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NOTE ON THE STOKES STRUCTURE OF FOURIER TRANSFORM

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Abstract. We study the Stokes structure of the Fourier transform of a meromorphic flat bundle on a projective line. We describe it in terms of the rapid decay homology introduced by S. Bloch and H. Esnault.

1. Introduction

Let (V, ∇) be a meromorphic flat bundle on $\mathbb{P}^1$. It is interesting to study a D-module $\mathfrak{Four}(V)$ on $\mathbb{P}^1$, obtained as the Fourier transform of (V, ∇). S. Bloch–H. Esnault [4] and R. García López [10] introduced the local Fourier transform of D-modules, to describe the formal completion of $\mathfrak{Four}(V)$ at ∞ in terms of the formal completion of (V, ∇) at the poles. See also the work of D. Arinkin [1]. The explicit formula was proved by J. Fang [8] and C. Sabbah [19]. (See also the work due to L. Fu [9] on the explicit formula for ℓ-adic local Fourier transform, and the previous influential work due to G. Laumon [12] and B. Malgrange [16].)

Then, it is natural to study the Stokes structure of $\mathfrak{Four}(V)$ at ∞. We should mention that Malgrange described it in his comprehensive work [16]. In this paper, we would like to give another apparently different description in terms of the homology theory introduced by Bloch-Esnault [5]. It would be desirable to have several ways to understand such a basic object, and the method using the Bloch-Esnault homology seems more elementary and direct in some aspects.

We should also mention that the argument in this paper is based on a method in an unpublished manuscript by A. Beilinson-S. Bloch-P. Deligne-H. Esnault [3]. Namely, we can investigate the asymptotic behaviour of periods by using the steepest descent method. (Such an estimate of periods seems to have also essentially appeared in [16].) And, we use the non-degeneracy of Vandermonde matrices to deduce that some tuple of flat sections gives a frame compatible with the Stokes filtration.

In this small note, we would like to add details on a construction of a family of cycles to [3], i.e., we choose the paths for integration in more explicit ways,

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and indicate how to modify the cycles when the Stokes structure of (V, ∇) is non-trivial, instead of using an observation due to C. Skiadas in [3] on the general existence of a path along which the behaviour of a given harmonic function can be controlled. It would make the Stokes structure of $\mathfrak{Four}(V)$ more visible, at least. In some concrete cases, we can take specific flat frames of $\mathfrak{Four}(V)$ on sectors around ∞, which seems useful to understand its Stokes structure in an explicit way.

As already suggested, it is our main purpose to understand the part of [3] related with the Stokes structure. The following is in our mind. Let k and K be subfields of $\mathbb{C}$. For simplicity, we assume that k is algebraically closed. Let (V, ∇) be a meromorphic flat bundle on $\mathbb{P}^1$, defined over k. Then, $\mathfrak{Four}(V)$ is also defined over k. Assume that the local system associated to (V, ∇) is equipped with a K-structure, compatible with the Stokes structure. Then, the space $\psi\big(\mathrm{Gr}\,\mathfrak{Four}(V)\big)$ of the multi-valued flat sections of $\mathrm{Gr}\,\mathfrak{Four}(V)$ is equipped with a (K, k)-structure. We will observe that it is described in terms of the data around the poles of (V, ∇). (See Corollary 5.11.) It seems a small refinement of a result in [3], where the (K, k)-structure of the determinant line of $\psi\big(\mathrm{Gr}\,\mathfrak{Four}(V)\big)$ is studied in relation with "ϵ-factor".

Let us give an outline of the paper. In Subsection 2.1 we recall basic facts on meromorphic flat bundles on curves. In particular, we review the notions of the Stokes filtration and the associated graded meromorphic flat bundle $\mathrm{Gr}(V)$. Subsection 2.2 is devoted to a review of basic facts on Fourier transform and local Fourier transform. We observe a property on the lattice of the Fourier transform induced by a good lattice pair in Lemma 2.6 based on [8], which is useful for our argument in Subsection 5.3, although not essential. In Subsection 2.3, we prepare a lemma to study the asymptotic behaviour of a family of the pairings between 1-forms and paths, following [3], which will be used in Subsections 3.3 and 4.3. In Subsection 2.4, after a brief description of homology theories due to Bloch-Esnault [5], we explain some procedures to make a 1-chain for a meromorphic flat bundle V from a chain for $\mathrm{Gr}(V)$. It will be used in Subsection 5.1. In Subsection 2.5, we recall that a flat section of $\mathfrak{Four}(V)^\vee$ is associated to a family of cycles of the dual of $V \otimes L(t\tau)$. (See Subsection 2.2 for the notation.) In Subsection 2.6, we recall that a (K, k)-structure of a meromorphic flat bundle V induces a (K, k)-structure on the space of the multi-valued flat sections of V.

In Subsections 3.1 and 4.1, for the functions F in (3.1) and (4.1), we give concrete and elementary ways to choose paths Γ such that (i) each Γ contains a critical point P of $\mathrm{Re}\,F$, (ii) $\mathrm{Re}\,F_{|\Gamma}$ has the maximum at P, and it rapidly decays in moving away from P. In Subsections 3.2 and 4.2, we describe a construction of cycles for the dual of $V \otimes L(t\tau)$ by using the perturbation of the above paths Γ, where V is elementary in the sense that $V \simeq \mathrm{Gr}(V)$. Subsections 3.3 and 4.3 are devoted to estimates for the pairings for the cycles and V-valued one forms. As a result, we obtain a quite concrete description of the Stokes structure of the Fourier transform of elementary meromorphic flat bundles. In Subsection 3.4, a construction of cycles and an estimate of pairings are recalled for the regular singular case.

In Subsection 5.1, we explain the construction of cycles for the dual of $V \otimes L(t\tau)$ from the cycles in Subsections 3.2 and 4.2, where V is a general meromorphic flat bundle. Subsection 5.2 is devoted to estimates of the cycles with a V-valued one forms. In Subsection 5.3, we give a description of the Stokes filtration of the Fourier transform $\mathfrak{Four}(V)$ in terms of the flat sections associated to the above cycles (Theorem 5.6). Then, we observe that the study on the induced (K,k)-structure on $\mathrm{Gr}(\mathfrak{Four}(V))$ can be reduced to those on the Fourier transform of the elementary meromorphic flat bundles associated to the poles of V (Corollary 5.10).

2. Preliminary

2.1. **Meromorphic flat bundles on curves.** We recall some basic facts on meromorphic flat bundles on curves. See [3], [7], [16], [18], [21] and [17], for example.

2.1.1. *Formal meromorphic flat bundle.* We implicitly use the bijection $\mathbb{C}(\!(z)\!)/z^m \mathbb{C}[\![z]\!] \simeq z^{m-1}\mathbb{C}[z^{-1}]$. Recall Hukuhara-Levelt-Turrittin theorem. Let $(\widehat{V},\widehat{\nabla})$ be a formal meromorphic flat bundle on $\mathbb{C}(\!(z)\!)$, i.e., $\widehat{V}$ is a $\mathbb{C}(\!(z)\!)$-vector space of finite rank equipped with a connection $\widehat{\nabla}:\widehat{V}\longrightarrow \widehat{V}\otimes\Omega^1_{\mathbb{C}(\!(z)\!)/\mathbb{C}}$. There exists a positive integer e with the following property:

- Let ζ be an e-th root of z. Then, there exist a subset $\mathrm{Irr}(\widehat{\nabla})\subset\mathbb{C}(\!(\zeta)\!)/\mathbb{C}[\![\zeta]\!]$ and a $\widehat{\nabla}$-flat decomposition

$$(\widehat{V},\widehat{\nabla})\otimes_{\mathbb{C}(\!(z)\!)}\mathbb{C}(\!(\zeta)\!)=\bigoplus_{\mathfrak{a}\in\mathrm{Irr}(\widehat{\nabla})}(\widehat{V}_{\mathfrak{a}},\widehat{\nabla}_{\mathfrak{a}}), \tag{2.1}$$

  such that $\widehat{\nabla}_{\mathfrak{a}}-d\widetilde{\mathfrak{a}}$ are regular singular for any $\mathfrak{a}\in\mathrm{Irr}(\widehat{\nabla})$, where $\widetilde{\mathfrak{a}}$ are lifts of $\mathfrak{a}$ to $\mathbb{C}(\!(\zeta)\!)$.

If $(\widehat{V},\widehat{\nabla})$ itself has such a decomposition, it is called unramified. We call (2.1) the Hukuhara-Levelt-Turrittin decomposition of $(\widehat{V},\widehat{\nabla})$ in this paper.

Let $(\widehat{V},\widehat{\nabla})$ be unramified. For a negative integer m, let $\mathrm{Irr}(\nabla,m)$ denote the image of $\mathrm{Irr}(\nabla)$ via the natural map $\eta_m:\mathbb{C}(\!(z)\!)/\mathbb{C}[\![z]\!]\longrightarrow\mathbb{C}(\!(z)\!)/z^{m+1}\mathbb{C}[\![z]\!]$. We have the coarser decompositions:

$$(\widehat{V},\widehat{\nabla})=\bigoplus_{\mathfrak{b}\in\mathrm{Irr}(\nabla,m)}(\widehat{V}^{(m)}_{\mathfrak{b}},\widehat{\nabla}^{(m)}_{\mathfrak{b}}),\quad \widehat{V}^{(m)}_{\mathfrak{b}}=\bigoplus_{\eta_m(\mathfrak{a})=\mathfrak{b}}\widehat{V}_{\mathfrak{a}}$$

In the case $\bigl|\mathrm{Irr}(\nabla)\bigr|>1$, we have the number $m_0\in\mathbb{Z}_{<0}$ determined by the conditions $\bigl|\mathrm{Irr}(\nabla,m_0-1)\bigr|=1$ and $\bigl|\mathrm{Irr}(\nabla,m_0)\bigr|>1$. We set $\mathrm{dec}(V):=-m_0$.

For a given $\mathfrak{a}\in\mathbb{C}(\!(z)\!)$, we will often use the symbol $L(\mathfrak{a})$ to denote the line bundle $\mathbb{C}(\!(z)\!)\,e$ with the connection $\nabla e=e\cdot d\mathfrak{a}$.

Let $(V,\widehat{\nabla})$ be not necessarily unramified. Let $\overline{\mathrm{Irr}}(\widehat{\nabla})$ denote the quotient of $\mathrm{Irr}(\widehat{\nabla})$ by the action of the Galois group of the extension $\mathbb{C}(\!(\zeta)\!)/\mathbb{C}(\!(z)\!)$. Let $\mathfrak{a}$ be a representative of an element of $\overline{\mathrm{Irr}}(\widehat{\nabla})$. There exists the subfield $K_{\mathfrak{a}}$ of $\mathbb{C}(\!(\zeta)\!)$ determined by the conditions (i) $\mathfrak{a}\in K_{\mathfrak{a}}/K_{\mathfrak{a}}\cap\mathbb{C}[\![\zeta]\!]$, (ii) $g^*\mathfrak{a}\neq\mathfrak{a}$ for any $g\in\mathrm{Gal}\big(K_{\mathfrak{a}}/\mathbb{C}(\!(z)\!)\big)\setminus\{1\}$. We take a lift $\widetilde{\mathfrak{a}}$ of $\mathfrak{a}$ to $K_{\mathfrak{a}}$. Let $q_{\mathfrak{a}}$ denote $\mathrm{Spec}\,K_{\mathfrak{a}}\longrightarrow\mathrm{Spec}\,\mathbb{C}(\!(z)\!)$. Then, there exists a regular singular connection $R_{\mathfrak{a}}$ on $K_{\mathfrak{a}}$ for each $\mathfrak{a}\in\overline{\mathrm{Irr}}(\widehat{\nabla})$ such that $(V,\nabla)\simeq\bigoplus_{\mathfrak{a}\in\overline{\mathrm{Irr}}(\nabla)}q_{\mathfrak{a}*}\Big(L(\widetilde{\mathfrak{a}})\otimes R_{\mathfrak{a}}\Big)$. We set

$$\mathrm{Gr}_{\mathfrak{a}}(V,\widehat{\nabla}):=q_{\mathfrak{a}*}\Big(L(\widetilde{\mathfrak{a}})\otimes R_{\mathfrak{a}}\Big). \tag{2.2}$$

2.1.2. *Stokes structure.* We set $\Delta:=\big\{z\in\mathbb{C}\,\big|\,|z|<1\big\}$ and $\Delta^*:=\Delta\setminus\{0\}$. The point $0\in\Delta$ is often denoted by O. When we distinguish a variable such as z, we use the symbol Δ_z and Δ_z^*. Let $M(\Delta,O)$ denote the space of meromorphic functions on Δ, whose poles are contained in $\{O\}$. Let $H(\Delta)$ be the space of holomorphic functions on Δ. We implicitly use the natural bijection $M(\Delta,O)/z^m\,H(\Delta)\simeq\mathbb{C}(\!(z)\!)/z^m\mathbb{C}[\![z]\!]$. Any $\mathfrak{a}\in M(\Delta,O)$ has the Laurent expansion $\mathfrak{a}=\sum\mathfrak{a}_j\,z^j$. For a non-zero $\mathfrak{a}$, let $\mathrm{ord}(\mathfrak{a})$ be the minimum j such that $\mathfrak{a}_j\neq 0$. We formally set $\mathrm{ord}(0):=0$. For any $\mathfrak{a}\in M(\Delta,O)/H(\Delta)$, we set $\mathrm{ord}(\mathfrak{a}):=\min\big\{\mathrm{ord}(\widetilde{\mathfrak{a}}),0\big\}$, where $\widetilde{\mathfrak{a}}$ is any lift of $\mathfrak{a}$ to $M(\Delta,O)$.

Let $\pi:\widetilde{\Delta}(O)\longrightarrow\Delta$ be the real blow up of Δ at O. The fiber $\pi^{-1}(O)$ is naturally identified with $S^1=\big\{e^{\sqrt{-1}\theta}\,\big|\,\theta\in\mathbb{R}/2\pi\mathbb{Z}\big\}$. In this paper, "sector" of Δ^* means a closed sector, i.e., a subset of the form $\big\{re^{\sqrt{-1}\theta}\,\big|\,0<r\leq r_0,\ \theta_0\leq\theta\leq\theta_1\big\}=:S[r_0;\theta_0,\theta_1]$ for some $r_0>0$ and $\theta_0<\theta_1$. The number $\theta_1-\theta_0$ is called the angle of S. For a sector S in Δ^*, its closure in $\widetilde{\Delta}(O)$ is denoted by $\overline{S}$. The intersection $\overline{S}\cap\pi^{-1}(O)$ is denoted by Z. The completion along Z is denoted by $\widehat{Z}$.

For a non-zero $\mathfrak{a}\in M(\Delta,O)/H(\Delta)$, we have a C^∞-function $F_{\mathfrak{a}}:=|z|^{-\mathrm{ord}(\mathfrak{a})}\,\mathrm{Re}(\widetilde{\mathfrak{a}})$ on $\widetilde{\Delta}(O)$, where $\widetilde{\mathfrak{a}}$ is a lift of $\mathfrak{a}$ to $M(\Delta,O)$. The set $\pi^{-1}(O)\cap F_{\mathfrak{a}}^{-1}(0)$ is independent of the choice of $\widetilde{\mathfrak{a}}$.

Let $\mathcal{I}$ be a finite subset of $M(\Delta,O)/H(\Delta)$, and let $S=S[r_0;\theta_0,\theta_1]$ be a sector in Δ^*. For each distinct pair $\mathfrak{a}_i\in\mathcal{I}$ $(i=1,2)$, we implicitly assume that $Z\cap(F_{\mathfrak{a}_1-\mathfrak{a}_2})^{-1}(0)$ is contained in the interior of Z, regarded as a subset of $\pi^{-1}(O)$. Recall that an order $\leq_S$ on $\mathcal{I}$ is associated to S, namely $\mathfrak{a}_1\leq_S\mathfrak{a}_2$ for $\mathfrak{a}_1,\mathfrak{a}_2\in\mathcal{I}$, if $F_{\mathfrak{a}_1-\mathfrak{a}_2}\geq 0$ on Z. It means that there exists $r_1>0$ such that $-\mathrm{Re}(\mathfrak{a}_1(Q))\leq-\mathrm{Re}(\mathfrak{a}_2(Q))$ for any $Q\in S[r_1;\theta_0,\theta_1]$. For two sectors $S'\subset S$, $\mathfrak{a}_1\leq_S\mathfrak{a}_2$ clearly implies $\mathfrak{a}_1\leq_{S'}\mathfrak{a}_2$, but the converse is not true in general.

Full Stokes filtration in the unramified case. Let (V,∇) be a meromorphic flat bundle on (Δ,O). It is called unramified, if the induced formal meromorphic flat bundle $(V,\nabla)_{|\widehat{O}}:=(V,\nabla)\otimes_{\mathcal{O}}\mathbb{C}(\!(z)\!)$ is unramified. The formal decomposition

(2.1) for $(V,\nabla)_{|\widehat{O}}$ induces the decomposition:

$$\pi^{-1}(V,\nabla)_{|\widehat{\pi^{-1}(O)}} = \bigoplus_{\mathfrak{a}\in \operatorname{Irr}(\nabla)} \pi^{-1}\big(\widehat{V}_{\mathfrak{a}}, \nabla_{\mathfrak{a}}\big)$$

Let P be any point of $\pi^{-1}(O)$. According to a classical asymptotic analysis (see [23], for example), there exists a sector S such that (i) P is contained in the interior of $\overline{S}$, (ii) we have a flat decomposition

$$\pi^{-1}(V,\nabla)_{|\overline{S}} = \bigoplus_{\mathfrak{a}\in \operatorname{Irr}(\nabla)} (V_{\mathfrak{a},S}, \nabla_{\mathfrak{a},S}) \tag{2.3}$$

satisfying $V_{\mathfrak{a},S|\widehat{Z}} = \pi^{-1}(\widehat{V}_{\mathfrak{a}})_{|\widehat{Z}}$. Although such a splitting (2.3) is not unique, it is easy to show that the flat subbundles

$$\widetilde{\mathcal{F}}^S_{\mathfrak{a}}(V_{|\overline{S}}) := \bigoplus_{\mathfrak{b}\leq_S \mathfrak{a}} V_{\mathfrak{b},S}$$

are independent of the choice of (2.3). Thus, we obtain a filtration $\widetilde{\mathcal{F}}^S$ of $V_{|\overline{S}}$ indexed by the ordered set $\big(\operatorname{Irr}(\nabla), \leq_S\big)$, which is called the full Stokes filtration of $V_{|\overline{S}}$. If $S' \subset S$, we have a compatibility $\widetilde{\mathcal{F}}^{S'}_{\mathfrak{a}} = \widetilde{\mathcal{F}}^{S'}_{<\mathfrak{a}} + \widetilde{\mathcal{F}}^S_{\mathfrak{a}}$, which induces an isomorphism $\operatorname{Gr}^{\widetilde{\mathcal{F}}^{S'}}_{\mathfrak{a}} \simeq \operatorname{Gr}^{\widetilde{\mathcal{F}}^S}_{\mathfrak{a}}$ on $\overline{S}'$.

By the restriction, we obtain filtrations $\widetilde{\mathcal{F}}^S(V_{|S})$ of $(V,\nabla)_{|S}$ for small sectors S. The flat subbundle $\widetilde{\mathcal{F}}^S_{\mathfrak{b}}(V_{|S})$ is characterized by the following condition:

(A): Let $v_1,\ldots,v_r$ be a meromorphic frame of V. Let f be a flat section of $V_{|S}$. We have the expression $f = \sum f_i\, v_{i|S}$. We set $\boldsymbol{f} := (f_i \,|\, i=1,\ldots,r)$. Then, f is contained in $\widetilde{\mathcal{F}}^S_{\mathfrak{b}}(V_{|S})$ if and only if the following holds for some $C>0$:

$$\big|\boldsymbol{f}\, \exp(\mathfrak{b})\big| = O\Big(|z|^{-C}\Big)$$

Moreover, f is contained in $\widetilde{\mathcal{F}}^S_{<\mathfrak{b}}(V_S)$ if and only if $\big|\boldsymbol{f}\, \exp(\mathfrak{b})\big| = O\big(\exp(-\epsilon|z|^{-\delta})\big)$ for some $\epsilon,\delta>0$.

The system of filtrations $\big\{\widetilde{\mathcal{F}}^S\big\}$ satisfies the above compatibility. Such a system of filtrations is called a Stokes data. It is known that the meromorphic flat bundle (V,∇) on (Δ,O) can be reconstructed from a flat bundle $(V,\nabla)_{|\Delta^*}$ with the Stokes data. (See [7], [15], [16] and [21]. See also Chapter 7 of [17].)

Remark 2.1. If the angle of S is smaller than $\pi/\operatorname{dec}(V)$, we have a splitting of $(V,\nabla)_{|S}$ as in (2.3). See [23], for example. □

Stokes filtration in the level m. Let m be a negative integer. For a given decomposition (2.3), we set

$$V^{(m)}_{\mathfrak{c},S} := \bigoplus_{\eta_m(\mathfrak{a})=\mathfrak{c}} V_{\mathfrak{a},S}, \quad \widetilde{\mathcal{F}}^{(m)\,S}_{\mathfrak{b}}(\pi^{-1}V_{|\overline{S}}) := \bigoplus_{\mathfrak{c}\leq_S \mathfrak{b}} V^{(m)}_{\mathfrak{c},S}.$$

Then, we obtain a filtration $\mathcal{F}^{(m)\,S}$ of $V_{|\overline{S}}$ by flat subbundles, which is independent of the choice of a splitting (2.3). The flat subbundle $\mathcal{F}^{(m)\,S}_{\mathfrak{a}}(V_{|\overline{S}})$ is characterized by $\mathcal{F}^{(m)\,S}_{\mathfrak{a}}(V_{|\overline{S}})_{|\widehat{Z}} = \bigoplus_{\mathfrak{b}\leq_S\mathfrak{a}} \pi^{-1}(\widehat{V}^{(m)}_{\mathfrak{b}})$. Since the order $\leq_S$ depends on S, we do not have a global filtration. However, the system of filtrations $\{\mathcal{F}^{(m)\,S}\}$ satisfies the above compatibility, and we obtain the associated graded bundle, denoted by $\mathrm{Gr}^{(m)}(V_{|\Delta^*})$ on Δ^*. Since it is equipped with an induced Stokes data, it is naturally prolonged to a meromorphic flat bundle, denoted by $\mathrm{Gr}^{(m)}(V)$. In the case $m=-1$, $\mathrm{Gr}^{(-1)}(V)$ is denoted by $\mathrm{Gr}(V)$.

