}(c_{11}(f, b_1)) & \cdots & \mathcal{D}^{\alpha_{11}}(c_{1t_1}(f, b_{t_1})) \\ \mathcal{D}^{\alpha_{12}}(c_{11}(f, b_1)) & \cdots & \mathcal{D}^{\alpha_{12}}(c_{1t_1}(f, b_{t_1})) \\ \vdots & \vdots & \vdots \\ \mathcal{D}^{\alpha_{1t_1}}(c_{11}(f, b_1)) & \cdots & \mathcal{D}^{\alpha_{1t_1}}(c_{1t_1}(f, b_{t_1})) \end{vmatrix} \not\equiv 0.$$

Now consider $i \geq 2$. By the choice of the set I_i there exist meromorphic mappings $c_{ij} \not\equiv 0$, $c_{ij} \in \mathcal{R}(\mathcal{B})$ $(t_{i-1}+1 \leqslant j \leqslant t_i)$ such that $\sum_{j=t_{i-1}+1}^{t_i} c_{ij} \cdot b_j \in \left(\bigcup_{j=1}^{i-1} I_j\right)_K$. Then, there exist meromorphic mappings $c_{ij} \in K$ $(0 \leqslant j \leqslant t_{i-1})$ such that $\sum_{j=0}^{t_i} c_{ij} \cdot b_j = 0$. Define $c_{ij} = 0$ for all $j > t_i$. Then $\sum_{j=0}^{t_k} c_{ij} \cdot (f, b_j) = 0$. Since $\{c_{ij}(f, b_j)\}_{j=t_{i-1}+1}^{t_i}$ is **C**-linearly independent, there exists $\{\alpha_{ij}\}_{j=t_{i-1}+1}^{t_i} \subset \mathbf{Z}_+^n$ $(|\alpha_{ij}| \leqslant t_i - t_{i-1} - 1 \leqslant N)$ such that

$$A_i = \det\Big(\mathcal{D}^{\alpha_{ij}}\Big(c_{is}(f, \tilde{a}_s)\Big)\Big)_{j,s=t_{i-1}+1}^{t_i} \not\equiv 0.$$

Consider an $t_k \times (t_k + 1)$ minor matrixes $\mathcal{T}$ given by

$$\mathcal{T} = \begin{bmatrix} \mathcal{D}^{\alpha_{11}}(c_{10}(f, b_0)) & \cdots & \mathcal{D}^{\alpha_{11}}(c_{1t_k}(f, b_{t_k})) \\ \mathcal{D}^{\alpha_{12}}(c_{10}(f, b_0)) & \cdots & \mathcal{D}^{\alpha_{12}}(c_{1t_k}(f, b_{t_k})) \\ \vdots & \vdots & \vdots \\ \mathcal{D}^{\alpha_{1t_1}}(c_{10}(f, b_0)) & \cdots & \mathcal{D}^{\alpha_{1t_1}}(c_{1t_k}(f, b_{t_k})) \\ \mathcal{D}^{\alpha_{2t_1+1}}(c_{20}(f, b_0)) & \cdots & \mathcal{D}^{\alpha_{2t_1+1}}(c_{2t_k}(f, b_{t_k})) \\ \mathcal{D}^{\alpha_{2t_1+2}}(c_{20}(f, b_0)) & \cdots & \mathcal{D}^{\alpha_{2t_1+2}}(c_{2t_k}(f, b_{t_k})) \\ \vdots & \vdots & \vdots \\ \mathcal{D}^{\alpha_{2t_2}}(c_{20}(f, b_0)) & \cdots & \mathcal{D}^{\alpha_{2t_2}}(c_{2t_k}(f, b_{t_k})) \\ \vdots & \vdots & \vdots \\ \mathcal{D}^{\alpha_{kt_{k-1}+1}}(c_{k0}(f, b_0)) & \cdots & \mathcal{D}^{\alpha_{kt_{k-1}+1}}(c_{kt_k}(f, b_{t_k})) \\ \mathcal{D}^{\alpha_{kt_{k-1}+2}}(c_{k0}(f, b_0)) & \cdots & \mathcal{D}^{\alpha_{kt_{k-1}+2}}(c_{kt_k}(f, b_{t_k})) \\ \vdots & \vdots & \vdots \\ \mathcal{D}^{\alpha_{kt_k}}(c_{k0}(f, b_0)) & \cdots & \mathcal{D}^{\alpha_{kt_k}}(c_{kt_k}(f, b_{t_k})) \end{bmatrix}.$$