Pull back via a ramified covering. Let $\varphi : (\Delta', O') \longrightarrow (\Delta, O)$ be a ramified covering of order p. The pull back induces the bijection $\mathrm{Irr}(\nabla) \simeq \mathrm{Irr}(\varphi^*\nabla)$. For a small sector $S \subset \Delta^*$, we have the isomorphism of the Stokes filtrations $\mathcal{F}^{(pm)\,\varphi^{-1}(S)} \simeq \varphi^*\mathcal{F}^{(m)\,S}$. We also have a natural isomorphism

$$\varphi^* \mathrm{Gr}^{(m)}(V,\nabla) \simeq \mathrm{Gr}^{(pm)}\varphi^*(V,\nabla).$$

Ramified case. Let (V,∇) be a meromorphic flat bundle on (Δ, O), which is not necessarily unramified. Let $\varphi : (\Delta', O') \longrightarrow (\Delta, O)$ be a ramified covering such that $\varphi^*(V,\nabla)$ is unramified. Taking Gr with respect to the full Stokes filtration, we obtain the graded bundle $\mathrm{Gr}(\varphi^*V) = \bigoplus_{\mathfrak{a}\in\mathrm{Irr}(\varphi^*\nabla)} \mathrm{Gr}_{\mathfrak{a}}(\varphi^*V)$. Let $\overline{\mathrm{Irr}}(\nabla)$ denote the quotient of $\mathrm{Irr}(\varphi^*\nabla)$ by the action of the Galois group $\mathrm{Gal}(\varphi)$. Let $q : \mathrm{Irr}(\varphi^*\nabla) \longrightarrow \overline{\mathrm{Irr}}(\nabla)$ be the projection. For each $\mathfrak{c} \in \overline{\mathrm{Irr}}(\nabla)$, the bundle $\bigoplus_{\mathfrak{a}\in q^{-1}(\mathfrak{c})} \mathrm{Gr}_{\mathfrak{a}}\big(\varphi^*V\big)$ is $\mathrm{Gal}(\varphi)$-equivariant. The descent is denoted by $\mathrm{Gr}_{\mathfrak{c}}(V)$. We have the natural isomorphism $\mathrm{Gr}_{\mathfrak{c}}(V)_{|\widehat{O}} \simeq \mathrm{Gr}_{\mathfrak{c}}(V_{|\widehat{O}})$, where the right hand is as in (2.2). We set $\mathrm{Gr}(V) := \bigoplus_{\mathfrak{c}\in\overline{\mathrm{Irr}}(\nabla)} \mathrm{Gr}_{\mathfrak{c}}(V)$. If $(V,\nabla) \simeq \mathrm{Gr}(V,\nabla)$, we say that (V,∇) is elementary, by following [18].

Remark 2.2. We can regard $\mathrm{Gr}_{\mathfrak{c}}(V)$ as a meromorphic flat bundle on $(\mathbb{P}^1, \{0,\infty\})$ in a natural way, which is regular singular at ∞. □

2.1.3. *A condition.* Let G be the Galois group of the extension of fields $\mathbb{C}(\!(u)\!)/\mathbb{C}(\!(t)\!)$, where $t=u^p$. Let M be a G-equivariant free $\mathbb{C}[\![u]\!]$-module with a meromorphic connection ∇ such that

- M is an unramifiedly good lattice of $(M(*u),\nabla)$, i.e., it has the Hukuhara-Levelt-Turrittin decomposition $(M,\nabla) = \bigoplus(M_{\mathfrak{a}},\nabla_{\mathfrak{a}})$ compatible with the decomposition (2.1) for $(M(*u),\nabla)$, and $\nabla_{\mathfrak{a}} - d\mathfrak{a}$ are logarithmic with respect to $M_{\mathfrak{a}}$.
- There exist numbers $\mathfrak{r}(\mathfrak{a}) \in \mathbb{R}$ such that the eigenvalues α of $\mathrm{Res}(\nabla_{\mathfrak{a}})$ satisfy $\mathfrak{r}(\mathfrak{a}) \leq \mathrm{Re}(\alpha) < \mathfrak{r}(\mathfrak{a})+1$.

Let M' be the $k[\![t]\!]$-module obtained as the descent of M, i.e., M' is the G-invariant part of M, which is equipped with an induced connection ∇.

Lemma 2.3. *Let φ' be an endomorphism of (M',∇') such that $\varphi'_{|t=0}$ is the identity. Then, φ' is the identity.*

Proof. We have the induced endomorphism φ of (M,∇). It is easy to observe that $\varphi_{|u=0}$ is the identity. By a standard argument, we obtain φ is the identity, and hence φ' is also the identity. □

2.2. **Fourier transform and local Fourier transform.** We recall basic facts on Fourier transform and local Fourier transform. See [1], [4], [8], [10], [16], [19] and the references therein for more details.

2.2.1. *Fourier transform.* Let M be a $\mathbb{C}[t]\langle\partial_t\rangle$-module. Recall that we have a $\mathbb{C}[\tau]\langle\partial_\tau\rangle$-module $\mathfrak{Four}(M)$, called the Fourier transform of M. We set $\mathfrak{Four}(M) := M$ as a $\mathbb{C}$-vector space, and the actions of τ and ∂_τ are given by $\tau\, m = -\partial_t m$ and $\partial_\tau m = t\, m$, respectively. It is known to be isomorphic to the cokernel of $M\otimes\mathbb{C}[\tau] \xrightarrow{\partial_t+\tau} M\otimes\mathbb{C}[\tau]$, where the $\mathbb{C}[\tau]\langle\partial_\tau\rangle$-module structure on $M\otimes\mathbb{C}[\tau]$ is given by $\partial_\tau(m\otimes\tau^\ell) = t\,m\otimes\tau^\ell + \ell\, m\otimes\tau^{\ell-1}$ and $\tau(m\otimes\tau^\ell) = m\otimes\tau^{\ell+1}$. The Fourier transform has the sheaf theoretic version. Namely, for a D-module $\mathcal{M}$ on $\mathbb{P}^1_t$, its Fourier transform $\mathfrak{Four}(\mathcal{M})$ on $\mathbb{P}^1_\tau$ is defined by the sheaf-theoretic version of the above procedure, which we briefly describe. (See [16] for example.) Let π_i be the projection of $\mathbb{P}^1_t\times\mathbb{P}^1_\tau$ onto the i-th component. We put $\mathcal{D} := \big(\{\infty\}\times\mathbb{P}^1_\tau\big)\cup\big(\mathbb{P}^1_t\times\{\infty\}\big)$. Let $L(t\tau)$ be the meromorphic flat bundle $\mathcal{O}_{\mathbb{P}^1_t\times\mathbb{P}^1_\tau}(*\mathcal{D})\,e$ with the flat connection $\nabla e = e\,d(t\tau)$. Then, $\mathfrak{Four}(\mathcal{M})$ is given as follows:

$$\mathfrak{Four}(\mathcal{M}) := \pi_{2\dagger}\big(\pi_1^*\mathcal{M}\otimes L(t\tau)\big) = R\pi_{2*}\Big(\mathrm{DR}_{\mathbb{P}^1_t\times\mathbb{P}^1_\tau/\mathbb{P}^1_\tau}\big(\pi_1^*\mathcal{M}\otimes L(t\tau)\big)[1]\Big)$$

Here, $\pi_{2\dagger}$ denotes the push-forward of D-modules via π_2, $\mathrm{DR}_{\mathbb{P}^1_t\times\mathbb{P}^1_\tau/\mathbb{P}^1_\tau}\big(\pi_1^*\mathcal{M}\otimes L(t\tau)\big)$ denotes the relative de Rham complex of $\pi_1^*\mathcal{M}\otimes L(t\tau)$ over $\mathbb{P}^1_\tau$, and [1] denotes the shift of degree.

2.2.2. *Local Fourier transform.* The local Fourier transforms $\mathcal{F}^{(0,\infty)}$ and $\mathcal{F}^{(\infty,\infty)}$ for D-modules were introduced in [4], which are functors from $\mathbb{C}(\!(T)\!)$-connections to $\mathbb{C}(\!(Z)\!)$-connections, where T and Z are just formal variables. (See also [1] and [10].)

Let (V,∇) be a meromorphic flat bundle on $\mathbb{P}^1_t$. Let $\mathrm{Sing}(V,\nabla)$ be the set of poles of (V,∇). For each $c\in\mathrm{Sing}(V,\nabla)\setminus\{\infty\}$, we regard $(V,\nabla)_{|\widehat{t=c}}$ as a $\mathbb{C}(\!(T)\!)$-connection by the coordinate change $T = t-c$. If $\infty\in\mathrm{Sing}(V,\nabla)$, we regard $(V,\nabla)_{|\widehat{t=\infty}}$ as a $\mathbb{C}(\!(T)\!)$-connection by the coordinate change $T = t^{-1}$. We set $Z = \tau^{-1}$. Then, we are given the following isomorphism:

$$\mathfrak{Four}(V)_{|\widehat{\{\tau=\infty\}}} \simeq \mathcal{F}^{(\infty,\infty)}\Big((V,\nabla)_{|\widehat{\{t=\infty\}}}\Big)\oplus\bigoplus_{c\in\mathrm{Sing}(V,\nabla)\setminus\{\infty\}}\mathcal{F}^{(0,\infty)}\Big((V,\nabla)_{|\widehat{\{t=c\}}}\Big)\otimes L(c/Z)$$

Remark 2.4. Local Fourier transform $\mathcal{F}^{(\infty,0)}$ is also introduced in [4]. □

2.2.3. *Explicit stationary phase formula.* Fu [9] proved an explicit formula to describe the local Fourier transform of ℓ-adic sheaves up to isomorphisms, which is called the explicit stationary phase formula. (See also the influential works by Laumon [12] and Malgrange [16].) Fang [8] and Sabbah [19] computed the explicit stationary phase formula for meromorphic flat bundles. Here, we follow Sabbah's description.

Any $\rho \in U\,\mathbb{C}[\![U]\!]$ gives a ramified covering $\operatorname{Spec}\mathbb{C}(\!(U)\!) \longrightarrow \operatorname{Spec}\mathbb{C}(\!(T)\!)$ by $T = \rho(U)$. For any $\mathfrak{a} \in \mathbb{C}(\!(U)\!)$, let $L(\mathfrak{a})$ be a meromorphic flat line bundle on $\operatorname{Spec}\mathbb{C}(\!(U)\!)$ given by $L(\mathfrak{a}) = \mathbb{C}(\!(U)\!)\, e$ with $\nabla e = e\, d\mathfrak{a}$. Let R be a $\mathbb{C}(\!(U)\!)$-regular connection. We obtain a $\mathbb{C}(\!(T)\!)$-connection $\rho_*\bigl(L(\mathfrak{a}) \otimes R\bigr)$. Any $\mathbb{C}(\!(T)\!)$-connection can be obtained as the direct sum of such connections. (See [19] for more details on ambiguity in the classification.)

The case of $\mathcal{F}^{(0,\infty)}$. If $\mathfrak{a} \not\equiv 0$ in $\mathbb{C}(\!(U)\!)/\mathbb{C}[\![U]\!]$, we set

$$\widehat{\rho}^{(0)}(U) := -\frac{\rho'(U)}{\mathfrak{a}'(U)}, \quad \widehat{\mathfrak{a}}^{(0)}(U) := \mathfrak{a}(U) - \frac{\rho(U)}{\rho'(U)}\mathfrak{a}'(U),$$

$$\widehat{R} = R \otimes L_n, \quad L_n = \Bigl(\mathbb{C}(\!(U)\!),\, d - \frac{n}{2}\frac{dU}{U}\Bigr).$$

Here, $n = -\operatorname{ord}_U(\mathfrak{a})$. Then, according to [8] and [19], the local Fourier transform $\mathcal{F}^{(0,\infty)}\Bigl(\rho_*\bigl(L(\mathfrak{a}) \otimes R\bigr)\Bigr)$ is isomorphic to $\widehat{\rho}^{(0)}_*\bigl(L(\widehat{\mathfrak{a}}^{(0)}) \otimes \widehat{R}\bigr)$, where $\widehat{\rho}^{(0)}_*$ is the push-forward $\operatorname{Spec}\mathbb{C}(\!(U)\!) \longrightarrow \operatorname{Spec}\mathbb{C}(\!(Z)\!)$ given by $Z = \widehat{\rho}^{(0)}(U)$. In the case $\mathfrak{a} = 0$, it is easy to see $\mathcal{F}^{(0,\infty)}R \simeq R$ for a regular singular $\mathbb{C}(\!(T)\!)$-connection R, as remarked in [8] and [19].

The case of $\mathcal{F}^{(\infty,\infty)}$. Let ρ, $\mathfrak{a}$ and R be as above. We set $p := \operatorname{ord}_U(\rho)$ and $n := -\operatorname{ord}_U(\mathfrak{a})$. Assume $n > p$. We set

$$\widehat{\rho}^{(\infty)}(U) := -\frac{\rho'(U)}{\mathfrak{a}(U)\,\rho(U)^2}, \quad \widehat{\mathfrak{a}}^{(\infty)}(U) := \mathfrak{a}(U) + \frac{\rho(U)}{\rho'(U)}\mathfrak{a}'(U), \quad \widehat{R} = R \otimes L_n.$$

Then, $\mathcal{F}^{(\infty,\infty)}\bigl(\rho_*(L(\mathfrak{a}) \otimes R)\bigr) \simeq \widehat{\rho}^{(\infty)}_*\bigl(L(\widehat{\mathfrak{a}}^{(\infty)}) \otimes \widehat{R}\bigr)$. In the case $n \leq p$, we have $\mathcal{F}^{(\infty,\infty)}\bigl(\rho_*(L(\mathfrak{a}) \otimes R)\bigr) = 0$.

Remark 2.5. In [8] and [19], an explicit formula is given also for $\mathcal{F}^{(\infty,0)}$. □

2.2.4. *Good lattice pair.* We recall the notion of good lattice pair for a meromorphic flat bundle on a complex curve. (See [6] and [4].) Let X be a smooth complex curve. Let D be a finite subset of X. Let (V, ∇) be a meromorphic flat bundle on (X, D), i.e., V is a locally free $\mathcal{O}_X(*D)$-module of finite rank with a connection ∇. Let $\mathcal{V}$ and $\mathcal{W}$ be lattices of V such that (i) $\mathcal{V} \subset \mathcal{W} \subset V$, (ii) $\nabla(\mathcal{V}) \subset \omega_X(D) \otimes \mathcal{W}$, (iii) the following morphism is a quasi-isomorphism:

$$\Bigl(\mathcal{V} \xrightarrow{\nabla} \omega_X(D) \otimes \mathcal{W}\Bigr) \longrightarrow \Bigl(V \xrightarrow{\nabla} \omega_X(D) \otimes V\Bigr)$$

Such a pair $(\mathcal{V}, \mathcal{W})$ is called a good lattice pair. A similar notion for a meromorphic flat bundle on $\mathbb{C}(\!(t)\!)$ is also defined. We know a rather canonical construction of such a pair given in Lemma 3.3 of [4], which we describe in the case $X = \Delta$ and $D = \{O\}$ for simplicity:

- Let (V,∇) be an unramified meromorphic flat bundle on (Δ,O). Let E be the Deligne-Malgrange lattice of V, i.e., we have the flat decomposition $(E,\nabla)_{|\widehat{O}}=\bigoplus(\widehat{E}_{\mathfrak{a}},\nabla_{\mathfrak{a}})$ such that (i) $\nabla_{\mathfrak{a}}-d\mathfrak{a}$ are logarithmic with respect to $\widehat{E}_{\mathfrak{a}}$, (ii) any eigenvalues α of $\mathrm{Res}(\nabla_{\mathfrak{a}})$ satisfy $0\leq\mathrm{Re}(\alpha)<1$. We set $\mathcal{V}:=E$. We put $\widehat{\mathcal{W}}:=\bigoplus z^{\mathrm{ord}\,\mathfrak{a}}\widehat{E}_{\mathfrak{a}}$, and let $\mathcal{W}$ be the corresponding lattice of V. Then, $(\mathcal{V},\mathcal{W})$ is a good lattice pair.
- Let (V,∇) be a meromorphic flat bundle on (Δ,O), which is not necessarily unramified. Take a ramified covering $\varphi:(\Delta',O')\longrightarrow(\Delta,O)$ such that $\varphi^*(V,\nabla)$ is unramified. Let $(\mathcal{V}',\mathcal{W}')$ be the good lattice pair for $\varphi^*(V,\nabla)$ as above, which is $\mathrm{Gal}(\varphi)$-equivariant. The descent $(\mathcal{V},\mathcal{W})$ is a good lattice pair for (V,∇).

The above is called the BDE-good lattice pair for (V,∇) in this paper. Note that $\big(z^{-n}\mathcal{V},\,z^{-n}\mathcal{W}\big)$ is also a good lattice pair for each $n\geq 0$.

2.2.5. *The induced lattice of the local Fourier transform.* Recall that the local Fourier transform can be described in terms of any good lattice pair [4]. Moreover, any good lattice pair induces a lattice of the local Fourier transform.

The case of $\mathcal{F}^{(0,\infty)}$. Let $(\widehat{V},\widehat{\nabla})$ be a $\mathbb{C}(\!(T)\!)$-connection. Let $(\mathcal{V},\mathcal{W})$ be a good lattice pair for $(\widehat{V},\widehat{\nabla})$. Then, according to [4], the local Fourier transform $\mathcal{F}^{(0,\infty)}(\widehat{V})$ is naturally isomorphic to

$$\mathrm{Cok}\Big(Z\widehat{\nabla}+dT:\mathcal{V}(\!(Z)\!)\longrightarrow\mathcal{W}(\!(Z)\!)\,dT/T\Big).$$

We will often identify dT with 1. The action of ∂_Z on $\mathcal{F}^{(0,\infty)}(\widehat{V})$ is induced by $\partial_Z(v\,Z^{\ell})=-Tv\,Z^{\ell-2}+\ell v\,Z^{\ell-1}$. It is naturally equipped with the lattice

$$\mathcal{F}^{(0,\infty)}(\widehat{V};\mathcal{V},\mathcal{W}):=\mathrm{Cok}\Big(\mathcal{V}[\![Z]\!]\stackrel{Z\widehat{\nabla}+dT}{\longrightarrow}\mathcal{W}[\![Z]\!]\,dT/T\Big)\subset\mathcal{F}^{(0,\infty)}(\widehat{V}),\tag{2.4}$$

which is preserved by the action of $Z^2\partial_Z$. We have the naturally defined map $T^{-1}\mathcal{W}\longrightarrow\mathcal{F}^{(0,\infty)}(\widehat{V};\mathcal{V},\mathcal{W})$. For any $v\in\widehat{V}$, we can choose a good lattice pair $(\mathcal{V},\mathcal{W})$ such that v is contained in $T^{-1}\mathcal{W}$. Hence, we have an induced map $\iota:\widehat{V}\longrightarrow\mathcal{F}^{(0,\infty)}(\widehat{V})$. If $\{v\in\widehat{V}\,|\,\widehat{\nabla}v=0\}=0$, then ι is an isomorphism of $\mathbb{C}$-vector spaces. We have the relations $\iota\circ\partial_T=-Z^{-1}\circ\iota$ and $\iota\circ T=-Z^2\partial_Z\circ\iota$. For a given N, if L is a sufficiently large number, we have $\iota\big(T^L\mathcal{W}\big)\subset Z^N\,\mathcal{F}^{(0,\infty)}\big(\widehat{V};\mathcal{V},\mathcal{W}\big)$, which is obvious from the definition (2.4). See [4] for more details.

The case of $\mathcal{F}^{(\infty,\infty)}$. Let $(\widehat{V},\widehat{\nabla})$ be a $\mathbb{C}(\!(T)\!)$-connection such that each direct summand has slope >1. Let $(\mathcal{V},\mathcal{W})$ be a good lattice pair for $(\widehat{V},\widehat{\nabla})$. According to [4], the local Fourier transform $\mathcal{F}^{(\infty,\infty)}(\widehat{V})$ is naturally isomorphic to

$$\mathrm{Cok}\Big(Z\widehat{\nabla}-T^{-2}dT:\mathcal{V}(\!(Z)\!)\longrightarrow\mathcal{W}(\!(Z)\!)\,dT/T\Big).$$

The action of ∂_Z is induced by $\partial_Z(v\,Z^\ell) = -T^{-1}v\,Z^{\ell-2} + \ell\,v\,Z^{\ell-1}$. It is naturally equipped with the lattice

$$\mathcal{F}^{(\infty,\infty)}(\widehat{V};\mathcal{V},\mathcal{W}) := \mathrm{Cok}\Big(Z\widehat{\nabla} - T^{-2}dT : \mathcal{V}[\![Z]\!] \longrightarrow \mathcal{W}[\![Z]\!]\,dT/T\Big) \subset \mathcal{F}^{(\infty,\infty)}(\widehat{V}).$$

We have the naturally defined map $T\,\mathcal{W} \longrightarrow \mathcal{W}\,dT/T$ given by $w \longmapsto w(-dT/T^2)$. We also have $\mathcal{W}\,dT/T \longrightarrow \mathcal{F}^{(\infty,\infty)}(V;\mathcal{V},\mathcal{W})$. As in the case of $\mathcal{F}^{(0,\infty)}$, we obtain an induced map $\iota : \widehat{V} \longrightarrow \mathcal{F}^{(\infty,\infty)}(\widehat{V})$, which is shown to be an isomorphism of $\mathbb{C}$-vector spaces. We have the relations $\iota\circ(T^2\partial_T) = Z^{-1}\circ\iota$ and $\iota\circ T^{-1} = -Z^2\partial_Z\circ\iota$. For a given N, if L is a sufficiently large number, we have $\iota\big(T^L\mathcal{W}\big) \subset Z^N\,\mathcal{F}^{(\infty,\infty)}(\widehat{V};\mathcal{V},\mathcal{W})$, which is obvious from the definition. See [4] for more details.

2.2.6. *A property of the induced lattice.* Let $(\widehat{V},\widehat{\nabla})$ be a $\mathbb{C}(\!(T)\!)$-connection. Let $(\widetilde{\mathcal{V}},\widetilde{\mathcal{W}})$ be the BDE-good lattice pair for $(\widehat{V},\widehat{\nabla})$. For any non-negative integer N, we set $\widetilde{\mathcal{V}}^{(N)} := t^{-N}\mathcal{V}_0$ and $\widetilde{\mathcal{W}}^{(N)} := t^{-N}\mathcal{W}_0$. Then, $\big(\widetilde{\mathcal{V}}^{(N)},\widetilde{\mathcal{W}}^{(N)}\big)$ is also a good lattice pair.

Lemma 2.6. *$\mathcal{F}^{(0,\infty)}(\widehat{V};\widetilde{\mathcal{V}}^{(N)},\widetilde{\mathcal{W}}^{(N)})$ and $\mathcal{F}^{(\infty,\infty)}(\widehat{V};\widetilde{\mathcal{V}}^{(N)},\widetilde{\mathcal{W}}^{(N)})$ are the descent of lattices as in Subsection* 2.1.3.

Proof. This is essentially contained in [8]. We give only an outline. See [8] for details of the computation in the following argument, although there are small minor changes for our convenience. It is easy to observe that we have only to consider the case $\widehat{V} \simeq \rho_*\big(L(\mathfrak{a})\otimes R\big)$, where $\rho(u) = u^p$, $\mathfrak{a} \in u^{-1}\mathbb{C}[u^{-1}]$, and R is a regular singular connection of rank one. So we set $\mathcal{L} := \mathbb{C}(\!(u)\!)\,e$ equipped with a connection

$$u\partial_u e = e\,p\left(-a_0u^{-n} - a_1u^{-n+1} - \cdots - a_{n-1}u^{-1} - a_n\right), \qquad a_0 \neq 0.$$

We may assume $M \leq -\mathrm{Re}\,a_n < M + 1/p$ for some M. We may also assume the irreducibility, i.e., $a_j \neq 0$ for some j such that $j \not\equiv 0$ modulo p. Let $\mathcal{V} := \mathbb{C}[\![u]\!]\,e$ and $\mathcal{W} := \mathbb{C}[\![u]\!]\,u^{-n}\,e$. We have only to consider the lattice associated to $(\mathcal{V},\mathcal{W})$.