Denote by $\mathcal{D}_i$ the minor of the matrix obtained by deleting the i-th column of the minor matrix $\mathcal{T}$. Since the sum of each row of $\mathcal{T}$ is zero, we actually have

$$\mathcal{D}_i = (-1)^i \mathcal{D}_0 = (-1)^i \prod_{i=1}^{k} A_i \not\equiv 0.$$

Without loss of generality, we may assume that $t_k = q - 1$. It is easy to see that $A_i \in \Gamma_{\mathrm{rat}}(R, [(\sum_{j=t_{i-1}+1}^{t_i} |\alpha_{ij}|)J] \otimes L^{t_i - t_{i-1}})$. Hence

$$D_i \in \Gamma_{\mathrm{rat}}(R, [(\sum_{ij} |\alpha_{ij}|)J] \otimes L^{q-1}).$$

This yields

$$\frac{D_i}{\prod_{j \neq i}(f, b_j)} \in \Gamma_{\mathrm{rat}}(R, [(\sum_{i,j} |\alpha_{ij}|)J]), \quad 0 \leqslant \forall i \leqslant q-1. \tag{5}$$

On the other hand, we have

$$\sum_{i=0}^{q-1} |(f,b_i)|^2 = \left(\sum_{i=0}^{q-1} \left| \frac{D_i}{\prod_{j\neq i}(f,b_j)} \right|^2 \right) \frac{\prod_{i=0}^{q-1}|(f,b_i)|^2}{|D_0|^2}.$$

By transmiting current equations, it implies that

$$\begin{aligned} &dd^c\left[\log\left(\sum_{i=0}^{q-1}|(f,b_i)|^2\right)\right] \\ &= dd^c\left[\log\left(\sum_{i=0}^{q-1}\left|\frac{D_i}{\prod_{j\neq i}(f,b_j)}\right|^2\right)\right] + dd^c\left[\log\left(\frac{\prod_{i=0}^{q-1}|(f,b_i)|^2}{|D_0|^2}\right)\right]. \end{aligned}$$

By (5) we also have

$$\int_R dd^c\left[\log\left(\sum_{i=0}^{q-1}\left|\frac{D_i}{\prod_{j\neq i}(f,b_j)}\right|^2\right)\right]\wedge\omega^{N-1} = \left(\sum_{i,j}|\alpha_{ij}|\right)N(J;\omega).$$

Since $(f,b_i)\in\Gamma_{\mathrm{rat}}(R,L)$, it implies that

$$\int_R dd^c\left[\log\left(\sum_{i=0}^{q-1}|(f,b_i)|^2\right)\right]\wedge\omega^{N-1} = N(\mathrm{div}\sigma_0;\omega) = T(f;\omega).$$

Denote by ν the divisor given by the section $\dfrac{\prod_{i=0}^{q-1}(f,b_i)}{D_0}$. Then

$$\int_R dd^c\left[\log\left(\frac{\prod_{i=0}^{q-1}|(f,b_i)|^2}{|D_0|^2}\right)\right]\wedge\omega^{N-1} = N(\nu;\omega).$$

This yields

$$T(f;\omega) = N(\nu;\omega) + \left(\sum_{i,j}|\alpha_{ij}|\right)N(J;\omega).$$

We also see that

$$\sum_{i,j}|\alpha_{ij}| \leqslant \sum_{i=1}^{k}\frac{(t_i-t_{i-1})(t_i-t_{i-1}+1)}{2} \leqslant \frac{m(m+1)}{2}.$$