The case of $\mathcal{F}^{(0,\infty)}$. If $\mathfrak{a} = 0$, the claim is clear from the relation $(Z\partial_Z)\circ\iota = \iota\circ(\partial_T T)$. Hence, we have only to consider the case $\mathfrak{a} \neq 0$. For $i = 1,\ldots,p+n$, we set $v_i := \iota\big(u^{-i}e\big)$, which give a frame of $\mathcal{F}^{(0,\infty)}\big(\rho_*\mathcal{L};\rho_*\mathcal{V},\rho_*\mathcal{W}\big)$. Let us consider the extension of fields $\mathbb{C}(\!(\zeta)\!)/\mathbb{C}(\!(Z)\!)$, where $\zeta^{n+p} = Z$. We set $w_i := \zeta^i\,Z^{-1}\,v_i$ for $i = 1,\ldots,p+n$. We have the following:

$$Z\partial_Z w_i = \left(\frac{i}{p+n} - \frac{i}{p}\right)w_i - \zeta^{-n}\sum_{0\leq j\leq n} a_j\,\zeta^j\,w_{i+n-j}, \quad (1 \leq i \leq p)$$

$$Z\partial_Z w_i = \left(\frac{i}{p+n} - 1\right)w_i - \zeta^{-n}\,w_{i-p}, \quad (p+1 \leq i \leq p+n)$$

Let E be the lattice of $\mathbb{C}(\!(\zeta)\!)\otimes\mathcal{F}^{(0,\infty)}(\rho_*\mathcal{L})$ generated by $w_1,\ldots,w_{p+n}$. Note that $\mathcal{F}^{(0,\infty)}(\rho_*\mathcal{L};\rho_*\mathcal{V},\rho_*\mathcal{W})$ is the descent of E. Let us show that E is as in Subsection 2.1.3.

Let us introduce some $(n+p)$-square matrices. We indicate only the (possibly) non-zero entries. Let J be given by $J_{i,j}=1$ in the case $j=i{+}1$. Let Θ be given by $\Theta_{p+i,1}=a_{n-i}\zeta^{n-i}$ for $i=0,\ldots,n$. Let Θ_0 be given by $(\Theta_0)_{p+n,1}=a_0$. Let C_0 be given by $(C_0)_{i,i}=-in/p$ for $i=1,\ldots,p$. Let C_1 be given by $(C_1)_{i,i}=i-(p+n)$ for $i=1,\ldots,p+n$. We set $\Gamma=J+\Theta$ and $\Gamma_0=J+\Theta_0$. We have the expansion $\Gamma=\Gamma_0+\sum_{j=1}^{n}\Gamma_j\,\zeta^j$, where Γ_j are constant matrices.

Then, the action of $\zeta\partial_\zeta$ is expressed by the matrix $\mathcal{A}=-(p+n)\,\zeta^{-n}\Gamma^n+(p+n)\,C_0+C_1$ with respect to the frame $\boldsymbol{w}=(w_1,\ldots,w_{p+n})$. Since Γ_0 has distinct eigenvalues, there exists a matrix $G\in\mathrm{GL}_{n+p}(\mathbb{C}[\![\zeta]\!])$ such that $G^{-1}\Gamma^nG$ is diagonal. Then, the matrix $G^{-1}\mathcal{A}G+G^{-1}\zeta\partial_\zeta G$ expresses the action of $\zeta\partial_\zeta$ with respect to $\boldsymbol{w}\,G$. Note that $G^{-1}\mathcal{A}G+G^{-1}\zeta\partial_\zeta G$ and $G^{-1}\Gamma^nG$ have the same polar parts. Then, by using a well-established argument in [13], we can show that E is an unramifiedly good lattice.

We obtain the irregular decomposition $E_{\widehat{|\zeta=0}}=\bigoplus_{\mathfrak{b}\in\mathrm{Irr}(\widehat{\nabla})}\widehat{E}_{\mathfrak{b}}$. Since we have assumed the irreducibility, the action of Gal on $\mathrm{Irr}(\widehat{\nabla})$ is transitive, and hence the eigenvalues of the residues are independent of $\mathfrak{b}$, which we denote by α. By the above construction, we can observe that α depends on the coefficients a_i continuously. Moreover, according to the explicit stationary phase formula recalled in Subsection 2.2.3, α is determined by a_n up to ambiguity of integers. Hence, we can take $\mathfrak{r}(\mathfrak{b})$ such that $\mathfrak{r}(\mathfrak{b})\leq\mathrm{Re}(\alpha)<\mathfrak{r}(\mathfrak{b})+1$ for any choice of a_i as above with $M\leq\mathrm{Re}(a_n)<M+1/p$. We can check it also by a direct calculation of the trace of the connection form. Thus, we obtain the claim of Lemma 2.6 for $\mathcal{F}^{(0,\infty)}$.

The case of $\mathcal{F}^{(\infty,\infty)}$. For any integer m, we put

$$\mathcal{V}_m:=u^{-m}\mathbb{C}[\![u]\!]\,e,\qquad \mathcal{W}_m:=u^{-m-n}\mathbb{C}[\![u]\!]\,e.$$

We set $v_i:=\iota\big(u^{-i}\,e\big)$. Then, v_i $(i=m+1,\ldots,m+n-p)$ give a frame of the induced lattice $\mathcal{F}^{(\infty,\infty)}\big(\widehat{V};\mathcal{V}_m,\mathcal{W}_m\big)$. Note that v_m is a section of $\mathcal{F}^{(\infty,\infty)}(\widehat{V};\mathcal{V}_m,\mathcal{W}_m)$, and we have $v_{m|Z=0}=0$ in $\mathcal{F}^{(\infty,\infty)}(\widehat{V};\mathcal{V}_m,\mathcal{W}_m)_{|Z=0}$. Hence, we have the expression:

$$v_m=\sum_{j=m+1}^{m+n-p}Z\,a_j(Z)\,v_j,\quad a_j(Z)\in\mathbb{C}[\![Z]\!]\tag{2.5}$$

Let us consider the extension $\mathbb{C}(\!(\zeta)\!)/\mathbb{C}(\!(Z)\!)$ given by $\zeta^{n-p}=Z$. We set $w_i:=\zeta^i\,Z^{-1}\,v_i$. We obtain the lattice $\mathfrak{V}_m$ generated by $w_{m+1},\ldots,w_{m+n-p}$ in $\mathcal{F}^{(\infty,\infty)}(\widehat{V})\otimes\mathbb{C}(\!(\zeta)\!)$. From (2.5), we obtain that $w_m\in\mathfrak{V}_m$, and hence $\mathfrak{V}_{m-1}\subset\mathfrak{V}_m$. In particular, we have $w_j\in\mathfrak{V}_m$ for any $j\leq m+n-p$, which we will implicitly use in the following argument.

Because $T^2\partial_T\big(u^{-i}e\big)=-ip^{-1}u^{-i+p}\,e-\big(a_0u^{-i-n+p}\,e+a_1u^{-i-n+p+1}\,e+\cdots+a_nu^{-i+p}\,e\big)$, we have the following relation:

$$w_i=-\frac{1}{a_0}w_{i-n+p}-\frac{(i-n+p)\,\zeta^n}{a_0\,p}w_{i-n}-\sum_{j=1}^{n}\frac{a_j\,\zeta^j}{a_0}w_{i-j}$$

Hence, we have the relation between tuples of vectors

$$(w_{i+1},\ldots,w_{i+n})=(w_i,\ldots,w_{i+n-1})\,A^{(i)},$$

where $A^{(i)}$ are the n-square matrices such that $A^{(i)}_{k,l}$ are as follows:

$$A^{(i)}_{k,k-1}=1\quad(k=2,\ldots,n)\qquad A^{(i)}_{k,n}=-\frac{a_{n-k+1}}{a_0}\zeta^{n-k+1}\quad(k\neq 1,p+1)$$

$$A^{(i)}_{1,n}=-\frac{a_n\zeta^n}{a_0}-\frac{(i+p)\zeta^n}{a_0}\qquad A^{(i)}_{p+1,n}=-\frac{a_{n-p}\zeta^{n-p}}{a_0}-\frac{1}{a_0}\qquad A^{(i)}_{k,l}=0\ \ \text{(otherwise)}$$

Because $Z\partial_Z v_i=-Z^{-1}v_{i+p}$, we have $Z\partial_Z w_i=-\zeta^{-n}\,w_{i+p}+\big(-1+i/(n-p)\big)\,w_i$. Hence, we obtain the following relation

$$Z\partial_Z(w_{-p+1},\ldots,w_{n-p})=(w_{-p+1},\ldots,w_{n-p})\Big(\Gamma-\zeta^{-n}A^{(-p+1)}\cdots A^{(-1)}\,A^{(0)}\Big),$$

where Γ is the diagonal matrix whose (j,j)-th entries are $(j-n)/(n-p)$.

Let C be the n-square matrix such that $C_{k,l}=A^{(i)}_{k,l}$ for $(k,l)\neq(1,n)$ and $C_{1,n}=0$. Then, the polar part of $\Gamma-\zeta^{-n}A^{(-p+1)}\cdots A^{(0)}$ is equal to the polar part of $-\zeta^{-n}C^p$. There exists $G\in\mathrm{GL}_n(\mathbb{C}[\![\zeta]\!])$ such that the following holds:

- $G^{-1}CG$ is the direct sum of p-square matrix U_{11} and $(n-p)$-square matrix U_{22}, where U_{22} is diagonal.
- For the expansion $G=\sum_{j=0}^{\infty}G_j\,\zeta^j$, the 0-th term G_0 is of the form

$$G_0=\begin{pmatrix}G_{0;11}&0\\G_{0;21}&G_{0;22}\end{pmatrix},$$

  where $G_{0;11}$ is a p-square matrix.

We set $(\widetilde{w}_{-p+1},\ldots,\widetilde{w}_{n-p}):=(w_{-p+1},\ldots,w_{n-p})\,G$. We can observe that $\widetilde{w}_1,\ldots,\widetilde{w}_{n-p}$ give a frame of $\mathcal{F}^{(\infty,\infty)}(V;\mathcal{V}_0,\mathcal{W}_0)$. We also have

$$Z\partial_Z(\widetilde{w}_1,\ldots,\widetilde{w}_{n-p})=(\widetilde{w}_1,\ldots,\widetilde{w}_{n-p})\,\big(\mathcal{A}+U_{22}\big),$$

where the entries of $\mathcal{A}$ are contained in $\mathbb{C}[\![\zeta]\!]$. Then, we can conclude the claim of Lemma 2.6 for $\mathcal{F}^{(\infty,\infty)}$ by using the argument in the case of $\mathcal{F}^{(0,\infty)}$. □

2.3. **Asymptotic behaviour of some integrals.** We prepare a lemma to study the asymptotic behaviour of a family of the pairings between 1-forms and paths, following [3]. Let Δ_z be a disc $\big\{z\,\big|\,|z|<1\big\}$. Let U be an open subset of $\mathbb{C}$. Let $G(z,u)$ be a holomorphic function on $\Delta_z\times U$. The restriction to $\{z\}\times U$ is denoted by G_z. We set $\mathcal{Z}:=\big\{(z,u)\,\big|\,\partial_uG(z,u)=0\big\}$. We assume that (i) $\mathcal{Z}$ is smooth, (ii) the naturally induced morphism $\mathcal{Z}\longrightarrow\Delta_z$ is an isomorphism.

We obtain the section $\nu : \Delta_z \simeq \mathcal{Z} \subset \Delta_z \times U$. We also assume that $\partial_u^2 G(z,u)$ is nowhere vanishing on $\mathcal{Z}$.

Let $\gamma : \Delta_z \times [-\epsilon, \epsilon] \longrightarrow \Delta_z \times U$ be a C^∞-map over Δ_z such that $\gamma(z,0) = \big(z, \nu(z)\big)$. Let γ_z denote the restriction to $\{z\} \times [-\epsilon,\epsilon]$. Assume the following on γ_0:

- The image of γ_0 is contained in $\big\{\operatorname{Re} G_0(u) \geq \operatorname{Re} G_0\big(\nu(0)\big)\big\}$, and $\operatorname{Re} G_0\big(\gamma_0(s)\big) = \operatorname{Re} G_0\big(\nu(0)\big)$ holds if and only if $s = 0$.
- $\gamma_0'(0)$ satisfies $\operatorname{Re}\big[G_0''\big(\gamma_0(0)\big)\, \gamma_0'(0)^2\big] > 0$.

We can take a holomorphic function w on $\Delta_z \times U$ such that $w^2 = G(z,u) - G\big(z,\nu(z)\big)$. We determine the signature of w by the condition that $\operatorname{Re} w \circ \gamma_0$ is increasing around 0. For simplicity, we assume that $(\mathrm{id}, w) : \Delta_z \times U \longrightarrow \Delta_z \times \mathbb{C}$ is injective. The image is denoted by $\mathcal{U}$. We set $\mathcal{U}_z := \mathcal{U} \times_{\Delta_z} \{z\}$. Let $\widetilde{\gamma}$ denote the induced C^∞-map $\Delta_z \times [-\epsilon,\epsilon] \longrightarrow \mathcal{U}$, and $\widetilde{\gamma}_z$ denote the induced map $\{z\} \times [-\epsilon, \epsilon] \longrightarrow \mathcal{U}_z$.

There exists $r_1 > 0$ such that the following holds for any $|z| < r_1$.

- The image of $\widetilde{\gamma}_z$ is contained in $\big\{\operatorname{Re} w^2 \geq 0\big\}$, and $\operatorname{Re}\big(\widetilde{\gamma}_z(s)^2\big) = 0$ holds if and only if $s = 0$.
- There exists $\epsilon_1 > 0$ such that (i) the restriction of $\operatorname{Re} \widetilde{\gamma}_z$ to $[-\epsilon_1, \epsilon_1]$ is a diffeomorphism into $\mathbb{R}$, (ii) there exist $\eta_1 > 0$ and $\theta_1 > 0$ such that $\operatorname{Re}\big(\widetilde{\gamma}_z(s)^2\, e^{\sqrt{-1}\varphi}\big) > \eta_1$ for any $|s| > \epsilon_1$ and $|\varphi| < \theta_1$.

Let ℓ be a positive number. Let $\theta_2 > 0$ be such that $\theta_2\, \ell < \theta_1$. We put $S := \big\{r e^{\sqrt{-1}\theta} \,\big|\, r < r_1,\ |\theta| \leq \theta_2\big\}$. We have $\operatorname{Re}\big(\widetilde{\gamma}_z(s)^2\, z^{-\ell}\big) > \eta_1\, |z|^{-\ell}$ for any $|s| > \epsilon_1$ and for any $z \in S$.

Let $f(z,u)\, du$ be a holomorphic section of $\Omega^1_{S\times U/S}$. Assume that $|f(z,u)|$ is dominated by $|z|^{N_1}(-\log|z|)^{N_2}$ for some $N_1, N_2 \in \mathbb{R}$. We consider the asymptotic behaviour of the following pairings for $z \longrightarrow 0$ in S:

$$F(z) := \int_{\gamma_z} e^{-z^{-\ell} G(z,u)} f(z,u) du \tag{2.6}$$

Lemma 2.7. *We have the following estimate:*

(2.7)
$$\left| e^{z^{-\ell} G(z,\nu(z))}\, F(z) - \Big(\frac{2\pi}{\partial_z^2 G\big(z,\nu(z)\big)}\Big)^{1/2} f\big(z,\nu(z)\big)\, z^{\ell/2} \right| = O\Big(|z|^{N_1+\ell}\, \big(-\log|z|\big)^{N_2}\Big)$$

Here, the signature of $\big(\partial_z^2 G\big(z,\nu(z)\big)\big)^{1/2}$ *is determined by the condition*

$$\operatorname{Re}\Big(\big(\partial_z^2 G\big(z,\nu(z)\big)\big)^{1/2}\, \gamma_z'(0)\Big) > 0.$$

In particular, if $f \sim z^{N_1}(\log z)^{N_2}$*, i.e.,* $fz^{-N_1}(\log z)^{-N_2}$ *and* $f^{-1}z^{N_1}(\log z)^{N_2}$ *are bounded on* $S \times U$*, then we have*

$$F(z) \sim e^{-z^{-\ell}G(z,\nu(z))}\Big(\frac{2\pi}{\partial_z^2 G\big(z,\nu(z)\big)}\Big)^{1/2} f\big(z,\nu(z)\big)\, z^{\ell/2}$$

Proof. We have only to use a standard argument. By the change of variables, (2.6) is rewritten as follows:

$$e^{-z^{-\ell}G(z,\nu(z))}\int_{\widetilde{\gamma}_z} e^{-z^{-\ell}w^2} g(z,w)\,dw =: e^{-z^{-\ell}G(z,\nu(z))}\,H(z)$$

The contribution of $\big\{s\,\big|\,\epsilon_1 < |s| < \epsilon\big\}$ to $H(z)$ can be dominated by $e^{-|z|^{-\ell}\eta_2}$ for some $\eta_2 > 0$. Let us consider the contribution of $\big\{|s| \le \epsilon_1\big\}$. Let $a_\pm(z) := \mathrm{Re}\big(\widetilde{\gamma}_z(\pm\epsilon_1)\big)$. Let $L_\pm(z)$ be the segment connecting $a_\pm(z)$ and $\widetilde{\gamma}_z(\pm\epsilon_1)$. Then, the contribution of $\big\{|s| \le \epsilon_1\big\}$ is the same as the sum of the integrals over $L_\pm(z)$ and $\big[a_-(z), a_+(z)\big]$. The contribution of $L_\pm(z)$ can be dominated by $e^{-|z|^{-\ell}\eta_3}$ for some $\eta_3 > 0$.

Note that $\big|a_\pm(z) - a_\pm(0)\big| < C\,|z|$ for some $C > 0$. By a standard argument, we obtain the uniform estimate

$$\Big|H(z) - g(z,0)\sqrt{\pi}\,z^{\ell/2}\Big| = O\big(|z|^{N_1+\ell}\big|\log|z|\big|^{N_2}\big).$$

By using the relation $g(z,0) = 2^{1/2}\Big(\partial_u^2 G\big(z,\nu(z)\big)\Big)^{-1/2} f\big(z,\nu(z)\big)$, we obtain (2.7). □

2.4. **Some operations for 1-chains.** Let X be a compact complex curve with a locally finite subset D, which may have a boundary. Bloch and Esnault [5] introduced an appropriate homology theory $H_*\big(X, D; (V,\nabla)\big)$ for a meromorphic flat bundle (V,∇) on (X, D). We are especially interested in the 1-homology group. Roughly, it is made of 1-cycles of (X, D) equipped with rapid decay flat sections, modulo the boundary of 2-chains equipped with rapid decay flat sections. They showed that the naturally defined pairing

$$H^*_{DR}(X \setminus D, (V,\nabla)) \otimes H_*\big(X, D; (V^\vee,\nabla)\big) \longrightarrow \mathbb{C} \tag{2.8}$$

is perfect, if X is a compact Riemann surface. Here H^*_{DR} denotes the algebraic de Rham cohomology. (See [5] for the precise definition and more results.) Hien [11] generalized their homology theory for meromorphic flat bundles on higher dimensional varieties.

For simplicity, we will sometimes impose the following connectedness condition on 1-chains $\sum_{i=1}^N s_i \otimes \Gamma_i$.

(Connected): There exist $R_1, \ldots, R_{N+1} \in X$ such that (i) $\partial\Gamma_i = \{R_i, R_{i+1}\}$, (ii) $R_i \notin D$ for $i = 2, \ldots, N$, (iii) $s_i(R_i + 1) = s_{i+1}(R_i)$.

2.4.1. *Lift and descent for a ramified covering.* Let $\pi : (X', D') \longrightarrow (X, D)$ be a ramified covering. We have the following standard operations for 1-chains with values in meromorphic flat bundles on $\mathbb{P}^1_t$ or $\mathbb{P}^1_u$. We mention them just for reference in our later argument.

- **(Push-forward):** Let (V', ∇') be a meromorphic flat bundle on (X', D'). A 1-chain for (V', ∇') naturally induces a 1-chain for $\pi_*(V', \nabla')$ on (X, D).
- **(Descent):** Let (V, ∇) be a meromorphic flat bundle on (X, D). A 1-chain for $\pi^*(V, \nabla)$ on (X', D') naturally induces a 1-chain for (V, ∇) on (X, D).
- **(Lift):** Let $\sum s_i \otimes \Gamma_i$ be a connected 1-chain for (V, ∇) on (X, D). Take $x_0 \in \Gamma_1 \setminus D$. A choice of $\widetilde{x}_0 \in \pi^{-1}(x_0)$ determines the lift of $\sum s_i \otimes \Gamma_i$ to that for $\pi^*(V, \nabla)$ on (X', D').

2.4.2. *Lift with respect to Stokes filtration for meromorphic flat bundles on* $(\mathbb{P}^1, \{0, \infty\})$. We prepare some notation. For two points $P, Q \in \mathbb{C}$, let $\mathrm{Seg}(P, Q)$ denote the segment connecting P and Q. Let $\mathrm{Ray}(P, \infty)$ denote the ray connecting P and ∞, i.e., $\mathrm{Ray}(P, \infty) := \{t\,P \mid t \geq 1\}$. We also use the symbol $\mathrm{Ray}(0, P)$ to denote $\mathrm{Seg}(0, P)$. For two points $P, Q \in \mathbb{C}$ such that $|P| = |Q| =: r$, let $\mathrm{Arc}(P, Q)$ denote the shorter arc connecting P and Q in the circle $\{|z| = r\}$, where we will not consider the case $P = -Q$.

We introduce some operations for lifting of 1-chains with respect to Stokes filtrations, which will be used in Subsection 5.1. Let (V, ∇) be a meromorphic flat bundle on $(\mathbb{P}^1, \{0, \infty\})$ such that (i) unramified at 0, (ii) regular singular at ∞. Let $\mathcal{L}$ be a meromorphic flat bundle on $(\mathbb{P}^1, \{0, \infty\})$ with $\operatorname{rank} \mathcal{L} = 1$.

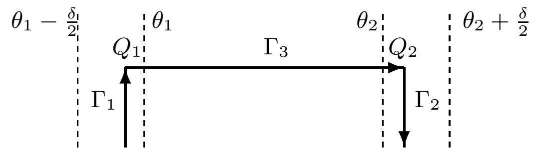
P1. We put $\mathcal{T} := \{\theta_0 < \arg u < \theta_1\}$ for some $\theta_i \in \mathbb{R}$ $(i = 0, 1)$ such that $|\theta_1 - \theta_0| < \pi / \operatorname{dec}(V)$. We can take a flat splitting $(V, \nabla)_{|\mathcal{T}} = \bigoplus_{\mathfrak{b} \in \mathrm{Irr}(\nabla)} (V_{\mathfrak{b}, \mathcal{T}}, \nabla_{\mathfrak{b}, \mathcal{T}})$ of the full Stokes filtration.

Let $\mathfrak{a} \in \mathrm{Irr}(\nabla)$. Let $\sum s_i \otimes \Gamma_i$ be a 1-chain for $\mathrm{Gr}_{\mathfrak{a}}(V) \otimes \mathcal{L}$ such that $\Gamma_i \subset \mathcal{T}$ for any i. By taking the lift $\widetilde{s}_i$ of s_i to $V_{\mathfrak{a}, \mathcal{T}} \otimes \mathcal{L}$ via the splitting, we obtain a 1-chain $\sum \widetilde{s}_i \otimes \Gamma_i$ for $V \otimes \mathcal{L}$.