Now we compute ν. Let z be a fixed point of M. Then there exists a neighbourhood U of z in M such that the restriction to U of the section σ_0 can be viewed as a holomorphic function on U. We also assume that there is a meromorphic function h on U such that $\nu_h = -\min_{i,j}\{\nu_{c_{i,j}}\}$ on U and there is a unique analytic subset S of pure codimension 1 such that $S = \bigcup_i \mathrm{supp}\ \mathrm{div}(f,b_i)\bigcup\bigcup_{i,j}\mathrm{supp}\ \mathrm{div}\, c_{i,j}$. Without loss of generality we may assume that z is a regular point of S.

Put $m_i = \operatorname{div}(f, b_i)(z)$ $(0 \leqslant i \leqslant q-1)$. Without loss of generality we may assume that $m_0 \leqslant m_1 \leqslant \cdots \leqslant m_q - 1$. Then

$$\begin{aligned}\operatorname{div}\mathcal{D}^{\alpha_{i,t_{i-1}+j}}\left(\frac{hc_{iv}(f,b_v)}{(f,b_0)}\right)(z) &\geq \max\{0, m_v - m_0 - |\alpha_{i,t_{i-1}+j}|\}\\ &\geq \max\{0, m_v - m_0 - m\}.\end{aligned}$$

On the other hand, we have

$$\frac{\prod_{i=1}^{q-1}(f,b_i)}{D_0} = h^{q-1}\frac{\prod_{i=1}^{q-1}((f,b_i)/(f,b_0))}{(\frac{h}{(f,b_0)})^{q-1}D_0}.$$

Hence

$$\begin{aligned}&\operatorname{div}\frac{\prod_{i=1}^{q-1}(f,b_i)}{D_0}(z)\\ &\leqslant \sum_{i=1}^{q-1}(m_i - m_0 - \max\{0, m_i - m_0 - m\}) + (q-1)\operatorname{div} h(z)\\ &\leqslant \sum_{i=1}^{q-1}\min\{m_i - m_0, m\} + (q-1)\operatorname{div} h(z)\\ &\leqslant \begin{cases}\sum_{i=1}^{q-1} \min\{m_i, m\} + (q-1)\operatorname{div} h(z) & \text{if } m_0 \geq 0,\\ \sum_{i=1}^{q-1} \min\{m_i, m\} - (q-1)m_0 + (q-1)\operatorname{div} h(z) & \text{if } m_0 \leqslant 0.\end{cases}\end{aligned}$$

Since (f, b_i) does have a multiplicity greater than m_0 at z, it implies that $m_0 \leqslant \nu_1(z) - \nu_2(z)$ if $m_0 \geq 0$ and $m_0 \geq \nu_2(z)$ if $m_0 \leqslant 0$. Moreover, since $\nu_{c_{ij}} \geq \nu_2 - \nu_1$, we have $\nu_h \leqslant \nu_1 - \nu_2$. This implies that

$$\begin{aligned}\operatorname{div}\frac{\prod_{i=0}^{q-1}(f,b_i)}{D_0}(z) \leqslant \sum_{i=0}^{q-1}&\min\{m_i, m\} + \nu_1(z) - (q-1)\nu_2(z)\\ &+ (q-1)(\nu_1(z) - \nu_2(z)).\end{aligned}$$

Hence

$$\nu \leqslant \sum_{i=0}^{q-1}\min\{\operatorname{div}(f,b_i), m\} + \nu_1 - (q-1)\nu_2 + (q-1)(\nu_1 - \nu_2).$$

Integrating both sides of the above inequality, we have

$$N(\nu;\omega) \leqslant \sum_{i=0}^{q-1} N^{(m)}(\operatorname{div}(f,b_i);\omega) + qN(\nu_1;\omega) + 2(q-1)N(\nu_2;\omega).$$

Combining the above assertions, we deduce that

$$\operatorname{ht}(f;\omega) \leqslant \sum_{i=0}^{q-1} N^{(m)}(\operatorname{div}(f,b_i);\omega) + \frac{m(m+1)}{2}N(J;\omega)$$

$$+ qN(\nu_1;\omega) + 2(q-1)N(\nu_2;\omega). \qquad \blacksquare$$

(e) (Curve case) The following definition is based on J.-T. Wang [16] §3.

Definition 4.4. Let V_i $(1 \leqslant i \leqslant 3)$ be $\mathbf{C}$-vector spaces. Assume that a $\mathbf{C}$-bilinear homomorphism

$$(u,v) \in V_1 \times V_2 \to uv \in V_3$$

is given. An element $\beta = (v_1, \ldots, v_l) \in V_2^l$ is said to be *nondegenerate for* V_1 if $\sum_{i=1}^{l} u_i v_i = 0$ with $u_i \in V_1$ implies $u_i = 0$ $(1 \leqslant i \leqslant l)$; otherwise, it is said to be degenerate for V_1.

The nondegeneracy of β for V_1 is equivalent to that $u_i v_j$ $(1 \leqslant i \leqslant k, 1 \leqslant j \leqslant l)$ are $\mathbf{C}$-linearly independent for all $\mathbf{C}$-linearly independent $u_i \in V_1$ $(1 \leqslant i \leqslant k)$.

Let R be a complex algebraic curve of genus g, and let $L_i, i = 1, 2$, be line bundles on R. The tensor product implies the natural $\mathbf{C}$-bilinear homomorphism

$$(u,v) \in H^0(R, L_1) \times H^0(R, L_2) \to uv \in H^0(R, L_1 \otimes L_2).$$

Let $f = (\sigma_0 : \ldots : \sigma_n) : R \to \mathbf{P}^n(\mathbf{C})$ be a rational mapping with $\sigma_j \in H^0(R, L_2)$. We say that f is nondegenerate for $H^0(R, L_1)$ if (σ_j) is nondegenerate for $H^0(R, L_1)$; this is clearly independent of the choice of the representation of f.

Let L be a line bundle on R with degree $\deg L$. For an arbitrarily given $\epsilon > 0$ we set

$$k_1(\epsilon) = \max\left\{ \left[\frac{2N-n+1}{\epsilon} + \frac{g-1}{\deg L} \right],\ 2g-2 \right\} + 1.$$

We prove the following.

Theorem 4.5. *Let H_j, $1 \leqslant j \leqslant q$, be linear forms in N-subgeneral position on $\mathbf{P}^n(\mathbf{C})$ with coefficients in $H^0(R, L)$. Let $\epsilon > 0$ be an arbitrary number. Then for an arbitrary integer $k(\epsilon) \geq k_1(\epsilon)$ and a holomorphic mapping $x : R \to \mathbf{P}^n(\mathbf{C})$ that is nondegenerate for $H(R, L^{k(\epsilon)+1})$ we have*

$$(q - 2N + n - 1 - \epsilon)\mathrm{ht}(x) \leqslant \sum_{j=1}^{q} N^{(k_2(\epsilon))}(\mathrm{div} H_j(x)) + C(k(\epsilon), g, \deg L, N, n),$$

where $k_2(\epsilon) = (n+1)((k(\epsilon)+1)\deg L - g + 1) - 1$ and $C(k(\epsilon), g, \deg L, N, n)$ is a constant depending on $k(\epsilon), g, \deg L, N$ and n.

Proof. Let $s = \dim H^0(R, L^{k(\epsilon)})$ and $t = \dim H^0(R, L^{k(\epsilon)+1})$ as in the proof of Theorem 3.5. Then the Riemann-Roch theorem implies

$$s = k(\epsilon)\deg L - g + 1,$$
$$t = s + \deg L.$$

We fix a global section $\sigma_0 \in H^0(R, L) \setminus \{0\}$. Let $(b_1, \ldots, b_s)$ be bases of $H^0(R, L^{k(\epsilon)})$ over $\mathbf{C}$. We take bases

$$c_1 = (\sigma_0 b_1, \ldots, c_s) = (\sigma_0 b_s, c_{s+1}, \ldots, c_t)$$

of $H^0(R, L^{k(\epsilon)+1})$ over $\mathbf{C}$.