P2. Assume that we are given a connected 1-chain $\sum_{i=1}^N s_i \otimes \Gamma_i$ for $\mathrm{Gr}_0^{(m)}(V) \otimes \mathcal{L}$ such that Γ_i are contained in regions $\mathcal{T}_i = \{\theta_1^{(i)} < \arg(u) < \theta_2^{(i)}\}$, on which we have flat splittings $V_{|\mathcal{T}_i} = \bigoplus_{\mathfrak{b} \in \mathrm{Irr}(\nabla, m)} V^{(m)}_{\mathfrak{b}, \mathcal{T}_i}$ of the Stokes filtrations in the level m. Let $\widetilde{s}_i$ be the lift of s_i to $V^{(m)}_{0, \mathcal{T}_i} \otimes \mathcal{L}$ by the splitting. We set $\delta_i := \widetilde{s}_i - \widetilde{s}_{i+1}$ on $\mathcal{T}_i \cap \mathcal{T}_{i+1}$, which naturally induce flat sections of $\mathcal{F}^{(m)\,\mathcal{T}_i \cap \mathcal{T}_{i+1}}_{<0}$. We have $R_1, \ldots, R_{N+1} \in \mathbb{P}^1$ such that $\partial \Gamma_i = \{R_i, R_{i+1}\}$. We put $T_i := \mathrm{Ray}[R_i, 0]$ $(i = 2, \ldots, N)$, which is contained in $\mathcal{T}_i \cap \mathcal{T}_{i+1}$. By taking appropriate orientation for T_i, we obtain a 1-chain $\sum_{i=1}^N \widetilde{s}_i \otimes \Gamma_i + \sum_{i=2}^N \delta_i \otimes T_i$ for $V \otimes \mathcal{L}$.

P3. We set $n := \mathrm{dec}(V,\nabla)$. Assume that we have an element $\mathfrak{a} \in \mathrm{Irr}(\nabla, -n)$ of the form $\mathfrak{a} = \alpha\, u^{-n}$. Let θ_i $(i = 1,2)$ satisfy (i) $\mathrm{Re}(\mathfrak{a})(e^{\sqrt{-1}\theta_i}) = 0$, (ii) $\theta_2 - \theta_1 = \pi/n$, (iii) we have $\mathrm{Re}(\mathfrak{a})(e^{\sqrt{-1}\theta}) > 0$ for $\theta_1 < \theta < \theta_2$.

Take a small $\delta > 0$ such that $\mathrm{Re}(\mathfrak{b}_1 - \mathfrak{b}_2)(e^{\sqrt{-1}\theta}) \neq 0$ for any θ with $0 < |\theta - \theta_i| < \delta$, where $\mathfrak{b}_l$ are distinct elements of $\mathrm{Irr}(\nabla, -n)$. We set $T_1 := \bigl\{\theta_1 - \delta/2 < \arg(u) < \theta_1\bigr\}$ and $T_2 := \bigl\{\theta_2 < \arg(u) < \theta_2 + \delta/2\bigr\}$. Take $Q_i \in T_i$. Let Γ_i $(i = 1,2)$ be paths connecting 0 and Q_i in T_i. Let Γ_3 be a path connecting Q_i in $\{\theta_1 - \delta/2 < \arg(u) < \theta_2 + \delta/2\}$.



Let s_i $(i = 1,2,3)$ be flat sections along Γ_i such that $s_i = s_3$ at Q_i. They give a 1-chain $\sum_{i=0,1,2} s_i \otimes \Gamma_i$ for $\mathrm{Gr}^{(-n)}_{\mathfrak{a}}(V) \otimes \mathcal{L}$.

We set $\mathcal{T}_1 := \bigl\{\theta_1 - \delta/2 < \arg(u) < \theta_2 - \delta/2\bigr\}$ and $\mathcal{T}_2 := \bigl\{\theta_2 - \delta < \arg(u) < \theta_2 + \delta\bigr\}$. We may have flat splittings of the Stokes filtrations in the level $-n$:

$$V_{|\mathcal{T}_i} = \bigoplus_{\mathfrak{b} \in \mathrm{Irr}(\nabla, -n)} V^{(-n)}_{\mathfrak{b}, \mathcal{T}_i} \tag{2.9}$$

Let $\widetilde{s}_1$ be the lift of s_1 to $V_{|\mathcal{T}_1}$ by the splitting (2.9) with $i = 1$. It induces a flat section $\widetilde{s}_3$ of V along Γ_3. We have the decomposition $\widetilde{s}_{3|Q_2} = \sum s_{3,\mathfrak{b}}$ corresponding to the decomposition (2.9) with $i = 2$. Note that $s_{3,\mathfrak{b}} = 0$ unless $-\mathrm{Re}(\mathfrak{b})(e^{\sqrt{-1}\theta_2}) \leq 0$.

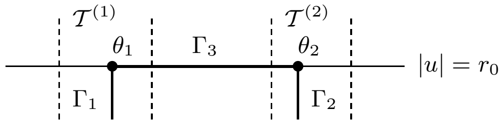
Let $\mathfrak{b} = \beta\, u^{-n}$ be such that $-\mathrm{Re}(\mathfrak{b})(e^{\sqrt{-1}\theta_2}) \leq 0$. Take $P_{\mathfrak{b}} \in \bigl\{0 < |\arg(u) - \theta_2| < \delta\bigr\}$ such that $-\mathrm{Re}(\mathfrak{b})(P_{\mathfrak{b}}) < 0$. Let $\Gamma_{\mathfrak{b}}$ be the union of the segments $\mathrm{Seg}(P_{\mathfrak{b}}, Q_2)$ and $\mathrm{Ray}(0, P_{\mathfrak{b}})$ with the appropriate orientation. Let $s_{2,\mathfrak{b}}$ be the flat section of V along $\Gamma_{\mathfrak{b}}$ induced by $s_{3,\mathfrak{b}}$. Then, we obtain the following 1-chain for $V \otimes \mathcal{L}$:

$$\sum_{i=1,3} \widetilde{s}_i \otimes \Gamma_i + \sum s_{2,\mathfrak{b}} \otimes \Gamma_{\mathfrak{b}}$$

We can clearly exchange the role of θ_1 and θ_2.

P4. We restrict ourselves to the case $\mathcal{L} = L(-u^{-p}/z)$ for some $z \in \mathbb{C} \setminus \{0\}$, i.e., $\mathcal{L} = \mathcal{O}_{\mathbb{P}^1}\bigl(*\{0,\infty\}\bigr)\, e$ with a connection $\nabla e = e\, d(-u^{-p}/z)$. We set $n := \mathrm{dec}(V,\nabla)$. Assume $p < n$. Assume that we have an element $\mathfrak{a} \in \mathrm{Irr}(\nabla, -n)$ of the form $\mathfrak{a} = \alpha\, u^{-n}$ with $\alpha \neq 0$. Let $\sum_{i=1,2,3} s_i \otimes \Gamma_i$ be a 1-chain for $\mathrm{Gr}^{(-n)}_{\mathfrak{a}}(V) \otimes L(-u^{-p}/z)$ of the following form:

$$\Gamma_1 = \mathrm{Ray}\bigl(0, r_0\, e^{\sqrt{-1}\theta_1}\bigr) \quad \Gamma_2 = \mathrm{Ray}\bigl(0, r_0\, e^{\sqrt{-1}\theta_2}\bigr) \quad \Gamma_3 = \mathrm{Arc}\bigl(r_0\, e^{\sqrt{-1}\theta_1},\ r_0\, e^{\sqrt{-1}\theta_2}\bigr)$$



We assume $\mathrm{Re}\big(e^{-p\sqrt{-1}\theta_2}/z\big) < 0$.

For $j = 1, 2$, we take sectors $\mathcal{T}^{(j)} = S[2r_0; \theta_1^{(j)}, \theta_2^{(j)}] \supset \Gamma_j$ with the following property:

- We have a flat splitting of the full Stokes filtration of $V_{|\mathcal{T}^{(1)}}$:

$$V_{|\mathcal{T}^{(1)}} = \bigoplus_{\mathfrak{b}\in\mathrm{Irr}(\nabla)} V_{\mathfrak{b},\mathcal{T}^{(1)}} \tag{2.10}$$

- $\theta_2^{(2)} - \theta_1^{(2)}$ is sufficiently close to π/n, and we have a flat splitting of the Stokes filtration in the level $-n$:

$$V_{|\mathcal{T}^{(2)}} = \bigoplus_{\mathfrak{b}\in\mathrm{Irr}(\nabla,-n)} V^{(-n)}_{\mathfrak{b},\mathcal{T}^{(2)}} \tag{2.11}$$

  Moreover, we have $\mathrm{Re}(e^{-p\sqrt{-1}\theta}/z) < 0$ for any $\theta \in]\theta_1^{(2)}, \theta_2^{(2)}[= \{\theta_1^{(2)} < \theta < \theta_2^{(2)}\}$.

Let $\widetilde{s}_1$ be the lift of s_1 to $V_{\mathfrak{a},\mathcal{T}^{(1)}}$ on $\mathcal{T}^{(1)}$ by the splitting (2.10). It induces a flat section $\widetilde{s}_3$ of V along Γ_3. We obtain the decomposition corresponding to (2.11):

$$\widetilde{s}_{3|r_0e^{\sqrt{-1}\theta_2}} = \sum_{\mathfrak{b}\in\mathrm{Irr}(\nabla,-n)} \widetilde{s}_{3,\mathfrak{b}}$$

Let $\widetilde{s}_{2,0}$ be the flat section along $\mathcal{P}_0 := \Gamma_2$ induced by $\widetilde{s}_{3,0}$. For each non-zero irregular value $\mathfrak{b} \in \mathrm{Irr}(\nabla, -n)$, we take $\varphi(\mathfrak{b}) \in]\theta_1^{(2)}, \theta_2^{(2)}[$ such that $\mathrm{Re}\big(\beta\, e^{-n\sqrt{-1}\varphi(\mathfrak{b})}\big) < 0$, where $\beta\, u^{-n}$ is the top term of $\mathfrak{b}$. Let $\mathcal{P}_{\mathfrak{b}}$ be the union of the arc $\mathrm{Arc}\big(r_0\, e^{\sqrt{-1}\theta_2}, r_0\, e^{\sqrt{-1}\varphi(\mathfrak{b})}\big)$ and the ray $\mathrm{Ray}\big(r_0\, e^{\sqrt{-1}\varphi(\mathfrak{b})}, 0\big)$ with the appropriate orientation. Let $\widetilde{s}_{2,\mathfrak{b}}$ be the flat section of V along $\mathcal{P}_{\mathfrak{b}}$ induced by $\widetilde{s}_{3,\mathfrak{b}}$. Then, we obtain a family of cycles:

$$\widetilde{s}_1 \otimes \Gamma_1 + \sum_{\mathfrak{b}\in\mathrm{Irr}(\nabla,-n)} \widetilde{s}_{2,\mathfrak{b}} \otimes \mathcal{P}_{\mathfrak{b}} + \widetilde{s}_3 \otimes \Gamma_3$$

2.4.3. *Translation.* For $z \in \mathbb{C}$, let $L(-t/z)$ denote a line bundle $\mathcal{O}_{\mathbb{P}^1}(*\infty)\, e$ with a connection $\nabla e = e\, d(-t/z)$. For $c \in \mathbb{C}$, let $\phi_c : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ be given by $\phi_c(t) := t - c$. We have a flat isomorphism $\phi_c^* L(-t/z) \simeq L(-t/z)$ induced by the correspondence $\phi_c^*\big(\exp(t/z)\, e\big) \longleftrightarrow \exp(t/z)\, e$ of flat sections.

Let (V, ∇) be a meromorphic flat bundle on $\mathbb{P}^1$. When we are given a 1-chain C for $(V, \nabla) \otimes L(-t/z)$, it naturally induces a 1-chain $\phi_c^* C$ for $\phi_c^*(V, \nabla) \otimes L(-t/z)$.

2.5. **Flat family of 1-cycles.** We recall how to use the duality (2.8) in construction of flat sections of meromorphic flat bundles obtained as the push-forward. We restrict ourselves to the case of Fourier transform. We use the notation in Subsection 2.2.1. The dual of $L(t\tau)$ is denoted by $L(-t\tau)$.

Let (V,∇) be a meromorphic flat bundle on $\mathbb{P}^1_t$. Let H be the pole of V. For simplicity, we assume $\infty\in H$. For each $h\in H$, we take a frame $\boldsymbol{v}_h=(v_{h,j})$ of V around h.

Let $U\subset\mathbb{C}_\tau$ be an open subset. Let $\Gamma_i:[0,1]\times U\longrightarrow\mathbb{P}^1_t\times U$ $(i=1,\dots,N)$ be C^∞-embeddings such that (i) $\pi_2\circ\Gamma_i$ are equal to the projection onto U, (ii) $\Gamma_i^{-1}(H)$ is the union of some connected components of $\bigl(\{0\}\cup\{1\}\bigr)\times U$, i.e., $\emptyset$, $\{0\}\times U$, $\{1\}\times U$ or $(\{0\}\cup\{1\})\times U$. The restrictions of Γ_i to $[0,1]\times\{\tau\}$ $(\tau\in U)$ are denoted by $\Gamma_{i|\tau}$.

Let s_i $(i=1,\dots,N)$ be flat sections of $\Gamma_i^{-1}\bigl(V^\vee\otimes L(-t\tau)\bigr)$ on $([0,1]\times U)\setminus\Gamma_i^{-1}(H)$ such that s_i are of rapid decay around $\Gamma_i^{-1}(H)$. Namely, if h is contained in the image of Γ_i, for the expression $s_i=\sum a_j\Gamma_i^{-1}v_{h,j}$ around $\Gamma_i^{-1}(h)$, any derivative of a_j are 0 at $\Gamma_i^{-1}(h)$. The restrictions of s_i to $[0,1]\times\{\tau\}\setminus\Gamma_i^{-1}(H)$ are denoted by $s_{i|\tau}$. We assume that $\mathcal{P}_\tau:=\sum s_{i|\tau}\otimes\Gamma_{i|\tau}$ $(\tau\in U)$ are 1-cycles with rapid decay flat sections of $(V^\vee,\nabla)_{|\pi_2^{-1}(\tau)}$. According to the duality (2.8), $\mathcal{P}_\tau$ gives an element of $\mathfrak{Four}(V)^\vee_{|\tau}$.

Lemma 2.8. *The family $\mathcal{P}_\tau$ $(\tau\in U)$ gives a flat section of $\mathfrak{Four}(V)^\vee$ on U.*

Proof. It can be shown by a standard argument. We give only an indication. We set $\widetilde{V}:=V\otimes L(t\tau)$. The induced flat connection on $\widetilde{V}$ is denoted by $\widetilde{\nabla}$. Let f be a flat section of $\mathfrak{Four}(V)_{|U}$. We have a section $\widetilde{f}$ of $\Omega^1_{\mathbb{P}^1_t\times\mathbb{P}^1_\tau/\mathbb{P}^1_\tau}\otimes\widetilde{V}$ on $\mathbb{P}^1_t\times U$, which induces f. The restrictions of $\widetilde{f}$ to $\mathbb{P}^1_t\times\{\tau\}$ $(\tau\in U)$ are denoted by $\widetilde{f}_\tau$. We have only to show that the pairing $\bigl\langle\widetilde{f}_\tau,\,\mathcal{P}_\tau\bigr\rangle$ is constant with respect to τ. Let $\widetilde{\nabla}(\partial_\tau)$ denote the endomorphism of $\Omega^1_{\mathbb{P}^1_t\times\mathbb{P}^1_\tau/\mathbb{P}^1_\tau}\otimes\widetilde{V}$ induced by $\widetilde{\nabla}$. Let $d_{\widetilde{V},\mathrm{rel}}$ denote the differential of $\Omega^\bullet_{\mathbb{P}^1_t\times\mathbb{P}^1_\tau/\mathbb{P}^1_\tau}\otimes\widetilde{V}$. By the flatness of f, there exists a section g of $\widetilde{V}$ such that $\widetilde{\nabla}(\partial_\tau)f=d_{\widetilde{V},\mathrm{rel}}g$. Then, we have

$$\partial_\tau\bigl\langle\widetilde{f}_\tau,\,\mathcal{P}_\tau\bigr\rangle=\bigl\langle\widetilde{\nabla}(\partial_\tau)\widetilde{f},\,\mathcal{P}_\tau\bigr\rangle=\bigl\langle d_{\widetilde{V},\mathrm{rel}}g_\tau,\,\mathcal{P}_\tau\bigr\rangle=0.$$

Thus, we obtain Lemma 2.8. □

2.6. (K,k)**-structure.** Let K and k be subfields of $\mathbb{C}$. For simplicity, we assume that k is algebraically closed. Let V be a $\mathbb{C}$-vector space. Let V_K be a K-vector space with an isomorphism $F_K:V_K\otimes_K\mathbb{C}\simeq V$, and let V_k be a k-vector space with an isomorphism $F_k:V_k\otimes_k\mathbb{C}\simeq V$. Such a tuple $\bigl((V_K,F_K),(V_k,F_k)\bigr)$ is called a (K,k)-structure of V. This kind of structure has been studied in Hodge theory. See [3], for example.

Let (V,∇) be a meromorphic flat bundle on (Δ_z,O). Let $\mathrm{Loc}(V,\nabla)$ be the local system on Δ_z^* associated to (V,∇). Assume that we are given the following:

- A connection (V_k, ∇_k) on $k(\!(z)\!)$ with an isomorphism
$$(V,\nabla)_{|\widehat{O}} \simeq (V_k,\nabla_k)\otimes_{k(\!(z)\!)}\mathbb{C}(\!(z)\!).$$
- A K-structure of $\mathrm{Loc}(V,\nabla)$ compatible with the Stokes structure, i.e., the subbundles $\widetilde{\mathcal{F}}^S_{\mathfrak{a}}(V_{|S})$ are defined over K for any small sector $S\subset\Delta_z^*$.

Such a structure is called (K,k)-structure of (V,∇).

Let $\psi\big(\mathrm{Gr}(V,\nabla)\big)$ be the space of the multi-valued flat sections of $\mathrm{Gr}(V,\nabla)$, which is naturally equipped with the induced K-structure. Let us recall that a k-structure of $\psi\big(\mathrm{Gr}(V,\nabla)\big)$ is also naturally induced, i.e., a (K,k)-structure is induced on $\psi\big(\mathrm{Gr}(V,\nabla)\big)$. (Note that the coordinate z is fixed in this situation.)

Unramified elementary case. Let us consider the case that $\nabla':=\nabla-d\mathfrak{a}$ is regular singular for some $\mathfrak{a}\in z^{-1}\mathbb{C}[z^{-1}]$. Note $\mathfrak{a}\in z^{-1}k[z^{-1}]$ in this situation. We take a splitting $\tau:\mathbb{C}/\mathbb{Z}\longrightarrow\mathbb{C}$ of the projection $\mathbb{C}\longrightarrow\mathbb{C}/\mathbb{Z}$. We have the uniquely determined lattice $E\subset V$ such that (i) ∇' is logarithmic with respect to E, (ii) the eigenvalues α of $\mathrm{Res}(\nabla')$ are contained in $\tau(\mathbb{C}/\mathbb{Z})$. Note that $E_{|\widehat{O}}$ is compatible with the k-structure of $V_{|\widehat{O}}$. In particular, the $\mathbb{C}$-vector space $E_{|O}$ has the induced k-structure. Let $\mathcal{S}$ denote the set of eigenvalues of $\mathrm{Res}(\nabla')$. For any multivalued flat section f of (V,∇), we have the expansion $f=\exp(-\mathfrak{a})\sum_{\alpha\in\mathcal{S}}\sum_{j\geq 0}z^{-\alpha}(\log z)^j\,s_{\alpha,j}$, where $s_{\alpha,j}$ are holomorphic sections of E, and the branch of $\log z$ is the natural one. Then, we obtain the isomorphism $\Phi_V:\psi(V,\nabla)\simeq E_{|O}$ given by $\Phi_V(f)=\sum_\alpha s_{\alpha,0|O}$. It induces a k-structure on $\psi(V,\nabla)$. It is easy to see that the induced k-structure is independent of the choice of τ. However, in general, it depends on the choice of a coordinate z in taking $\mathfrak{a}$. If V is regular singular, it is standard that Φ_V depends only on the tangent vector $\partial_{z|O}$.

For a ramified covering $\varphi:(\Delta_\zeta,O)\longrightarrow(\Delta_z,O)$ given by $\varphi(\zeta)=\zeta^p$, we have a naturally induced (K,k)-structure on $\varphi^*(V,\nabla)$. It is easy to see that the natural isomorphism $\psi\big(\varphi^*(V,\nabla)\big)\simeq\psi(V,\nabla)$ preserves the induced (K,k)-structures.

General case. A k-structure of $\mathrm{Gr}(V,\nabla)_{|\widehat{O}}$ is induced by the isomorphism $\mathrm{Gr}(V,\nabla)_{|\widehat{O}}\simeq(V,\nabla)_{|\widehat{O}}$. Take the pull back via an appropriate ramified covering $\varphi:(\Delta_\zeta,O)\longrightarrow(\Delta_z,O)$ given by $\varphi(\zeta)=\zeta^p$ such that $\varphi^*(V,\nabla)$ is unramified, and apply the above procedure, then we obtain the k-structure of $\psi\big(\mathrm{Gr}(V,\nabla)\big)\simeq\psi\big(\varphi^*\,\mathrm{Gr}(V,\nabla)\big)$. It is independent of the choice of p. If we have $|\operatorname{ord}_\zeta\mathfrak{a}|<p$ for any $\mathfrak{a}\in\mathrm{Irr}(\nabla)\subset\mathbb{C}(\!(\zeta)\!)/\mathbb{C}[\![\zeta]\!]$, then the induced k-structure depends only on the tangent vector $\partial_{z|O}$. Note that we do not have the term z^0 in a series $z^{-m/p}\sum_{j\geq 0}a_jz^j$ if $m<p$.

3. Elementary cases around 0

For $\mathfrak{a} \in u^{-1}\mathbb{C}[u^{-1}]$, let $\mathcal{L}$ be a meromorphic flat line bundle $\mathcal{O}_{\mathbb{P}^1}\big(*\{0,\infty\}\big)\, e$ with $\nabla e = e\, d\mathfrak{a}$. Let R be a meromorphic flat bundle on $(\mathbb{P}^1_u, \{0,\infty\})$ with regular singularity. Let $q : \mathbb{P}^1_u \longrightarrow \mathbb{P}^1_t$ be given by $q(u) = u^p$. We set $V := q_*\big(\mathcal{L} \otimes R\big)$.

We construct some families of cycles with values in $\big(V \otimes L(t/z)\big)^{\vee}$, where z varies in a sector $\mathcal{S}$ around $\infty \in \mathbb{P}^1_\tau$, namely, flat sections of $\mathfrak{Four}(V)^{\vee}_{|\mathcal{S}}$. (See Subsection 2.5 for the relation between such families and flat sections.) The essential idea is given in [3]. We indicate a concrete way to choose paths, which seems useful for understanding of the Stokes structure of $\mathfrak{Four}(V)$ at ∞. In the following, ∞_τ denotes $\infty \in \mathbb{P}^1_\tau$.