Let $x = (x_0 : \ldots : x_n)$ be a reduced representation with $x_j \in H^0(R, H)$, where H is a line bundle over R. Set the Wronskian

$$W = \begin{vmatrix} c_1x_0 & \cdots & c_1x_n & c_2x_0 & \cdots & c_tx_n \\ d(c_1x_0) & \cdots & d(c_1x_n) & d(c_2x_0) & \cdots & d(c_tx_n) \\ \vdots & & \vdots & \vdots & & \vdots \\ d^{(n+1)t-1}(c_1x_0) & \cdots & \cdot & \cdot & \cdots & d^{(n+1)t-1}(c_tx_n) \end{vmatrix}.$$

Then the nondegeneracy condition for x gives

$$W \in H^0(R, L^{(k(\epsilon)+1)(n+1)t} \otimes H^{(n+1)t} \otimes K_R^{(n+1)t((n+1)t-1)/2}) \setminus \{0\}. \tag{6}$$

For $H_{j_1}, \ldots, H_{j_{n+1}}$ in general position, $b_iH_{j_k}(x) \in H^0(R, L^{k(\epsilon)+1} \otimes H)$, $1 \leqslant i \leqslant s, 1 \leqslant k \leqslant n+1$, are $\mathbf{C}$-linearly independent. Adding $(n+1)(t-s)$ elements

$$h_m = \sum_{\substack{1\leqslant k\leqslant t \\ 0\leqslant l\leqslant n}} \beta_m^{kl} c_k x_l, \qquad \beta_m^{kl} \in \mathbf{C},$$

we get bases $(b_iH_{j_k}(x), h_m)$ of $H^0(R, L^{k(\epsilon)+1}) \otimes_{\mathbf{C}} (\sum_l \mathbf{C} \cdot x_l)$ considered as a subspace of $H^0(R, L^{k(\epsilon)+1} \otimes H)$. Let $\tilde{W}$ be the Wronskian formed by those $b_iH_{j_k}(x)$ and h_m. Then there is a constant $C \in \mathbf{C}^*$ such that

$$\tilde{W} = CW.$$

Therefore, if $H_{j_k}(x)$ vanishes at a point $a \in R$ with order $\nu_k \geq (n+1)t-1$ for some k, then

$$\operatorname{div} W(a) \geq s \sum_{\text{those } k} (\nu_k - (n+1)t + 1). \tag{7}$$

Since H_{j_k} are in general position, the determinant Δ of the coefficients gives rise to a non-zero holomorphic section of L^{n+1}. Thus

$$\deg \Delta = (n+1) \deg L. \tag{8}$$

We claim the following estimate:

$$s\sum_{i=1}^{q} \omega_i (\operatorname{div} H_i(x)(a) - (n+1)t + 1)^+ \leqslant \operatorname{div} W(a), \quad a \in R, \tag{9}$$

where $(\cdot)^+$ denotes the positive part. Set

$$Q = \{1, \ldots, q\}.$$

Take a point $a \in R$ and set

$$S(a) = \{i \in Q; \operatorname{div} H_i(x)(a) \geq (n+1)t - 1\}.$$

Suppose that $\sharp S(a) \geq N+1$. Then there is a subset $S(a)^\circ \subset S(a)$ such that $\sharp S(a)^\circ = n+1$ and H_i, $i \in S(a)^\circ$, are in general position. Then the order of zero of the determinant formed by the coefficients of H_i, $i \in S(a)^\circ$ is at most

$(n+1)\deg L$ by (8). Since $(n+1)t-1 > (n+1)\deg L$, all $x_i(a)$ must be zero by Cramer's formula. This is a contradiction. Therefore we deduce $\sharp S(a) \leqslant N$.