3.1. Paths with nice some property.

We give a preliminary for a construction of cycles with values in meromorphic flat bundles. Let n and p be positive integers. For $\kappa \in \mathbb{R}$, let F be the holomorphic function on $\mathbb{C} \setminus \{0\}$ given as follows:

$$F(v) = e^{\sqrt{-1}\kappa}\left(\frac{v^{-n}}{n} + \frac{v^p}{p}\right) \tag{3.1}$$

Because $F'(v) = e^{\sqrt{-1}\kappa}\big(-v^{-n-1} + v^{p-1}\big)$, we have $F'(v) = 0$ if and only if

$$v = \exp\Big(\frac{2m}{n+p}\pi\sqrt{-1}\Big) =: [n,p,m] \qquad (m \in \mathbb{Z})$$

We set $[\![n,p,m]\!] := 2m\pi/(n+p)$, for which $[n,p,m] = \exp\big(\sqrt{-1}[\![n,p,m]\!]\big)$.

Take $\delta > 0$. For each $[n,p,m]$, we will take a continuous path $\Gamma_m : [0,1] \longrightarrow \mathbb{C}$ with the following property. Because a parametrization is not significant here, we will not distinguish Γ_m and its image.

- Γ_m contains $[n,p,m]$, the end points of Γ_m are contained in $\{0,\infty\}$, and Γ_m is the union of some arcs and segments.
- The restriction $\operatorname{Re} F_{|\Gamma_m}$ has the unique maximum, which is attained at $[n,p,m]$.
- There exist $\epsilon > 0$ and $\eta > 0$ such that
$$\operatorname{Re} F(v) \leq \operatorname{Re} F([n,p,m]) - \eta\,(1 + |v^{-n}| + |v^p|)$$
for any $v \in \Gamma_m \setminus B_{[n,p,m]}(\epsilon)$, where $B_{[n,p,m]}(\epsilon)$ denotes an ϵ-neighbourhood of $[n,p,m]$.
- Γ_m is contained in $\big\{v \neq 0 \,\big|\, \theta_m(\kappa) \leq \arg(v) \leq \theta_m(\kappa) + \pi/n + \delta\big\}$ for some $\theta_m(\kappa) \in \mathbb{R}$.

We will explain how to choose Γ_m in Subsection 3.1.2. A choice of the orientation of Γ_m will be indicated in Subsection 3.1.4.

3.1.1. *Preliminary.*

For the polar coordinate $v = re^{\sqrt{-1}\theta}$, we have

$$\operatorname{Re} F(r,\theta) = \frac{r^{-n}}{n}\cos\big(n\theta - \kappa\big) + \frac{r^p}{p}\cos\big(p\theta + \kappa\big).$$

If $\cos(p\theta^{(1)} + \kappa)\cos(n\theta^{(1)} - \kappa) < 0$ for some $\theta^{(1)}$, the function $\operatorname{Re} F(r, \theta^{(1)})$ is monotonous with respect to r.

Let us look at the critical points of the function $\operatorname{Re} F_{|r=1}$. Because $\partial_\theta \operatorname{Re} F(1,\theta) = -\sin(n\theta - \kappa) - \sin(p\theta + \kappa)$, we have $\partial_\theta \operatorname{Re} F(1,\theta) = 0$ if and only if either one of the following holds for some $m, q \in \mathbb{Z}$:

$$n\theta - \kappa = -(p\theta + \kappa) + 2m\pi \iff \theta = \frac{2m\pi}{n+p} = [\![n,p,m]\!] \tag{3.2}$$

$$n\theta - \kappa = p\theta + \kappa + (2q+1)\pi \iff \theta = \frac{2\kappa}{n-p} + \frac{(2q+1)\pi}{n-p} \tag{3.3}$$

(The second case can happen only in the case $n \neq p$.)

At $\theta_0 = 2m\pi/(n+p) = [\![n,p,m]\!]$, we have $\cos(n\theta_0 - \kappa) = \cos(p\theta_0 + \kappa)$ due to (3.2), and

$$\partial_\theta^2 \operatorname{Re} F(1,\theta_0) = -pn \operatorname{Re} F(1,\theta_0).$$

Hence, $\operatorname{Re} F(1,\theta_0)$ is maximal (resp. minimal) of the function $\operatorname{Re} F_{|r=1}$, if $\operatorname{Re} F(1,\theta_0) > 0$ (resp. $\operatorname{Re} F(1,\theta_0) < 0$). We also have

$$\operatorname{Re} F(r,\theta_0) = \left(\frac{r^{-n}}{n} + \frac{r^p}{p}\right)\cos(p\theta_0 + \kappa).$$

If $n \neq p$, we have $\cos(-n\theta_1 + \kappa) = -\cos(p\theta_1 + \kappa)$ at $\theta_1 = \big(2\kappa + (2q+1)\pi\big)/(n-p)$. Hence, we have $\cos(n\theta_1 - \kappa)\cos(p\theta_1 + \kappa) \leq 0$. Because $\partial_\theta^2 \operatorname{Re} F(1,\theta_1) = pn \operatorname{Re} F(1,\theta_1)$, $\operatorname{Re} F(1,\theta_1)$ is a maximal (resp. minimal) of the function $\operatorname{Re} F_{|r=1}$, if $\operatorname{Re} F(1,\theta_1) < 0$ (resp. $\operatorname{Re} F(1,\theta_1) > 0$).

Let θ_0 and θ_1 be as above. We have $\theta_0 = \theta_1$ only if $\cos(p\theta_0 + \kappa) = 0$ is satisfied. More precisely, we have the following lemma on the degenerated cases.

Lemma 3.1. *Let m be an integer. We have the following equivalence:*

$$\frac{2m\pi}{n+p} = \frac{2q+1}{n-p}\pi + \frac{2\kappa}{n-p} \quad (\exists q \in \mathbb{Z}) \iff \cos\left(p\frac{2m\pi}{n+p} + \kappa\right) = 0 \tag{3.4}$$

Proof. Assume the left hand side holds. Then, the following holds:

$$\frac{2mp\pi}{n+p} + \kappa = m\pi - \left(q + \frac{1}{2}\right)\pi$$

It implies the right hand side. The converse can also be shown easily. □

We prepare a lemma to use in Subsection 3.1.2 (the case $\cos(p\theta_0 + \kappa) > 0$). We remark that we have $-n\theta_2 = -(2m+1)\pi + p\theta_2$ for $\theta_2 = (2m+1)\pi/(n+p)$, and hence $\cos(p\theta_2 + \kappa)\cos(n\theta_2 - \kappa) \leq 0$.

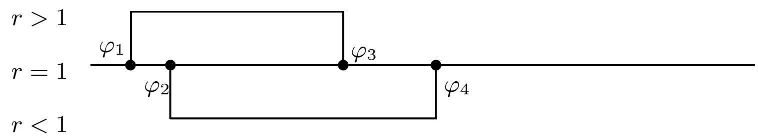
Lemma 3.2. *Assume $\cos(p\theta_0 + \kappa) \neq 0$ for $\theta_0 = [\![n,p,m]\!]$. We also assume $n \neq p$. We take $q \in \mathbb{Z}$ such that*

$$\theta_3 := \frac{(2q-1)\pi + 2\kappa}{n-p} < [\![n,p,m]\!] < \frac{(2q+1)\pi + 2\kappa}{n-p} =: \theta_4.$$

There exist φ_i $(i=1,2)$ *such that (i)* $\theta_3<\varphi_1<\theta_0<\varphi_2<\theta_4$*, (ii)* $\cos(p\varphi_i+\kappa)\cos(n\varphi_i-\kappa)=0$.

Proof. If $\cos(p\theta_3+\kappa)\cos(n\theta_3-\kappa)=0$ holds, θ_3 is of the form $[\![n,p,m']\!]$ for some $m'<m$. Hence, we have $\theta_5:=(2m-1)\pi/(n+p)>\theta_3$. If $\cos(p\theta_5+\kappa)=0$, we may take $\varphi_1=\theta_5$. If $\cos(p\theta_5+\kappa)\neq 0$, we can take the desired φ_1 in the interval $]\theta_5,\theta_0[$. We can take φ_2 with a similar consideration. □

3.1.2. *Paths.* We describe how to take a path Γ_m. Let $\theta_0=[\![n,p,m]\!]$. We use the notation in Subsection 2.4. For $a,b\in\mathbb{R}$, let $]a,b]:=\bigl\{t\in\mathbb{R}\bigm| a<t\leq b\bigr\}$. We use the symbols $]a,b[$, $[a,b]$ in similar meanings. In the following pictures, the horizontal central line describes a part of the circle $\{r=1\}$. The upper and lower spaces correspond to $\{r>1\}$ and $\{r<1\}$, respectively. A box in the upper space indicates an arc on which $\cos(p\theta+\kappa)\leq 0$. A box in the lower space indicates an arc on which $\cos(n\theta-\kappa)\leq 0$.

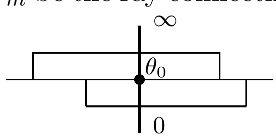


For example, in the above picture, we have the following:

$$\cos(p\varphi_1+\kappa)=\cos(p\varphi_3+\kappa)=\cos(n\varphi_2-\kappa)=\cos(n\varphi_4-\kappa)=0$$

$$\cos(p\theta+\kappa)\leq 0\ (\varphi_1\leq\theta\leq\varphi_3)\qquad \cos(n\theta-\kappa)\leq 0\ (\varphi_2\leq\theta\leq\varphi_4)$$

The case $\cos(p\theta_0+\kappa)<0$. Let Γ_m be the ray connecting 0 and ∞ through $e^{\sqrt{-1}\theta_0}$.



The case $\cos(p\theta_0+\kappa)>0$. We consider the following sets:

$$\mathcal{S}_+^{(1)}:=\bigl\{\theta>\theta_0\bigm|\cos(p\theta+\kappa)=0\bigr\},\mathcal{S}_+^{(2)}:=\bigl\{\theta>\theta_0\bigm|\cos(n\theta-\kappa)=0\bigr\},\mathcal{S}_+:=\mathcal{S}_+^{(1)}\cup\mathcal{S}_+^{(2)}$$

$$\mathcal{S}_-^{(1)}:=\bigl\{\theta<\theta_0\bigm|\cos(p\theta+\kappa)=0\bigr\},\mathcal{S}_-^{(2)}:=\bigl\{\theta<\theta_0\bigm|\cos(n\theta-\kappa)=0\bigr\},\mathcal{S}_-:=\mathcal{S}_-^{(1)}\cup\mathcal{S}_-^{(2)}$$

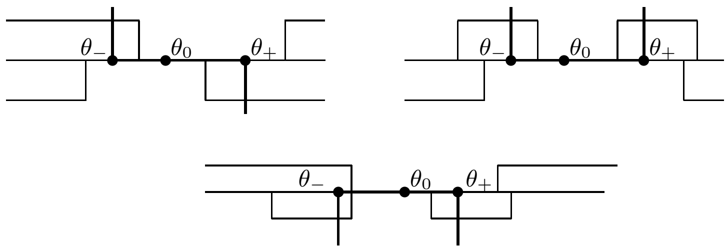
We can take $\theta_+>\min\mathcal{S}_+$ and $\theta_-<\max\mathcal{S}_-$ satisfying the following conditions:

- $\cos(p\theta+\kappa)\cos(n\theta-\kappa)\neq 0$ on the intervals $]\min\mathcal{S}_+,\theta_+]$ and $[\theta_-,\max\mathcal{S}_-[$.
- $\theta_+-\theta_-<\pi/n+\delta$.
- $\mathrm{Re}\,F(1,\theta)$ is monotonous on the intervals $[\theta_-,\theta_0]$ and $[\theta_0,\theta_+]$. (Recall Lemma 3.2.)

Let $x=\pm$. At least one of $\cos(p\theta_x+\kappa)$ or $\cos(n\theta_x-\kappa)$ is negative. We set

$$\Lambda_{\theta_x}:=\begin{cases}\mathrm{Ray}[0,e^{\sqrt{-1}\theta_x}] & \bigl(\cos(n\theta_x-\kappa)<0\bigr)\\ \mathrm{Ray}[e^{\sqrt{-1}\theta_x},\infty] & \bigl(\cos(n\theta_x-\kappa)>0\bigr)\end{cases}$$

Then, let Γ_m be the union of Λ_{θ_+}, Λ_{θ_-} and the arc $\mathrm{Arc}(e^{\sqrt{-1}\theta_-}, e^{\sqrt{-1}\theta_+})$. See the following pictures.



The following elementary lemma describes when the degenerate case happens.

Lemma 3.3. *If* $\min \mathcal{S}_+^{(1)} = \min \mathcal{S}_+^{(2)}$*, it is equal to* $(2m+1)\pi\big/(n+p)$*. Similarly, if* $\min \mathcal{S}_-^{(1)} = \min \mathcal{S}_-^{(2)}$*, it is equal to* $(2m-1)\pi\big/(n+p)$*. Such degeneration may happen when* $\cos\bigl(p(2q'+1)\pi/(n+p)+\kappa\bigr)=0$ *holds for some integer* q'*.*

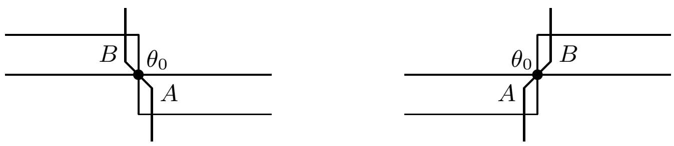
Proof. We indicate only an outline. We put $\theta_1 := \min \mathcal{S}_+^{(1)} = \min \mathcal{S}_+^{(2)}$. As in the proof of Lemma 3.2, we can easily observe that $\theta_1 \leq (2m+1)\pi/(n+p)$. Because $\cos(p\theta_1+\kappa) = \cos(n\theta_1-\kappa) = 0$, we obtain $p\theta_1+\kappa = \pm(n\theta_1-\kappa)+\ell\pi$ for some integer ℓ. By the argument in the proof of Lemma 3.2, we can show that θ_1 is not of the form $\bigl((2q+1)\pi+2\kappa\bigr)\big/(n-p)$. By a similar argument, we can show that θ_1 is not of the form $2q\pi/(n+p)$. If θ_1 is of the form $\bigl(2q\pi+2\kappa\bigr)\big/(n-p)$, it is easy to observe that $\cos(p\theta_1+\kappa)=0$ implies θ_1 is of the form $(2q'+1)\pi\big/(n+p)$ for some integer q', similarly to Lemma 3.1. Hence, we can conclude that $\theta_1 = (2m+1)\pi\big/(n+p)$. □

The case $\cos(p\theta_0+\kappa)=0$. We set $\varphi := \theta-\theta_0$. Because $-n\theta_0 = -2m\pi+p\theta_0$, we have $\cos\bigl(-n\theta_0-n\varphi+\kappa\bigr) = \cos(p\theta_0-n\varphi+\kappa)$, and hence

$$\operatorname{Re} F(r,\theta) = \frac{r^{-n}}{n}\cos(p\theta_0-n\varphi+\kappa)+\frac{r^p}{p}\cos(p\theta_0+p\varphi+\kappa).$$

If $|\varphi|$ is sufficiently small, $\cos(p\theta_0-n\varphi+\kappa)$ and $\cos(p\theta_0+p\varphi+\kappa)$ have the opposite signatures. Hence, if we take φ such that $\cos(p\theta_0+p\varphi+\kappa)>0$ (resp. $\cos(p\theta_0+p\varphi+\kappa)<0$), $\operatorname{Re} F(r,\theta_0+\varphi)$ is increasing (resp. decreasing) with respect to r.

We take A and B in a neighbourhood of $e^{\sqrt{-1}\theta_0}$ such that (i) $A\in\{r<1\}$ and $B\in\{r>1\}$, (ii) the segment $\mathrm{Seg}[A,B]$ is a steepest descent for $\operatorname{Re} F$ at $e^{\sqrt{-1}\theta_0}$. Then, let Γ_m be the union of $\mathrm{Seg}[A,B]$, $\mathrm{Ray}(0,A)$ and $\mathrm{Ray}(B,\infty)$.



3.1.3. *Perturbation.* We fix $a_j \in \mathbb{C}$ $(j = 1, \dots, n-1)$. We set

$$\widetilde{F}(\zeta, v) = e^{\sqrt{-1}\kappa} \left(\frac{v^{-n}}{n} + \frac{v^p}{p} \right) + \sum_{j=1}^{n-1} a_j \, v^{-j} \zeta^{n-j}$$

We fix a sufficiently small $\epsilon_0 > 0$. There exists $\delta_0 > 0$ with the following property:

- **(A1):** Recall that $[n, p, m]$ is a solution of $\partial_v F(v) = 0$. If $|\zeta| < \delta_0$, the ϵ_0-neighbourhood of $[n, p, m]$ contains one solution of $\partial_v \widetilde{F}(\zeta, v) = 0$, which is denoted by $\nu_m(\zeta)$.
- **(A2):** There exist a relatively compact subset $U \subset \mathbb{C}_v^*$ and $C_i > 0$ $(i = 1, 2)$ such that

$$\mathrm{Re}\big(\zeta^{-\ell} \widetilde{F}(\zeta, v)\big) \leq -C_1 \, |\zeta|^{-\ell} \left(|v|^{-n} + |v|^p \right)$$

  for any ζ with $0 < |\zeta| < \delta_0$ and $\big|\arg(\zeta)\big| < C_2$, and for any $v \in \Gamma_m \setminus U$.

We can use the path Γ_m to construct a family of cycles later. But, for the estimate of the asymptotic behaviour of the pairings, we take a family of paths $\widetilde{\Gamma}_m : \Delta_\zeta(\delta_1) \times [0, 1] \longrightarrow \Delta_\zeta(\delta_1) \times \mathbb{C}_v$ satisfying (B1–3) below for some $\delta_1 < \delta_0$, by slightly modifying Γ_m. The restriction $\widetilde{\Gamma}_{m|\zeta \times [0,1]}$ is denoted by $\widetilde{\Gamma}_{m,\zeta}$. Since a parametrization is not significant, we will not distinguish $\widetilde{\Gamma}_{m,\zeta}$ and its image.

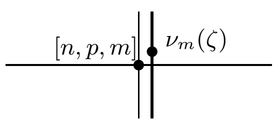
- **(B1):** $\widetilde{\Gamma}_{m,\zeta}$ is the union of arcs and segments for each $\zeta \in \Delta_\zeta(\delta_1)$, and $\nu_m(\zeta) \in \widetilde{\Gamma}_{m,\zeta}$.
- **(B2):** There exist a neighbourhood $\widetilde{U}$ of $[n, p, m]$ such that we can apply the result in Subsection 2.3 to the function $\widetilde{F}$ and the family of the paths $\widetilde{\Gamma}_m$ on $\Delta_\zeta(\delta_1) \times \widetilde{U}$.
- **(B3):** There exist $\widetilde{C}_i > 0$ $(i = 1, 2)$ such that

$$\mathrm{Re}\big(\zeta^{-\ell} \widetilde{F}(\zeta, v)\big) \leq -\widetilde{C}_1 \, |\zeta|^{-\ell} \left(|v|^{-n} + |v|^p \right) + \mathrm{Re}\Big(\zeta^{-\ell} \widetilde{F}\big(\zeta, \nu_m(\zeta)\big)\Big)$$

  for any ζ with $0 < |\zeta| < \delta_1$ and $|\arg(\zeta)| < \widetilde{C}_2$, and for any $v \in \widetilde{\Gamma}_{m,\zeta} \setminus \widetilde{U}$.

We modify Γ_m as follows. Let $\theta_0 := [\![n, p, m]\!]$.

The case $\cos(p\theta_0 + \kappa) < 0$. Let $\widetilde{\Gamma}_{m,\zeta}$ be the ray connecting 0 and ∞ through $\nu_m(\zeta)$.



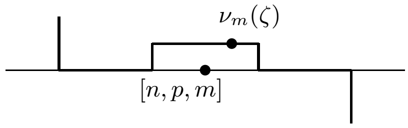
The case $\cos(p\theta_0 + \kappa) > 0$. Let $r_{10}(\zeta) := |\nu_m(\zeta)|$. Take a small $\epsilon_0 > 0$. The path $\widetilde{\Gamma}_{m,\zeta}$ is the union of the following:

- The restriction of Γ_m to $\big\{u \,\big|\, |\arg(u) - \theta_0| > \epsilon_0\big\}$.

- The segments

$$\mathrm{Seg}\big[e^{\sqrt{-1}(\theta_0-\epsilon_0)}, r_{10}(\zeta)\, e^{\sqrt{-1}(\theta_0-\epsilon_0)}\big], \quad \mathrm{Seg}\big[e^{\sqrt{-1}(\theta_0+\epsilon_0)}, r_{10}(\zeta)\, e^{\sqrt{-1}(\theta_0+\epsilon_0)}\big].$$

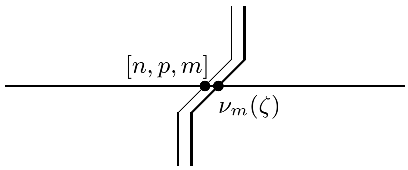
- The arc $\mathrm{Arc}\big(r_{10}(\zeta)\, e^{\sqrt{-1}(\theta_0-\epsilon_0)},\ r_{10}(\zeta)\, e^{\sqrt{-1}(\theta_0+\epsilon_0)}\big)$.



The case $\cos(p\theta_0+\kappa)=0$. We perturb A and B as follows:

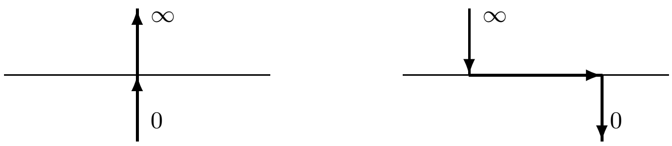
$$A(\zeta) := A + \big(\nu_m(\zeta) - [n,p,m]\big), \qquad B(\zeta) := B + \big(\nu_m(\zeta) - [n,p,m]\big)$$

Then, $\widetilde{\Gamma}_{m,\zeta}$ is the union of the segments $\mathrm{Seg}[B(\zeta),\infty]$, $\mathrm{Seg}[B(\zeta),A(\zeta)]$ and $\mathrm{Seg}[A(\zeta),0]$.



3.1.4. *Orientation.* For each $[n,p,m]$, we may choose an orientation of the path Γ_m by the following rule. If Γ_m is a ray connecting 0 and ∞, the orientation is from 0 to ∞. Otherwise, Γ_m contains an arc or a segment around $[n,p,m]$, which can be parameterized by θ, and the orientation of Γ_m is given by the parametrization.

Since $\widetilde{\Gamma}_m$ are obtained as small perturbation of Γ_m, the orientations of $\widetilde{\Gamma}_m$ are also induced.



3.2. **Cycles of elementary meromorphic flat bundle.**

3.2.1. *Construction.* We set $\mathfrak{a} := \alpha\, u^{-n} + \sum_{j=1}^{n-1} \alpha_j\, u^{-j}$ for some $\alpha, \alpha_j \in \mathbb{C}$. We assume $\alpha \neq 0$. Let $\mathcal{L}$ be a meromorphic flat line bundle $\mathcal{O}_{\mathbb{P}^1}\big(*\{0,\infty\}\big)\, e$ with $\nabla e = e\, d\mathfrak{a}$. Let R be a meromorphic flat bundle on $(\mathbb{P}^1_u, \{0,\infty\})$ with regular singularity. Let $q : \mathbb{P}^1_u \longrightarrow \mathbb{P}^1_t$ be given by $q(u) = u^p$. We set $V := q_*\big(\mathcal{L} \otimes R\big)$.

Let $z = \tau^{-1}$ be the coordinate of $\mathbb{P}^1_\tau$ around ∞_τ. We would like to construct a tuple of flat sections of $\mathfrak{Four}(V)^\vee$ on small sectors in Δ_z^*. We take a ramified covering $\psi_\zeta : \Delta_\zeta \longrightarrow \Delta_z$ given by $z = \zeta^{n+p}$. Let $\pi : \widetilde{\Delta}_\zeta(0) \longrightarrow \Delta_\zeta$ be the real blow up. We consider a sector $\mathcal{S}$ in Δ_ζ^* around $e^{\sqrt{-1}\Theta} \in \pi^{-1}(0)$.

We fix an $(n+p)$-th root $\beta e^{\sqrt{-1}\mu}$ of α, where $\beta > 0$ and $\mu \in \mathbb{R}$. We take an isomorphism $\Phi : \Delta_\zeta \times \mathbb{C}_v \simeq \Delta_\zeta \times \mathbb{C}_u$ given by

$$v := \Big(\frac{p}{n}\Big)^{1/(n+p)} \zeta^{-1}\, \beta^{-1}\, e^{-\sqrt{-1}\mu}\, u.$$

We set $\zeta_1 = \zeta\, e^{-\sqrt{-1}\Theta}$. Then, we have

$$(3.5)\quad \alpha\, u^{-n} + \sum_{j=1}^{n-1} \alpha_j u^{-j} + \frac{u^p}{z} = \beta^p\, p^{n/(n+p)}\, n^{p/(n+p)}\, \zeta_1^{-n} \times e^{\sqrt{-1}(p\mu - n\Theta)} \times \left(\frac{v^{-n}}{n} + \frac{v^p}{p} + \sum_{j=1}^{n-1} \alpha_j\, p^{(j-n)/(n+p)}\, n^{-(p+j)/(n+p)}\, e^{\sqrt{-1}(p+j)\mu + \sqrt{-1}(n-j)\Theta}\, \zeta_1^{n-j}\, v^{-j}\right)$$

We obtain the paths Γ_m and $\widetilde{\Gamma}_{m,\zeta}$ in $\mathbb{C}_v$, by applying the construction in Subsection 3.1.2, with

$$\kappa := p\mu - n\Theta, \qquad F = e^{\sqrt{-1}(p\mu - n\Theta)}\big(v^{-n}/n + v^p/p\big),$$

$$\widetilde{F} = e^{\sqrt{-1}(p\mu - n\Theta)} \Bigg(\frac{v^{-n}}{n} + \frac{v^p}{p} + \sum_{j=1}^{n-1} \alpha_j\, p^{(j-n)/(n+p)}\, n^{-(p+j)/(n+p)}\, e^{\sqrt{-1}(p+j)\mu + \sqrt{-1}(n-j)\Theta} \zeta_1^{n-j} v^{-j}\Bigg).$$

By the correspondence Φ, we obtain the families of paths $\Gamma_{m,\Theta,\mathfrak{a}}(\zeta)$ and $\widetilde{\Gamma}_{m,\Theta,\mathfrak{a}}(\zeta)$ in $\mathbb{C}_u$. They are equipped with the orientations by the rule in Subsection 3.1.4. Let $\widetilde{\nu}_{m,\Theta,\mathfrak{a}} : \mathcal{S} \longrightarrow \mathcal{S} \times \mathbb{C}_u$ denote the section corresponding to $\nu_m : \mathcal{S} \longrightarrow \mathcal{S} \times \mathbb{C}_v$.

If $\mathcal{S}$ is sufficiently small, there exists $\theta_1 \in \mathbb{R}$ and $\delta > 0$ such that the paths $\Gamma_{m,\Theta,\mathfrak{a}}(\zeta)$ and $\widetilde{\Gamma}_{m,\Theta,\mathfrak{a}}(\zeta)$ $(\zeta \in \mathcal{S})$ are contained in a region $\mathcal{T}_{m,\Theta,\mathfrak{a}}$ of the form $\big\{\theta_1 \le \arg(u) \le \theta_1 + \pi/n + \delta\big\}$ in $\mathbb{C}_u^*$.

Let s be a flat section of $(\mathcal{L} \otimes R)^\vee$ on $\mathcal{T}_{m,\Theta,\mathfrak{a}}$. It induces a flat section $\exp(u^p/z)\, s$ of $\big(\mathcal{L} \otimes R\big)^\vee \otimes L(-u^p/z)$ on $\mathcal{T}_{m,\Theta,\mathfrak{a}}$. We obtain the families of cycles $\exp(u^p/z)\, s \otimes \widetilde{\Gamma}_{m,\Theta,\mathfrak{a}}(\zeta)$ for $\big(\mathcal{L} \otimes R\big)^\vee \otimes L(-u^p/z)$. By the push-forward we obtain families of cycles with values in $V^\vee \otimes L(-t/z)$, which are denoted by $\Delta_{m,\Theta,\mathfrak{a}}(s;\zeta)$. Let $\Xi_{m,\Theta,\mathfrak{a}}(s)$ denote the induced flat section of $\mathfrak{Four}(V)^\vee$ on $\mathcal{S}$. (See Subsection 2.5 for the flat section induced by a family of cycles.)

Remark 3.4. Although $\widetilde{\Gamma}_{m,\Theta,\mathfrak{a}}(\zeta)$ will be useful for the estimate of pairings, we can use $\Gamma_{m,\Theta,\mathfrak{a}}(\zeta)$ for construction of $\Xi_{m,\Theta,\mathfrak{a}}(s)$. □

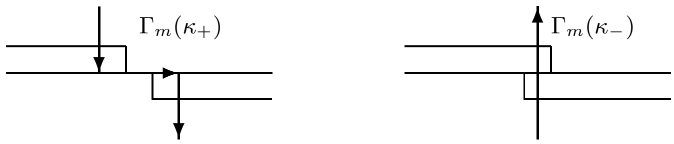
3.2.2. *Change of flat sections for variation of* Θ. Let s be a flat section of $V^\vee$ on $\mathcal{T}_{m,\Theta,\mathfrak{a}}$. Let us describe how the sections $\Xi_{m,\Theta,\mathfrak{a}}(s)$ change for variation of Θ. We only state the formulas, which can be checked easily. We set $\kappa_0 := p\mu - n\Theta_0$ and $\theta_0 = [\![n,p,m]\!]$. Let $\epsilon > 0$ be sufficiently small, and we set $\kappa_\pm = p\mu - n\Theta_0 \pm n\epsilon$.

We use the symbol $\Gamma_m(\kappa_\pm)$ denote the path Γ_m in Subsection 3.1 to indicate the dependence on $\kappa_\pm$.

First let us describe the case $n \neq p$. In the case $\cos(p\theta_0 + \kappa_0) = 0$ and $\sin(p\theta_0 + \kappa_0) = -1$, we have

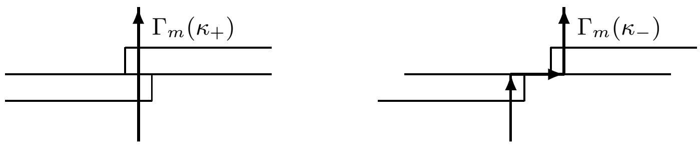
$$\Xi_{m,\Theta_0-\epsilon,\mathfrak{a}}(s) = -\Xi_{m,\Theta_0+\epsilon,\mathfrak{a}}(s)$$

See the following picture.



In the case $\cos(p\theta_0 + \kappa_0) = 0$ and $\sin(p\theta_0 + \kappa_0) = 1$, we have

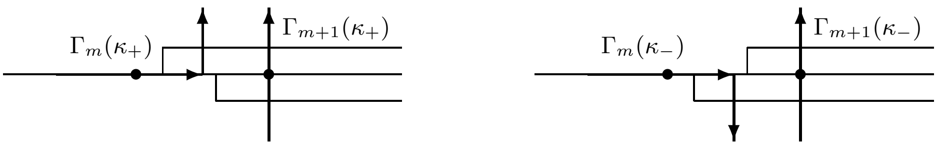
$$\Xi_{m,\Theta_0-\epsilon,\mathfrak{a}}(s) = \Xi_{m,\Theta_0+\epsilon,\mathfrak{a}}(s)$$



In the case $\cos\Big(p(2m+1)\pi/(n+p)+\kappa_0\Big) = 0$ and $\sin\Big(p(2m+1)\pi/(n+p)+\kappa_0\Big) = 1$, we have

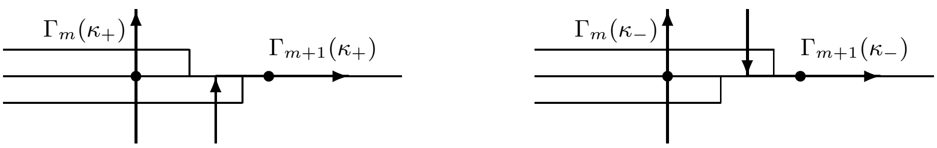
$$\Xi_{m+1,\Theta_0-\epsilon,\mathfrak{a}}(s') = \Xi_{m+1,\Theta_0+\epsilon,\mathfrak{a}}(s')$$
$$\Xi_{m,\Theta_0-\epsilon,\mathfrak{a}}(s) = \Xi_{m,\Theta_0+\epsilon,\mathfrak{a}}(s) + \Xi_{m+1,\Theta_0+\epsilon,\mathfrak{a}}(s')$$

Here, s' is the flat section on $\mathcal{T}_{m+1,\Theta_0,\mathfrak{a}}$ naturally obtained as the parallel transform of s.



In the case $\cos\Big(p(2m+1)\pi/(n+p)+\kappa_0\Big) = 0$ and $\sin\Big(p(2m+1)\pi/(n+p)+\kappa_0\Big) = -1$, we have

$$\Xi_{m,\Theta_0-\epsilon,\mathfrak{a}}(s) = \Xi_{m,\Theta_0+\epsilon,\mathfrak{a}}(s)$$
$$\Xi_{m+1,\Theta_0-\epsilon,\mathfrak{a}}(s') = \Xi_{m+1,\Theta_0+\epsilon,\mathfrak{a}}(s') + \Xi_{m,\Theta_0+\epsilon,\mathfrak{a}}(s)$$



Let us describe the case $n = p$. In the case $\cos(p\theta_0+\kappa_0) = 0$ and $\sin(n\theta_0+\kappa_0) = -1$, we have

$$\Xi_{m,\Theta_0-\epsilon,\mathfrak{a}}(s) = -\Xi_{m,\Theta_0+\epsilon,\mathfrak{a}}(s)$$

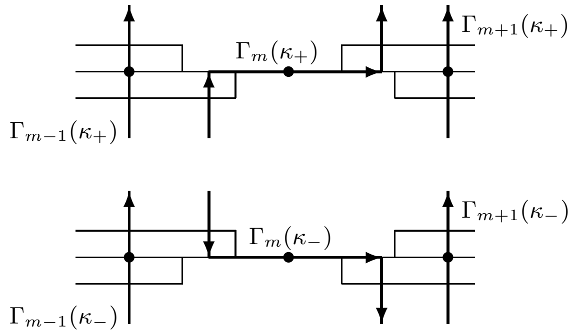
In the case $\cos(p\theta_0+\kappa_0)=0$ and $\sin(n\theta_0+\kappa_0)=1$, we have

$$\Xi_{m,\Theta_0-\epsilon,\mathfrak{a}}(s)=\Xi_{m,\Theta_0+\epsilon,\mathfrak{a}}(s)$$

In the case $\kappa\equiv\pm\pi/2$ modulo 2π, and $\cos(n\theta_0+\kappa)>0$, we have

$$\Xi_{m,\Theta_0-\epsilon,\mathfrak{a}}(s)=\Xi_{m,\Theta_0+\epsilon,\mathfrak{a}}(s)+\Xi_{m-1,\Theta_0+\epsilon,\mathfrak{a}}(s^{(m-1)})+\Xi_{m+1,\Theta_0+\epsilon,\mathfrak{a}}(s^{(m+1)})$$

Here, $s^{(j)}$ be the flat sections on $\mathcal{T}_{j,\Theta_0,\mathfrak{a}}$ naturally obtained as the parallel transport of s.



3.3. **Estimate of pairings.**

3.3.1. *Preliminary.* We continue to use the notation in Subsection 3.2. We give a preparation for the estimate of the pairings between the cycles $\Xi_{m,\Theta,\mathfrak{a}}(s)$ and a meromorphic section of V.

We will use the following standard estimates without mention:

$$\int_C^{|\zeta^{-1}|}\exp\bigl(-\epsilon r\,|\zeta^{-n}|\bigr)r^N\,dr\leq\int_C^{\infty}\exp\bigl(-\epsilon r\,|\zeta^{-n}|\bigr)\,r^N\,dr=O\Bigl(\exp\bigl(-\epsilon'|\zeta^{-n}|\bigr)\Bigr),$$

$$\int_0^C\exp\bigl(-\epsilon r^{-1}|\zeta^{-n}|\bigr)\,r^N\,dr=O\Bigl(\exp\bigl(-\epsilon'\,|\zeta^{-n}|\bigr)\Bigr)\quad(C>0)$$

Here, N is any real number.

Let $S=S[r_0;\theta_1,\theta_2]$ be a small sector in $\mathbb{C}_u^*$. Let f be a holomorphic function on S such that $|f|=O\bigl(u^M\,(\log u)^j\bigr)$ for some $M\in\mathbb{R}$ and $j\in\mathbb{Z}_{\geq 0}$. We set

$$h(u,\zeta):=u^p/\zeta^{n+p}+\mathfrak{a}(u),\qquad H_{S,f}(\zeta):=\int_{\widetilde{\Gamma}_{m,\Theta,\mathfrak{a}}\cap S}\exp\bigl(h(u,\zeta)\bigr)\,f\,du.$$

We also put $\widehat{\mathfrak{a}}_{m,\Theta}(\zeta):=h\bigl(\widetilde{\nu}_{m,\Theta,\mathfrak{a}}(\zeta),\zeta\bigr)$.

Lemma 3.5. *If $S\not\ni\widetilde{\nu}_{m,\Theta,\mathfrak{a}}(\zeta)$, there exists $\epsilon>0$ such that*

$$H_{S,f}(\zeta)=O\Bigl(\exp\bigl(\widehat{\mathfrak{a}}_{m,\Theta}(\zeta)-\epsilon|\zeta|^{-n}\bigr)\Bigr).\tag{3.6}$$

If $S \ni \widetilde{\nu}_{m,\Theta,\mathfrak{a}}(\zeta)$, *we have the following estimate:*

$$(3.7) \quad H_{S,f}(\zeta) = \exp\bigl(\widehat{\mathfrak{a}}_{m,\Theta}(\zeta)\bigr) \left(\Bigl(\frac{2\pi}{-\partial_u^2 h(u,\zeta)}\Bigr)^{1/2} f\right)_{|u=\widetilde{\nu}_{m,\Theta,\mathfrak{a}}(\zeta)} + \exp\bigl(\widehat{\mathfrak{a}}_{m,\Theta}(\zeta)\bigr)\, O\Bigl(\zeta^{M+n+1}(\log\zeta)^j\Bigr)$$

Note that $\widetilde{\nu}_{m,\Theta,\mathfrak{a}}(\zeta) \sim \zeta$ *and* $\partial_u^2 h\bigl(\widetilde{\nu}_{m,\Theta,\mathfrak{a}}(\zeta),\zeta\bigr)^{-1/2} \sim \zeta^{n/2+1}$. *Hence, if* $f \sim u^M(\log u)^j$, *we have*

$$H_{S,f}(\zeta) \sim \exp\bigl(\widehat{\mathfrak{a}}_{m,\Theta}(\zeta)\bigr)\, \zeta^{M+n/2+1}\bigl(\log\zeta\bigr)^j.$$

For any $N > 0$, *there exists* $M > 0$ *such that the following holds for any* f *with* $|f| = O\bigl(|u|^M\bigr)$:

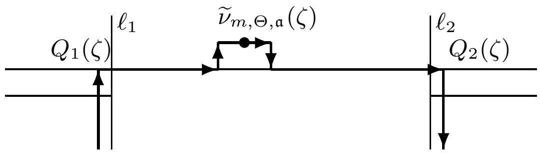
$$H_{S,f}(\zeta) = O\Bigl(\exp\bigl(\widehat{\mathfrak{a}}_{m,\Theta}(\zeta)\bigr)\,|\zeta|^N\Bigr)$$

Proof. If $S \not\ni \widetilde{\nu}_{m,\Theta,\mathfrak{a}}(\zeta)$, there exists $\epsilon' > 0$ such that the following holds for $u = v\,\zeta \in \widetilde{\Gamma}_{m,\Theta,\mathfrak{a}} \cap S$:

$$\mathrm{Re}\, h(u,\zeta) \le \mathrm{Re}\, \widehat{\mathfrak{a}}_{m,\Theta}(\zeta) - \epsilon'\,|\zeta|^{-n}\,\bigl(|v|^{-n} + |v|^p\bigr)$$

Then, we obtain (3.6). If $S \ni \widetilde{\nu}_{m,\Theta,\mathfrak{a}}(\zeta)$, we obtain (3.7) by using Lemma 2.7 and the condition for the family of paths $\widetilde{\Gamma}_m$. The other claims immediately follow. □

We give a related estimate for our later use (Section 5). Let $\theta_j \in \mathbb{R}$ $(j = 1,2)$ satisfy (i) $\mathrm{Re}\bigl(\alpha\, e^{-n\sqrt{-1}\theta_i}\bigr) = 0$, (ii) $\theta_2 - \theta_1 = \pi/n$, (iii) we have $\mathrm{Re}\bigl(\alpha\, e^{-n\sqrt{-1}\theta}\bigr) > 0$ for $\theta_1 < \theta < \theta_2$. Let us consider the case $\widetilde{\Gamma}_{m,\Theta,\mathfrak{a}}(\zeta)$ contains rays close to the half lines $\ell_i := \{\arg(u) = \theta_i\}$ $(i = 1,2)$. (See the case $\cos(p\theta_0 + \kappa) > 0$ in Subsection 3.1.2.)



For $\beta \in \mathbb{C}$ with $\mathrm{Re}(\beta\, e^{-n\sqrt{-1}\theta_1}) \le 0$, we take $\varphi_1(\beta) \in \mathbb{R}$ such that (i) $|\varphi_1(\beta) - \theta_1| \ne 0$ is sufficiently small, (ii) $\mathrm{Re}\bigl(\beta\, e^{-n\sqrt{-1}\varphi_1(\beta)}\bigr) < 0$. We set $r(\zeta) := |Q_i(\zeta)|$. Let $\Gamma^{(1)}_\beta$ be the union of the ray $\mathrm{Ray}[0, r(\zeta)\, e^{\sqrt{-1}\varphi_1(\beta)}]$ and the arc $\mathrm{Arc}\bigl(r(\zeta)\, e^{\sqrt{-1}\varphi_1(\beta)}, Q_1(\zeta)\bigr)$.

We set $\mathfrak{b} := \beta u^{-n} + \sum_{j=1}^{n-1} \beta_j u^{-j}$ for some $\beta_j \in \mathbb{C}$, and $\Gamma^{(1)}_{\mathfrak{b}} := \Gamma^{(1)}_\beta$. There exists $\epsilon > 0$ such that the following holds for any $u \in \Gamma^{(1)}_{\mathfrak{b}}$, if $|\zeta|$ is sufficiently small:

$$\mathrm{Re}\Bigl(\mathfrak{b}(u) + u^p/\zeta^{n+p}\Bigr) < \mathrm{Re}(\widehat{\mathfrak{a}}_{m,\Theta}(\zeta)) - \epsilon|u^{-n}|$$

Let $\mathcal{T}_1$ be a sector around ℓ_1. Let f be a holomorphic function on $\mathcal{T}_1$ such that $|f| = O\big(|u|^M\big)$ for some M. Then, there exists $\epsilon' > 0$ such that the following holds:

$$\int_{\mathcal{T}_1 \cap \Gamma^{(1)}_{\mathfrak{b}}} \exp(\mathfrak{b} + u^p/\zeta^{n+p})\, f\, du = O\Big(\exp\big(\widehat{\mathfrak{a}}_{m,\Theta}(\zeta) - \epsilon'|\zeta^{-n}|\big)\Big) \tag{3.8}$$

We have similar modifications and estimates for $\beta \in \mathbb{C}$ satisfying $\mathrm{Re}(\beta e^{\sqrt{-1}\theta_2}) \leq 0$.

3.3.2. *Estimate of pairings.* For simplicity, we assume that the monodromy of R has a unique eigenvalue ω. We set $r := \operatorname{rank} R$. We take flat frames $\boldsymbol{s}_m = \big(s_{m,i} \,\big|\, i = 1, \ldots, r\big)$ of $(\mathcal{L} \otimes R)^\vee$ on $\mathcal{T}_{m,\Theta,\mathfrak{a}}$. We obtain flat sections $U_{m,i} := \Xi_{m,\Theta,\mathfrak{a}}(s_{m,i})$ of $\mathfrak{Four}(V)^\vee$ on $\mathcal{S}$. We set $\boldsymbol{U}_m := \big(U_{m,i}\big)$.

Let E be the Deligne-Malgrange lattice of $\mathcal{L} \otimes R$. Let $\boldsymbol{w} = (w_1, \ldots, w_r)$ be a frame of E on $\mathbb{C}_u$. Let $\Upsilon_{k,l}$ denote the sections of $\mathfrak{Four}(V)$ around ∞_τ induced by $u^{-k}\, w_l$. The tuple

$$\boldsymbol{\Upsilon} = \big(\Upsilon_{k,l} \,\big|\, k = 1, \ldots, n+p,\ l = 1, \ldots, r\big)$$

gives a frame of $\mathfrak{Four}(V)$ around ∞_τ.

Let α be the eigenvalue of the residue $\mathrm{Res}(\nabla) \in \mathrm{End}(E_{|O})$. We set

$$a_{(m,i),(k,l)} := \frac{\big\langle U_{m,i}, \Upsilon_{k,l}\big\rangle}{\exp\big(\mathfrak{a}_m(\zeta)\big)\, \zeta^{\alpha-k+n/2+p}}.$$

By taking a bijection $\mu : \{1, \ldots, n+p\} \times \{1, \ldots, r\} \simeq \{1, \ldots, (n+p)r\}$, we obtain the matrix-valued function $\mathcal{A} := \det\big(a_{\mu(m,i),\mu(k,l)}\big)$ on $\mathcal{S}$.

Proposition 3.6. *The entries of $\mathcal{A}$ are bounded up to log order. There exist a non-zero complex number C and a positive number δ such that* $\det(\mathcal{A}) = C + O(|\zeta|^\delta)$.

Proof. The following argument is essentially given in [3]. We fix a branch of $\log u$ on $\mathcal{T}_{m,\Theta,\mathfrak{a}}$ for each m. For each m, i and l, let $f(u)$ be the function determined by $\big\langle s_{m,i}, w_l\big\rangle = \exp\big(-\mathfrak{a}(u)\big)\, u^\alpha\, f(u)$. Then, $f(u)$ is bounded up to log order around $u = 0$, and of polynomial order around $u = \infty$.