By making use of Lemma 3.2 and (7) we have a subset $S(a)^\circ$ of $S(a)$ such that $\sharp S(a)^\circ = \operatorname{rank} S(a)^\circ = \operatorname{rank} S(a)$, and deduce that

$$
\begin{aligned}
& s\sum_{j\in Q}\omega_j(\mathrm{div}H_j(x)(a)-(n+1)t+1)^+ \\
& = s\sum_{j\in S(a)}\omega_j(\mathrm{div}H_j(x)(a)-(n+1)t+1) \\
& \leqslant s\sum_{j\in S(a)^\circ}(\mathrm{div}H_j(x)(a)-(n+1)t+1) \\
& \leqslant \operatorname{div} W(a). \qquad (10)
\end{aligned}
$$

Thus we proved (9). Note that

$$\min\{\operatorname{div} H_j(x)(a),(n+1)t-1\}+(\operatorname{div} H_j(x)(a)-(n+1)t+1)^+ = \operatorname{div} H_j(x)(a).$$

It follows from this and (9) that

$$
\begin{aligned}
& s\sum_{j\in Q}\omega_j\mathrm{div}H_j(x)(a)-\mathrm{div}W(a) \\
& \leqq s\sum_{j\in Q}\omega_j\min\{\mathrm{div}H_j(x)(a),(n+1)t-1\}. \qquad (11)
\end{aligned}
$$

We write $\|x(a)\| = \sqrt{\sum_i |x_i(a)|^2}$, which defines a hermitian metric in H. Using the notation in Lemma 3.2, we set

$$\phi(a) = \frac{\|x(a)\|^{\tilde{\omega}(q-2N+n-1)s-(n+1)(t-s)}|W(a)|}{|H_1(a)|^{\omega_1 s}\cdots|H_q(a)|^{\omega_q s}},$$

which defines a singular hermitian metric in $L^{k(\epsilon)(n+1)t}\otimes K_R^{(n+1)t((n+1)t-1)/2}$. Taking the differential as current, one gets

$$\int_R dd^c[\log|\phi|^2] = k(\epsilon)(n+1)t\deg L+(n+1)t((n+1)t-1)(g-1).$$

It follows from this and (11) that

$$
\begin{aligned}
& (\tilde{\omega}(q-2N+n-1)s-(n+1)(t-s))\mathrm{ht}(x) \\
& = \sum_{a\in R}\left\{s\sum_{j\in Q}\omega_i\mathrm{div}H_i(x)(a)-\mathrm{div}W(a)\right\} \\
& \quad + k(\epsilon)(n+1)t\deg L+(n+1)t((n+1)t-1)(g-1) \\
& \leqslant s\sum_{a\in R}\sum_{j\in Q}\omega_j\min\{\mathrm{div}H_j(x)(a),(n+1)t-1\} \\
& \quad + k(\epsilon)(n+1)t\deg L+(n+1)t((n+1)t-1)(g-1)
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant s\tilde{\omega} \sum_{j\in Q} N^{((n+1)t-1)}(\mathrm{div} H_j(x)) \\
&\quad + k(\epsilon)(n+1)t \deg L + (n+1)t((n+1)t-1)(g-1).
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
&\left\{ q - 2N + n - 1 - \frac{(2N-n+1)\deg L}{s} \right\} \mathrm{ht}(x) \\
&\leqslant \sum_{j\in Q} N^{((n+1)t-1)}(\mathrm{div} H_j(x)) + \left(1 + \frac{\deg L}{s}\right) k(\epsilon)(2N-n+1)\deg L \\
&+ \left(1 + \frac{\deg L}{s}\right)(2N-n+1)((n+1)(s+\deg L)-1)(g-1).
\end{aligned}
$$

With the choices of s and t we have

$$
\begin{aligned}
\epsilon &> \frac{(2N-n+1)\deg L}{s}, \\
k_2(\epsilon) &= (n+1)t - 1.
\end{aligned}
$$