By using Lemma 3.5, we obtain the following estimate:

$$\begin{aligned} \big\langle U_{m,i},\ \Upsilon_{k,l}\big\rangle &= \int_{\widetilde{\Gamma}_{m,\Theta,\mathfrak{a}}(\zeta)} \exp(u^p/z) \langle s_{m,i},\, u^{-k}\, w_l\rangle\, p u^{p-1}\, du \\ &= \int_{\widetilde{\Gamma}_{m,\Theta,\mathfrak{a}}(\zeta)} p \exp\big(h(u,\zeta)\big) u^{\alpha-k+p-1} f(u)\, du \\ &= p \exp\big(\widehat{\mathfrak{a}}_{m,\Theta}(\zeta)\big) \left(\Big(\frac{2\pi}{-\partial_u^2 h(u,\zeta)}\Big)^{1/2} u^{\alpha-k+p-1} f(u) \right)_{|u=\widetilde{\nu}_{m,\Theta,\mathfrak{a}}(\zeta)} \\ &\qquad + \exp\big(\widehat{\mathfrak{a}}_{m,\Theta}(\zeta)\big)\, O\big(\zeta^{\alpha-k+n+p} (\log \zeta)^{-M}\big) \end{aligned} \tag{3.9}$$

Note that $\partial_u^2 h\big(\widetilde{\nu}_{m,\Theta,\mathfrak{a}}(\zeta),\zeta\big)^{-1/2} \sim \zeta^{n/2+1}$ and $\widetilde{\nu}_{m,\Theta,\mathfrak{a}}(\zeta) \sim \zeta$. Thus, we obtain the first claim.

To show the second claim, we have only to consider the case $r=1$. We may assume that (i) $\boldsymbol{w}$ is given by a section e of E, (ii) $\boldsymbol{s}_m$ are given by $\exp(\mathfrak{a})\, u^{\alpha}\, e^{\vee}$, where $e^{\vee}$ is the dual frame of $E^{\vee}$. Then, the above f is constantly 1. Hence, we obtain the following for some $\delta>0$:

$$(3.10)\quad \det\Big(\big\langle U_m,\ \Upsilon_k\big\rangle\ \Big|\ k,m=1,\dots,n+p\Big) \\ =\prod_{m=1}^{n+p}\Big(\frac{2\pi}{-h''\big(\widetilde{\nu}_{m,\Theta,\mathfrak{a}}(\zeta),\zeta\big)}\Big)^{1/2}\widetilde{\nu}_{m,\Theta,\mathfrak{a}}(\zeta)^{\alpha+p-1} \\ \times\det\Big(p\,\widetilde{\nu}_{m,\Theta,\mathfrak{a}}(\zeta)^{-k}\ \Big|\ k,m=1,\dots,n+p\Big)\times\Big(1+O(|\zeta|^{\delta})\Big)$$

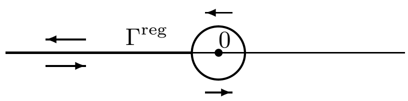
Then, the second claim follows. □

Corollary 3.7.

- *The tuple of flat sections* $\boldsymbol{U}:=\bigcup_m \boldsymbol{U}_m$ *is a flat frame of* $\mathfrak{Four}(V)^{\vee}_{|\mathcal{S}}$.
- *Let* $\mathfrak{Four}(V)^{\vee}_{m,\mathcal{S}}$ *denote the flat subbundle of* $\mathfrak{Four}(V)^{\vee}_{|\mathcal{S}}$ *generated by* $\boldsymbol{U}_m$. *Then, the decomposition* $\mathfrak{Four}(V)^{\vee}_{|\mathcal{S}}=\bigoplus\mathfrak{Four}(V)^{\vee}_{m,\mathcal{S}}$ *is a flat splitting of the full Stokes filtration of* $\mathfrak{Four}(V)^{\vee}_{|\mathcal{S}}$. *In particular,* $\boldsymbol{U}$ *is compatible with the full Stokes filtration.* □

3.4. **Regular singular case.** Let (V,∇) be a regular singular meromorphic flat bundle, whose poles are contained in $\{0,\infty\}$. We take cycles for $V^{\vee}\otimes L(-t/z)$, and construct a flat frame of $\mathfrak{Four}(V)^{\vee}$ on small sectors of Δ_z^*.

In a complex line $\mathbb{C}_v$, we take a path Γ^{reg} as follows. We come from ∞ to $-\epsilon$ along the ray, then we go around 0 along the circle $\{|z|=\epsilon\}$ in the counter-clockwise direction, then go back to ∞ from $-\epsilon$ along the ray. We also fix a branch of $\log v$ along Γ^{reg}. By $v=z^{-1}t$, we obtain a family of paths $\Gamma^{\mathrm{reg}}(z)$ in $\mathbb{C}_t$.



Let $\pi:\widetilde{\Delta}_z(0)\longrightarrow\Delta_z$ be the real blow up. Let $e^{\sqrt{-1}\Theta}$ be a point of $\pi^{-1}(0)$. Let $\mathcal{S}$ be a small sector around $e^{\sqrt{-1}\Theta}$. There exists a sector $\mathcal{T}_{1,\Theta,0}$ which contains $\big\{z\epsilon\,\big|\,z\in\mathcal{S}\big\}$. Let s be any flat section of $V^{\vee}$ on $\mathcal{T}_{1,\Theta,0}$. We have the induced flat section of $V^{\vee}$ along $\Gamma^{\mathrm{reg}}(z)$ for each $z\in\mathcal{S}$, which is also denoted by s. Then, we obtain a family of cycles $\exp\big(t/z\big)\,s\otimes\Gamma^{\mathrm{reg}}(z)$ for $V^{\vee}\otimes L(-t/z)$, which gives a flat section $\Xi_{1,\Theta,0}(s)$ of $\mathfrak{Four}(V)^{\vee}$ on $\mathcal{S}$. If we take a frame $\boldsymbol{s}=(s_i)$ of $V^{\vee}$ on $\mathcal{T}_{1,\Theta,0}$, the induced sections $U_i:=\Xi_{1,\Theta,0}(s_i)$ give a frame of $\mathfrak{Four}(V)^{\vee}_{|\mathcal{S}}$, which can be shown easily by using a general theory in [4]. The estimate of the pairing is remarked in [3], which we recall for our later use.

For simplicity, we assume that V has a unique eigenvalue. We take a frame $\boldsymbol{w} = (w_i)$ of the Deligne lattice E. Let Υ_i be the section of $\mathfrak{Four}(V)$ induced by $w_i\, dt/t$. Let α be the unique eigenvalue of $\mathrm{Res}(\nabla) \in \mathrm{End}(E_{|O})$. We set

$$a_{i,j} := \frac{\big\langle U_i, \Upsilon_j \big\rangle}{z^\alpha}$$

We obtain the matrix valued function $\mathcal{A} = (a_{i,j})$ on $\mathcal{S}$.

Lemma 3.8. *The entries of $\mathcal{A}$ are bounded up to log order. There exists a non-zero complex number C such that* $\det \mathcal{A} = C + O(|z|)$.

Proof. For each i, j, we have the expansion $\big\langle s_i, w_j \big\rangle = u^\alpha \sum a_l(z)\,(\log z)^l$, where $a_l(z)$ are polynomial of z. We obtain

$$\big\langle U_i, \Upsilon_j \big\rangle = z^\alpha \sum \int_{\Gamma^{\mathrm{reg}}} \exp(v)\, v^\alpha\, a_l(zv) \big(\log(zv)\big)^l\, dv/v$$

. Then, the first claim is clear. As for the second claim, we have only to consider the case $\mathrm{rank}\, V = 1$, which can be shown easily by a direct calculation. □

4. Elementary cases around ∞

4.1. Paths with some nice property.

Let $\kappa \in \mathbb{R}$. Let $n > p$. We consider the following holomorphic function on $\mathbb{C}_v^*$:

$$F(v) := e^{\sqrt{-1}\kappa} \left(\frac{v^{-n}}{n} - \frac{v^{-p}}{p} \right) \tag{4.1}$$

For the polar coordinate $v = re^{\sqrt{-1}\theta}$, we have

$$\mathrm{Re}\, F(r, \theta) = \frac{r^{-n}}{n} \cos\big(n\theta - \kappa\big) - \frac{r^{-p}}{p} \cos\big(p\theta - \kappa\big).$$

Because $F'(v) = e^{\sqrt{-1}\kappa}\big(-v^{-n-1} + v^{-p-1}\big)$, we have $F'(v) = 0$ if and only if

$$v = \exp\left(\frac{2\pi\sqrt{-1}m}{n-p}\right) =: [n, p, m]$$

For each $[n, p, m]$, we would like to take a continuous path Γ_m satisfying the following:

- Γ_m contains $[n, p, m]$, the end points of Γ_m are contained in $\{0\}$, and Γ_m is the union of some arcs and segments.
- The restriction $\mathrm{Re}\, F_{|\Gamma_m}$ has the unique maximum, which is attained at $[n, p, m]$.
- There exist $\epsilon > 0$ and $\eta > 0$ such that the following holds for any $v \in \Gamma_m \setminus B_{[n,p,m]}(\epsilon)$
$$\mathrm{Re}\, F(v) \leq \mathrm{Re}\, F([n, p, m]) - \eta\,(1 + |v^{-n}|),$$
where $B_{[n,p,m]}(\epsilon)$ denotes an ϵ-neighbourhood of $[n, p, m]$.

4.1.1. *Preliminary.* Because $\partial_\theta \operatorname{Re} F(1,\theta) = -\sin(n\theta-\kappa)+\sin(p\theta-\kappa)$, we have

$$\partial_\theta \operatorname{Re} F(1,\theta)=0$$

if and only if one of the following holds for some integers $m, q \in \mathbb{Z}$:

$$n\theta-\kappa = p\theta-\kappa+2m\pi \iff \theta=\frac{2m\pi}{n-p}$$

$$n\theta-\kappa=-p\theta+\kappa+(2q+1)\pi \iff \theta=\frac{(2q+1)\pi+2\kappa}{n+p}$$

We also have $\partial_\theta^2 \operatorname{Re} F(1,\theta) = -n\,\cos(n\theta-\kappa)+p\,\cos(p\theta-\kappa)$.

For $\theta_0 = 2m\pi/(n-p)$, we have the following:

$$\operatorname{Re} F(r,\theta_0)=\left(\frac{r^{-n}}{n}-\frac{r^{-p}}{p}\right)\cos(n\theta_0-\kappa)$$

$$\operatorname{Re} F(1,\theta_0)=\left(\frac{1}{n}-\frac{1}{p}\right)\cos(n\theta_0-\kappa)=\frac{1}{np}\partial_\theta^2 \operatorname{Re} F(1,\theta_0)$$

Hence, we have $\operatorname{Re} F(1,\theta_0)<0$ (resp. $\operatorname{Re} F(1,\theta_0)>0$) in the case $\cos(n\theta_0-\kappa)>0$ (resp. $\cos(n\theta_0-\kappa)<0$), and it is a maximal (resp. minimal) of the function $\operatorname{Re} F(1,\theta)$.

For $\theta_1=\big((2q+1)\pi+2\kappa\big)/(n+p)$, we have $\cos(p\theta_1-\kappa)=-\cos(n\theta_1-\kappa)$, and the following:

$$\operatorname{Re} F(r,\theta_1)=\left(\frac{r^{-n}}{n}+\frac{r^{-p}}{p}\right)\cos(n\theta_1-\kappa)$$

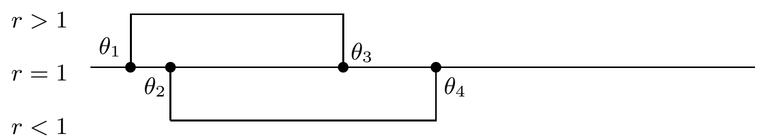
$$\operatorname{Re} F(1,\theta_1)=\left(\frac{1}{n}+\frac{1}{p}\right)\cos(n\theta_1-\kappa)=\frac{-1}{np}\partial_\theta^2 \operatorname{Re} F(1,\theta_1)$$

Hence, we have $\operatorname{Re} F(1,\theta_1)>0$ (resp. $\operatorname{Re} F(1,\theta_1)<0$) in the case $\cos(n\theta_1-\kappa)>0$ (resp. $\cos(n\theta_1-\kappa)<0$), and it is a maximal (minimal) of the function $\operatorname{Re} F(1,\theta)$.

The following lemma is similar to Lemma 3.1.

Lemma 4.1. *Let* $\theta_0=2m\pi/(n-p)$. *We have* $\cos(n\theta_0-\kappa)=0$ *if and only if* $\theta_0=\big((2q+1)\pi+2\kappa\big)/(n+p)$ *for some integer* q. □

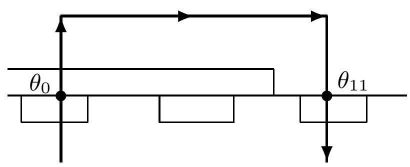
4.1.2. *Paths.* Let us describe how to choose paths Γ_m. In the following pictures, the horizontal central line describes a part of the circle $\{r=1\}$. The upper and lower spaces correspond to $\{r>1\}$ and $\{r<1\}$, respectively. A box in the upper space indicates an arc on which $\cos(p\theta-\kappa)\le 0$. A box in the lower space indicates an arc on which $\cos(n\theta-\kappa)\le 0$.



The case $\cos(n\theta_0-\kappa)<0$. Let $r_0>0$ be large such that $r_0^{-n}/n+r_0^{-p}/p<<\operatorname{Re}F(1,\theta_0)$. Let θ_{11} be the minimum of the following set:

$$\Big\{\theta=\big((2q+1)\pi+2\kappa\big)/(n+p)>\theta_0\,\Big|\,q\in\mathbb{Z},\ \cos(p\theta-\kappa)>0\Big\}$$

Note that it implies $\cos(n\theta_{11}-\kappa)<0$. Then, Γ_m is the union of $\operatorname{Ray}(0,r_0e^{\sqrt{-1}\theta_0})$, $\operatorname{Ray}(0,r_0e^{\sqrt{-1}\theta_{11}})$, and $\operatorname{Arc}\big(r_0e^{\sqrt{-1}\theta_0},\,r_0e^{\sqrt{-1}\theta_{11}}\big)$. An orientation is given as in the picture.



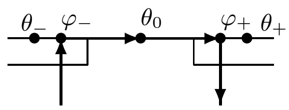
The case $\cos(n\theta_0-\kappa)>0$. We take the integer q such that

$$\theta_-=\frac{(2q-1)\pi+2\kappa}{n+p}<\theta_0<\theta_+=\frac{(2q+1)\pi+2\kappa}{n+p}. \tag{4.2}$$

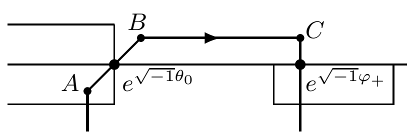
Since $\operatorname{Re}F_{|r=1}$ is maximal at θ_0, the function is not maximal at $\theta_{\pm}$. Hence, we obtain $\cos(n\theta_{\pm}-\kappa)\leq 0$. For any $\delta>0$, we take $\varphi_{\pm}$ such that

$$\varphi_-<\theta_0<\varphi_+,\quad \cos(n\varphi_{\pm}-\kappa)<0,\quad \varphi_+-\varphi_-\leq\frac{\pi}{n}+\delta.$$

Then, let Γ_m be the union of the rays $\operatorname{Ray}(0,e^{\sqrt{-1}\varphi_-})$, $\operatorname{Ray}(0,e^{\sqrt{-1}\varphi_+})$ and the arc $\operatorname{Arc}\big(e^{\sqrt{-1}\varphi_-},\,e^{\sqrt{-1}\varphi_+}\big)$. An orientation is given as in the picture.

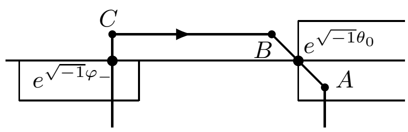


The case $\cos(p\theta_0-\kappa)=0$ *and* $\sin(p\theta_0-\kappa)=-1$. This can be regarded as the degenerated case $\theta_0=\theta_-$ in (4.2). We take θ_+ as in (4.2), for which we have $\cos(n\theta_+-\kappa)<0$. We take φ_+ such that (i) $\theta_0<\varphi_+<\theta_-$, (ii) $\cos(n\varphi_+-\kappa)<0$, (iii) $\varphi_+-\theta_0\leq\pi/n+\delta/2$. We take a small segment $\operatorname{Seg}(A,B)$ which is a steepest descent of $\operatorname{Re}F$ at $e^{\sqrt{-1}\theta_0}$. We put $C=|B|\,e^{\sqrt{-1}\varphi_+}$. Let Γ_m be the union of $\operatorname{Ray}(0,A)$, $\operatorname{Seg}(A,B)$, the arc $\operatorname{Arc}(B,C)$ and $\operatorname{Ray}(0,C)$.



An orientation is given as in the picture. It is easy to check that $\operatorname{Re}F_{|\operatorname{Ray}(0,A)}$ is monotonously increasing with respect to r, and $\operatorname{Re}F_{|\operatorname{Arc}(B,C)}$ is monotonously decreasing with respect to θ. Hence, the above Γ_m has the desired property.

The case $\cos(p\theta_0 - \kappa) = 0$ *and* $\sin(p\theta_0 - \kappa) = 1$. We take Γ_m symmetrically to the case $\cos(p\theta_0 - \kappa) = 0$ and $\sin(p\theta_0 - \kappa) = -1$, as in the following picture. An orientation is given as in the picture.



4.1.3. *Complement.* We give a complement in the case $\cos(n\theta_0 - \kappa) < 0$. We take $\beta \in \mathbb{C}$, and consider the following function

$$G(v) = \beta\, v^{-n} - e^{\sqrt{-1}\kappa}\frac{v^{-p}}{p}$$

on the region $\big\{|\arg(v) - \theta_{11}| < \pi/n\big\}$. Let θ_{11} be as in the case $\cos(n\theta_0 - \kappa) < 0$ of Subsection 4.1.2. We can take $\varphi \in \mathbb{R}$ such that (i) $|\varphi - \theta_{11}| < \pi/n$, (ii) $\mathrm{Re}\big(\beta\, e^{-n\sqrt{-1}\varphi}\big) < 0$, (iii) $\cos\big(p\varphi - \kappa\big) > 0$. Let Γ' be the union of the ray $\mathrm{Ray}[0, r_0 e^{\sqrt{-1}\varphi}]$ and the arc $\mathrm{Arc}\big(r_0 e^{\sqrt{-1}\theta_{11}}, r_0 e^{\sqrt{-1}\varphi}\big)$. If r_0 is sufficiently large, there exists $\eta' > 0$ such that the following holds for $v \in \Gamma'$:

$$\mathrm{Re}\, G(v) \leq \mathrm{Re}\, F([n,p,m]) - \eta'\left(1 + |v^{-n}|\right)$$

4.1.4. *Perturbation.* For some $a_j \in \mathbb{C}$ $(j = 1, \ldots, n-1)$, we set

$$\widetilde{F}(\zeta, v) := e^{\sqrt{-1}\kappa}\left(\frac{v^{-n}}{n} - \frac{v^{-p}}{p}\right) + \sum_{j=1}^{n-1} a_j\, v^{-j}\zeta^{n-j}$$

We fix $\epsilon_0 > 0$, which is sufficiently small. There exists $\delta_0 > 0$ with the following property:

- **(A1):** Recall that $[n,p,m]$ is a solution of $\partial_v F(v) = 0$. If $|\zeta| < \delta_0$, the ϵ_0-neighbourhood of $[n,p,m]$ contains one solution of $\partial_v \widetilde{F}(\zeta, v) = 0$, which is denoted by $\nu_m(\zeta)$.
- **(A2):** There exist a relatively compact subset $U \subset \mathbb{C}_v^*$ and $C_i > 0$ $(i = 1, 2)$ such that
$$\mathrm{Re}\big(\zeta^{-\ell}\widetilde{F}(\zeta, v)\big) \leq -C_1\, |\zeta|^{-\ell}|v|^{-n}$$
for any ζ with $0 < |\zeta| < \delta_0$ and $\big|\arg(\zeta)\big| < C_2$, and for any $v \in \Gamma_m \setminus U$.

As in Subsection 3.1.3, we have a perturbation of paths $\widetilde{\Gamma}_m$ for $\widetilde{F}$ such that the following holds for some $\delta_1 > 0$:

- **(B1):** $\widetilde{\Gamma}_{m,\zeta}$ is the union of arcs and segments for each $\zeta \in \Delta_\zeta(\delta_1)$, and $\nu_m(\zeta)$ is contained in $\widetilde{\Gamma}_{m,\zeta}$.
- **(B2):** There exists a neighbourhood $\widetilde{U}$ of $[n,p,m]$ such that we can apply the result in Subsection 2.3 to the function $\widetilde{F}$ and the family of the paths $\widetilde{\Gamma}_m$ on $\Delta_\zeta(\delta_1) \times \widetilde{U}$.

(B3): There exist $\widetilde{C}_i > 0$ $(i = 1, 2)$ such that

$$\mathrm{Re}\big(\zeta^{-\ell}\widetilde{F}(\zeta, v)\big) \leq -\widetilde{C}_1 \,|\zeta|^{-\ell}|v|^{-n} + \mathrm{Re}\Big(\zeta^{-\ell}\widetilde{F}\big(\zeta, \nu_m(\zeta)\big)\Big)$$

for any ζ with $0 < |\zeta| < \delta_1$ and $|\arg(\zeta)| < \widetilde{C}_2$, and for any $v \in \widetilde{\Gamma}_{m,\zeta} \setminus \widetilde{U}$.

We omit the details for construction.

4.2. Some cycles and 1-chains for elementary meromorphic flat bundles.

4.2.1. *Cycles.* We set $\mathfrak{a} = \alpha\, u^{-n} + \sum_{j=1}^{n-1} \alpha_j u^{-j}$ for some $\alpha, \alpha_j \in \mathbb{C}$. We assume $\alpha \neq 0$. Let $\mathcal{L}$ be a meromorphic flat line bundle $\mathcal{O}_{\mathbb{P}^1}\big(*\{0,\infty\}\big)\, e$ with $\nabla e = e\, d\mathfrak{a}$. Let R be a meromorphic flat bundle on $(\mathbb{P}^1_u, \{0,\infty\})$ with regular singularity. Let $q : \mathbb{P}^1_u \longrightarrow \mathbb{P}^1_t$ be given by $q(u) = u^{-p}$. We set $V := q_*\big(\mathcal{L} \otimes R\big)$.

Assume $p < n$. We would like to construct a tuple of flat sections of the Fourier transform $\mathfrak{Four}(V)$ on a small sector $\mathcal{S} \subset \Delta_z^*$. We take a ramified covering $\psi_\zeta : \Delta_\zeta \longrightarrow \Delta_z$ given by $z = \zeta^{n-p}$. Let $\pi : \widetilde{\Delta}_\zeta(0) \longrightarrow \Delta_\zeta$ be the real blow up. We consider a sector $\mathcal{S}$ of Δ_ζ^* around $e^{\sqrt{-1}\Theta} \in \pi^{-1}(0)$.