Set

$$
\begin{aligned}
&C(k(\epsilon), g, \deg L, N, n) \\
&= \left(1 + \frac{\deg L}{s}\right) k(\epsilon)(2N-n+1)\deg L \\
&\quad + \left(1 + \frac{\deg L}{s}\right)(2N-n+1)((n+1)(s+\deg L)-1)(g-1).
\end{aligned}
$$

We have

$$
(q - 2N + n - 1 - \epsilon)\mathrm{ht}(x) \leqslant \sum_{j\in Q} N^{(k_2(\epsilon))}(\mathrm{div} H_j(x)) + C(k(\epsilon), g, \deg L, N, n).
$$

This finishes the proof of our theorem. □

Remark 4.6. A reformulation of Theorem 4.5 with a weaker estimate was announced in [8], Theorem 4.1, where the nondegeneracy condition for x as in Theorem 4.5 must be added.

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References

[1] W. Chen, Defect relations for degenerate meromorphic maps, *Trans. Amer. Math. Soc.* **319** (1990), 499–515.

[2] H. Fujimoto, Non-integrated defect relation for meromorphic maps of complete Kähler manifolds into $\mathbf{P}^{N_1}(\mathbf{C}) \times \cdots \times \mathbf{P}^{N_k}(\mathbf{C})$, *Japanese J. Math.* **11** (1985), 233–264.

[3] H. Fujimoto, Value Distribution Theory of the Gauss Map of Minimal Surfaces in $\mathbf{R}^m$, *Aspects of Math.* **E21**, 1993.

[4] I. E. Nochka, Defect relations for meromorphic curves (in Russian), *Izv. Akad. Nauk. Moldav. SSR Ser. Fiz.-Tekhn. Math. Nauk* **1** (1982), 41–47.

[5] I. E. Nochka, On a theorem from linear algebra (in Russian), *Izv. Akad. Nauk. Moldav. SSR Ser. Fiz.-Tekhn. Math. Nauk* **3** (1982), 29–33.

[6] I. E. Nochka, On the theory of meromorphic functions, *Sov. Math. Dokl.* **27** (1983), 377–381.

[7] J. Noguchi, Nevanlinna-Cartan theory over function fields and a Diophantine equation, *J. reine angew. Math.* **487** (1997), 61–83; Correction to the paper, Nevanlinna-Cartan theory over function fields and a Diophantine equation, *J. reine angew. Math.* **497** (1998), 235.

[8] J. Noguchi, Intersection multiplicities of holomorphic and algebraic curves with divisors, *Proc. OKA 100 Conference Kyoto/Nara 2001*, *Advanced Studies in Pure Mathematics* **42**, pp. 243–248, *Japan Math. Soc.* Tokyo, 2004.

[9] J. Noguchi, A note on entire pseudo-holomorphic curves and the proof of Cartan-Nochka's theorem, *Kodai Math. J.* **28** (2005), 336–346.

[10] J. Noguchi and T. Ochiai, *Introduction to Geometric Function Theory in Several Complex Variables*, Trans. Math. Monogr. 80, Amer. Math. Soc., Providence, Rhode Island, 1990.

[11] M. Ru and W. Stoll, The second main theorem for moving targets, *J. Geom. Anal.* **1** (1991), 99–138.

[12] M. Ru and W. Stoll, The Cartan conjecture for moving targets, *Proc. Sympos. Pure Math.* **52** (1991), 99–138.

[13] M. Shirosaki, Another proof of the defect relation for moving targets, *Tôhoku Math. J.* **43** (1991), 355–360.

[14] W. Stoll, Value distribution theory of meromorphic maps, *Aspects of Mathematics* **E7** (1985), Vieweg, Braunschweig.

[15] J. T.-Y. Wang, The truncated second main theorem of function fields, *J. Number Theory* **58** (1996), 137–159.

[16] J. T.-Y. Wang, Cartan's conjecture with moving targets of same growth and effective Wirsing's theorem over function fields, *Math. Z.* **234** (2000), 739–754.

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