We fix an $(n-p)$-th root $(-\alpha)^{1/(n-p)}$. We change the variable

$$v := \Big(\frac{p}{n}\Big)^{1/(n-p)}\ \zeta^{-1}\,(-\alpha)^{1/(n-p)}\,u.$$

We set $\zeta_1 = \zeta\, e^{-\sqrt{-1}\Theta}$. Let $\beta > 0$ and $\mu \in \mathbb{R}$ satisfy $\beta e^{\sqrt{-1}\mu} = \alpha\,(-\alpha)^{-n/(n-p)}$. Then, we have

$$\alpha\, u^{-n} + \sum_{j=1}^{n-1} \alpha_j u^{-j} + \frac{u^p}{z} = \beta\, p^{n/(n-p)}\, n^{p/(n-p)}\, \zeta_1^{-n} \times$$
$$e^{\sqrt{-1}(\mu - n\Theta)}\ \left(\frac{v^{-n}}{n} - \frac{v^{-p}}{p} + \sum_{j=1}^{n-1} B_j\, \zeta_1^{n-j} v^{-j}\right).$$

We obtain the paths Γ_m in $\mathbb{C}_v$, by applying the procedure in Subsection 4.3 with

$$\kappa := \mu - n\Theta, \quad F = e^{\sqrt{-1}(\mu - n\Theta)}\big(v^{-n}/n + v^p/p\big).$$

We also take perturbation $\widetilde{\Gamma}_{m,\zeta}$ as in Subsection 4.1.4. For each $\zeta \in \mathcal{S}$, we obtain the corresponding paths $\Gamma^{(\infty)}_{m,\Theta,\mathfrak{a}}(\zeta)$ and $\widetilde{\Gamma}^{(\infty)}_{m,\Theta,\mathfrak{a}}(\zeta)$ in $\mathbb{C}_u$. They are equipped with the orientations. We obtain the section $\widetilde{\nu}_{m,\Theta,\mathfrak{a}} : \mathcal{S} \longrightarrow \mathcal{S} \times \mathbb{C}_u$ corresponding to $\nu_m : \mathcal{S} \longrightarrow \mathcal{S} \times \mathbb{C}_v$. We can take a sector $\mathcal{T}^{(\infty)}_{m,\Theta,\mathfrak{a}} = S[r_1, \theta_1, \theta_2]$ such that $\widetilde{\nu}_{m,\Theta,\mathfrak{a}}(\zeta) \in \mathcal{T}^{(\infty)}_{m,\Theta,\mathfrak{a}}$ for any $\zeta \in \mathcal{S}$. If we have $\cos(n\theta_0 - \kappa) \geq 0$, where $e^{\sqrt{-1}\theta_0} = [n, p, m]$, and if $\mathcal{S}$ is sufficiently small, we may assume the following: (i) $\theta_2 - \theta_1 \leq \pi/n + \delta$ for a given small $\delta > 0$, (ii) the paths $\widetilde{\Gamma}^{(\infty)}_{m,\Theta,\mathfrak{a}}(\zeta)$ can be contained in $\mathcal{T}^{(\infty)}_{m,\Theta,\mathfrak{a}}$ for any $\zeta \in \mathcal{S}$.

Let s be a flat section of $\big(\mathcal{L}\otimes R\big)^{\vee}$ on $\mathcal{T}^{(\infty)}_{m,\Theta,\mathfrak{a}}$. It induces a flat section $\exp(t/z)\,s$ of $\big(\mathcal{L}\otimes R\big)^{\vee}\otimes L(-u^p/z)$ along the paths $\widetilde{\Gamma}^{(\infty)}_{m,\Theta,\mathfrak{a}}(\zeta)$. Then, we obtain a family of cycles $\exp(t/z)\,s\otimes\widetilde{\Gamma}^{(\infty)}_{m,\Theta,\mathfrak{a}}(\zeta)$ for $\big(\mathcal{L}\otimes R\big)^{\vee}\otimes L(-u^p/z)$. The induced flat section of $\mathfrak{Four}(V)^{\vee}$ on the sector $\mathcal{S}$ is denoted by $\Xi^{(\infty)}_{m,\Theta,\mathfrak{a}}(s)$. (See Subsection 2.5 for the flat section induced by a family of cycles.)

Remark 4.2. We can describe the change of sections for variation of Θ, as in Subsection 3.2.2. However, we need to consider the contribution of the pole at $t=0$. See Subsection 4.4. □

4.2.2. *Auxiliary* 1*-chains.* We introduce auxiliary 1-chains for our later use (Section 5). We slightly change the setting. Let $e^{\sqrt{-1}\Theta}\in\pi^{-1}(0)\subset\widetilde{\Delta}_z(0)$. Let $\mathcal{S}$ be a small sector of Δ_z^* around $e^{\sqrt{-1}\Theta}$. Let $x=t^{-1}$. We take $x_\Theta\in\mathbb{C}_x\setminus\{0\}$ such that $\mathrm{Re}\big(x_\Theta^{-1}e^{-\sqrt{-1}\Theta}\big)<0$ and $\mathrm{Im}\big(x_\Theta^{-1}e^{-\sqrt{-1}\Theta}\big)=0$. Let $\varphi:\mathbb{C}_u\longrightarrow\mathbb{C}_x$ be the ramified covering given by $\varphi(u)=u^p$. Let $u_{j,\Theta}$ $(j=1,\ldots,p)$ be the inverse image of $\varphi^{-1}(x_\Theta)$.

Assume that we are given a finite subset $\mathcal{I}\subset u^{-1}\mathbb{C}[u^{-1}]$, which consists of elements of the form $\mathfrak{b}=\beta\,u^{-n}+\sum_{i=1}^{n-1}\beta_i\,u^{-i}$ for $\beta\neq 0$. We do not assume $p<n$. We would like to take a path $\mathcal{P}^{(\infty)}_{j,\Theta,\mathfrak{b}}$ for each $\mathfrak{b}\in\mathcal{I}$ and $j=1,\ldots,p$, connecting 0 and $u_{j,\Theta}$ in $\mathbb{C}_u$, such that the following holds for any $u\in\mathcal{P}^{(\infty)}_{j,\Theta,\mathfrak{b}}$ and $z\in\mathcal{S}$:

$$\big|e^{\mathfrak{b}(u)+z^{-1}u^{-p}}\big|=O\Big(\exp\big(-C|z^{-1}u^{-p}|\big)\Big)\tag{4.3}$$

In the case $n\leq p$, let $\mathcal{P}^{(\infty)}_{j,\Theta,\mathfrak{b}}$ be the ray $\mathrm{Ray}[0,u_{j,\Theta}]$. Let us consider the case $n>p$. For the description $u_{j,\Theta}=r_0\,e^{\sqrt{-1}\theta_j}$, we take φ_j such that (i) $\mathrm{Re}(\beta\,e^{-\sqrt{-1}n\varphi_j})<0$, (ii) $\mathrm{Re}(e^{-\sqrt{-1}(p\varphi_j+\Theta)})<0$ (iii) $|\varphi_j-\theta_j|\leq\pi/2n+\delta$ for any given $\delta>0$. Let $\mathcal{P}^{(\infty)}_{j,\Theta,\mathfrak{b}}$ be the union of $\mathrm{Ray}\big(0,r_0\,e^{\sqrt{-1}\varphi_j}\big)$ and $\mathrm{Arc}\big(r_0\,e^{\sqrt{-1}\varphi_j},\,r_0\,e^{\sqrt{-1}\theta_j}\big)$. We can take a sector $\mathcal{U}^{(\infty)}_{j,\Theta}=S\big[2r_0;\theta_j^{(1)},\theta_j^{(2)}\big]$ in $\mathbb{C}_u^*$ such that (i) $\mathcal{P}^{(\infty)}_{j,\Theta,\mathfrak{b}}\subset\mathcal{U}^{(\infty)}_{j,\Theta}$ for any $\mathfrak{b}\in\mathcal{I}$, (ii) $\theta_j^{(1)}-\theta_j^{(2)}<\pi/n$.

Lemma 4.3. *Let f be a holomorphic function on $\mathcal{U}^{(\infty)}_{j,\Theta}$ with $|f|=O\big(|u|^{-M}\big)$. We have the following estimate for any $\mathfrak{b}\in\mathcal{I}$ and for some $C>0$:*

$$\int_{\mathcal{P}^{(\infty)}_{j,\Theta,\mathfrak{b}}}\exp\big(z^{-1}u^{-p}+\mathfrak{b}(u)\big)\,f(u)\,du=O\Big(\exp\big(-C|z|^{-1}\big)\Big)\tag{4.4}$$

Proof. It follows from (4.3). □

Let s be an element of $\big(\mathcal{L}\otimes R\big)^{\vee}_{|u_{j,\Theta}}$, which induces a flat section of $\big(\mathcal{L}\otimes R\big)^{\vee}$ along the path $\mathcal{P}^{(\infty)}_{j,\Theta,\mathfrak{b}}$. We obtain a family of 1-chains $\exp(u^{-p}z^{-1})\,s\otimes\mathcal{P}^{(\infty)}_{j,\Theta,\mathfrak{b}}$ for $\big(\mathcal{L}\otimes R\big)^{\vee}\otimes L(-u^{-p}z^{-1})$.

4.3. **Estimate of some pairings.**

4.3.1. *Preliminary.* We continue to use the notation in Subsection 4.2. Let $S = S[r_0;\theta_1,\theta_2]$ be a small sector in $\mathbb{C}^*_u$. Let f be a holomorphic function on S such that $f = O\big(u^M(\log u)^j\big)$ for some $M\in\mathbb{R}$ and $j\in\mathbb{Z}_{\geq 0}$. We set

$$h(u,\zeta) := u^{-p}\zeta^{-n+p} + \mathfrak{a}(u), \qquad H_{S,f}(\zeta) := \int_{\widetilde{\Gamma}_{m,\Theta,\mathfrak{a}}\cap S} \exp\big(h(u,\zeta)\big)\, f\, du.$$

We also put $\widehat{\mathfrak{a}}^{(\infty)}_{m,\Theta}(\zeta) := h\big(\widetilde{\nu}^{(\infty)}_{m,\Theta,\mathfrak{a}}(\zeta),\zeta\big)$. The following lemma can be shown by the arguments in the proof of Lemma 3.5.

Lemma 4.4. *If $S\not\ni \widetilde{\nu}^{(\infty)}_{m,\Theta,\mathfrak{a}}(\zeta)$, there exists $\epsilon>0$ such that*

$$H_{S,f}(\zeta) = O\Big(\exp\big(\widehat{\mathfrak{a}}^{(\infty)}_{m,\Theta}(\zeta) - \epsilon|\zeta|^{-n}\big)\Big). \tag{4.5}$$

Assume $S\ni \widetilde{\nu}^{(\infty)}_{m,\Theta,\mathfrak{a}}(\zeta)$. We have the following estimate:

$$H_{S,f}(\zeta) = \exp\big(\widehat{\mathfrak{a}}^{(\infty)}_{m}(\zeta)\big)\left(\Big(\frac{2\pi}{-\partial_u^2 h(u,\zeta)}\Big)^{1/2} f\right)_{|u=\widetilde{\nu}_{m,\Theta,\mathfrak{a}}(\zeta)} + \exp\big(\widehat{\mathfrak{a}}^{(\infty)}_{m}(\zeta)\big)\, O\Big(\zeta^{M+n+1}(\log\zeta)^j\Big)$$

Note that $\widetilde{\nu}_{m,\Theta,\mathfrak{a}}(\zeta)\sim\zeta$ and $\partial_u^2 h\big(\widetilde{\nu}_{m,\Theta,\mathfrak{a}}(\zeta),\zeta\big)^{-1/2}\sim\zeta^{n/2+1}$. Hence, if $f\sim u^M(\log u)^j$, we have

$$H_{S,f}(\zeta)\sim \exp\big(\widehat{\mathfrak{a}}^{(\infty)}_{m,\Theta}(\zeta)\big)\,\zeta^{M+n/2+1}(\log\zeta)^j.$$

For any $N>0$, there exists $M>0$ such that $H_{S,f}(\zeta) = O\Big(\exp\big(\widehat{\mathfrak{a}}^{(\infty)}_{m}(\zeta)\big)\,\zeta^N\Big)$ for any f with $|f| = O\big(|u|^M\big)$. □

We give a related estimate for our later use (Section 5). Let $\theta_i\in\mathbb{R}$ $(i=1,2)$ satisfy (i) $\mathrm{Re}\big(\alpha\, e^{-n\sqrt{-1}\theta_i}\big)=0$, (ii) $\theta_2-\theta_1=\pi/n$, (iii) we have $\mathrm{Re}\big(\alpha\, e^{-n\sqrt{-1}\theta}\big)>0$ for $\theta_1<\theta<\theta_2$. Let us consider the case $\widetilde{\Gamma}^{(\infty)}_{m,\Theta,\mathfrak{a}}(\zeta)$ contains rays close to the half lines $\ell_i := \{\arg(u)=\theta_i\}$ $(i=1,2)$. (See the cases $\cos(p\theta_0-\kappa)>0$ and $\cos(p\theta_0-\kappa)=0$ in Subsection 4.1.2.)

Assume that θ_1 is not close to $\arg\widetilde{\nu}^{(\infty)}_{m,\Theta,\mathfrak{a}}(\zeta)$. (See the case $\cos(n\theta_0-\kappa)>0$, or the case $\cos(n\theta_0-\kappa)=0$ and $\sin(n\theta_0-\kappa)=1$ in Subsection 4.1.) For $\beta\in\mathbb{C}$ satisfying $\mathrm{Re}\big(\beta\, e^{-n\sqrt{-1}\theta_1}\big)\leq 0$ we take $\varphi(\beta)\in\mathbb{R}$ such that (i) $|\varphi(\beta)-\theta_1|\neq 0$ is sufficiently small, (ii) $\mathrm{Re}\big(\beta e^{-n\sqrt{-1}\varphi(\beta)}\big)<0$. We set $r(\zeta):=|Q_1(\zeta)|$, where $Q_1(\zeta)$ is the corner of $\widetilde{\Gamma}^{(\infty)}_{m,\Theta,\mathfrak{a}}(\zeta)$ near the half line ℓ_1. (See the picture in Subsection 3.3.1.) Let Γ_β be the union of the ray $\mathrm{Ray}[0,\, r(\zeta)\, e^{\sqrt{-1}\varphi(\beta)}]$ and the arc $\mathrm{Arc}\big(r(\zeta)\, e^{\sqrt{-1}\varphi(\beta)},\, Q_1(\zeta)\big)$.

We set $\mathfrak{b} := \beta u^{-n} + \sum_{j=1}^{n-1} \beta_j u^{-j}$ for some $\beta_j \in \mathbb{C}$, and $\Gamma_{\mathfrak{b}} := \Gamma_\beta$. There exists $\epsilon > 0$ such that the following holds for any $u \in \Gamma_{\mathfrak{b}}$:

$$\mathrm{Re}\Big(\mathfrak{b}(u) + u^{-p}/\zeta^{n-p}\Big) < \mathrm{Re}(\widehat{\mathfrak{a}}^{(\infty)}_{m,\Theta}(\zeta)) - \epsilon|u^{-n}| \tag{4.6}$$

Let $\mathcal{T}_1$ be a sector around ℓ_1. Let f be a holomorphic function on $\mathcal{T}_1$ such that $|f| = O\big(|u|^M\big)$ for some M. Then, there exists $\epsilon' > 0$ such that the following holds, if $|\zeta|$ is sufficiently small:

$$\int_{\mathcal{T}_1 \cap \Gamma_{\mathfrak{b}}} \exp\big(\mathfrak{b} + u^{-p}/\zeta^{n-p}\big)\, f\, du = O\Big(\exp\big(\widehat{\mathfrak{a}}^{(\infty)}_{m,\Theta}(\zeta) - \epsilon'|\zeta^{-n}|\big)\Big) \tag{4.7}$$

If θ_1 is close to $\arg \widetilde{\nu}^{(\infty)}_{m,\Theta,\mathfrak{a}}(\zeta)$, we have a similar estimate by exchanging the roles of θ_1 and θ_2.

4.3.2. *Estimate.* For simplicity, we assume that the monodromy of R has a unique eigenvalue ω. We set $r := \operatorname{rank} R$. We take flat frames $\boldsymbol{s}_m = \big(s_{m,i} \,\big|\, i = 1, \dots, r\big)$ of $(\mathcal{L} \otimes R)^\vee$ on $\mathcal{T}^{(\infty)}_{m,\Theta,\mathfrak{a}}$. We obtain flat sections $U_{m,i} := \Xi^{(\infty)}_{m,\Theta,\mathfrak{a}}(s_{m,i})$ of $\mathfrak{Four}(V)^\vee$ on $\mathcal{S}$. We set $\boldsymbol{U}_m := (U_{m,i})$. We put $\boldsymbol{U} := \bigcup_m \boldsymbol{U}_m$.

Let E be the Deligne-Malgrange lattice of $\mathcal{L} \otimes R$. Let $\boldsymbol{w} = (w_1, \dots, w_r)$ be a frame of E on $\mathbb{C}_u$. Let $\Upsilon_{k,l}$ denote the sections of $\mathfrak{Four}(V)$ around ∞_τ induced by $u^{-k}\, w_l$.

Let α be the eigenvalue of the residue $\mathrm{Res}(\nabla) \in \mathrm{End}(E_{|O})$. We consider the following:

$$a_{(m,i),(k,l)} := \frac{\big\langle U_{m,i}, \Upsilon_{k,l} \big\rangle}{\exp\big(\widehat{\mathfrak{a}}^{(\infty)}_{m,\Theta}(\zeta)\big)\, \zeta^{\alpha - k + n/2 - p}}$$

We take a bijection $\mu : \{1, \dots, n-p\} \times \{1, \dots, r\} \simeq \{1, \dots, (n-p)r\}$. For any $\rho \in \mathbb{Z}_{\geq 0}$, we also take a bijection $\mu_\rho : \{\rho+1, \dots, \rho+n-p\} \times \{1, \dots, r\} \simeq \{1, \dots, (n-p)r\}$. Then, we obtain the matrix-valued function $\mathcal{A}_\rho := \det\big(a_{\mu(m,i),\mu_\rho(k,l)}\big)$ on $\mathcal{S}$.

Proposition 4.5. *The entries of $\mathcal{A}_\rho$ are bounded up to log order. There exist a non-zero complex number C and a positive number δ such that* $\det \mathcal{A}_\rho = C + O(|\zeta|^\delta)$.

Proof. It can be shown by the argument in the proof of Proposition 3.6. □

4.4. **Change of sections.** We use the notation in Subsection 4.2. Let us give a complement on the contributions of the regular singularity at 0 for this elementary case.

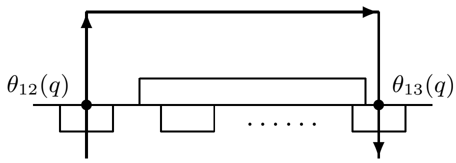
Additional sections. We take the following paths. Note that $\cos(p\theta - \kappa) < 0$ for any θ satisfying

$$\rho_1(q) := \frac{1}{p}\left(\frac{\pi}{2} + 2q\pi + \kappa\right) < \theta < \frac{1}{p}\left(\frac{3\pi}{2} + 2q\pi + \kappa\right) =: \rho_2(q).$$

We put $\mathcal{S} := \big\{\theta = \big((2j+1)\pi + 2\kappa\big)/(n+p) \,\big|\, j \in \mathbb{Z},\ \cos(p\theta - \kappa) > 0\big\}$. We put

$$\theta_{12}(q) := \max\big\{\theta \in \mathcal{S} \,\big|\, \theta < \rho_1(q)\big\}, \quad \theta_{13}(q) := \min\big\{\theta \in \mathcal{S} \,\big|\, \theta > \rho_2(q)\big\}.$$

We take r_0 such that $r_0^{-n}/n + r_0^{-p}/p$ is sufficiently small. Then, let Γ_q^{reg} be the union of $\mathrm{Ray}(0, r_0 e^{\sqrt{-1}\theta_{12}(q)})$, $\mathrm{Ray}(0, r_0 e^{\sqrt{-1}\theta_{13}(q)})$ and $\mathrm{Arc}\big(r_0 e^{\sqrt{-1}\theta_{12}(q)}, r_0 e^{\sqrt{-1}\theta_{13}(q)}\big)$. An orientation is given as in the picture.



As in Subsection 4.2, we obtain families of paths $\Gamma^{\mathrm{reg}}_{q,\Theta,\mathfrak{a}}(\zeta)$ in $\mathbb{C}_u$, corresponding to Γ^{reg}_q. Let $\mathcal{U}_{q,\Theta,\mathfrak{a}}$ be a simply connected region which contains $\Gamma^{\mathrm{reg}}_{q,\Theta,\mathfrak{a}}(\zeta)$ for any $\zeta \in \mathcal{S}$. Let s be a flat section of $\big(L(\mathfrak{a}) \otimes R_{\mathfrak{a}}\big)^{\vee}$ on $\mathcal{U}_{q,\Theta,\mathfrak{a}}$. We obtain a family of cycles of $\big(L(\mathfrak{a}) \otimes R_{\mathfrak{a}}\big)^{\vee} \otimes L(-t/z)$. The induced flat section of $\mathfrak{Four}(V)^{\vee}_{|\mathcal{S}}$ is denoted by $\Xi^{\mathrm{reg}}_{q,\Theta,\mathfrak{a}}(s)$.

We can easily give a frame of $\mathfrak{Four}(V)^{\vee}_{|\mathcal{S}}$ by using $\Xi^{\mathrm{reg}}_{q,\Theta,\mathfrak{a}}$ and $\Xi^{(\infty)}_{m,\Theta,\mathfrak{a}}$. We omit the details.

Change of flat sections. We set $\kappa_0 := \mu - n\Theta_0$, $\kappa_- := \mu - n(\Theta_0 + \epsilon)$ and $\kappa_+ = \mu - n(\Theta_0 - \epsilon)$. Let $e^{\sqrt{-1}\theta_0} = [n, p, m]$. Let us describe the change of flat sections for variation of Θ.

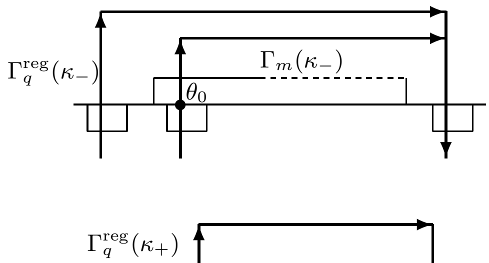
Assume that $\cos(n\theta_0 - \kappa_0) = 0$ and $\sin(n\theta_0 - \kappa_0) = -1$. We take $q \in \mathbb{Z}$ such that

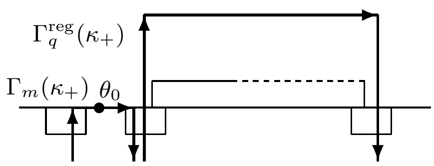
$$\frac{1}{p}\left(\frac{\pi}{2} + 2q\pi + \kappa_-\right) < \theta_0 < \frac{1}{p}\left(\frac{3\pi}{2} + 2q\pi + \kappa_-\right).$$

We have the following relation:

$$\Xi_{m,\Theta_0-\epsilon,\mathfrak{a}}(s) = -\Xi_{m,\Theta_0+\epsilon,\mathfrak{a}}(s) + \Xi^{\mathrm{reg}}_{q,\Theta_0+\epsilon,\mathfrak{a}}(s)$$
$$\Xi^{\mathrm{reg}}_{q,\Theta_0-\epsilon,\mathfrak{a}}(s) + \Xi_{m,\Theta_0-\epsilon,\mathfrak{a},\mathfrak{a}}(s) = \Xi^{\mathrm{reg}}_{q,\Theta_0+\epsilon,\mathfrak{a}}(s)$$

Here, the flat section s is extended appropriately. See the following pictures.





Assume that $\cos(n\theta_0 - \kappa_0) = 0$ and $\sin(n\theta_0 - \kappa_0) = 1$. We take $q \in \mathbb{Z}$ such that

$$\rho_1(q) := \frac{1}{p}\left(\frac{\pi}{2} + 2q\pi + \kappa_+\right) < \theta_0 < \frac{1}{p}\left(\frac{3\pi}{2} + 2q\pi + \kappa_+\right) =: \rho_2(q)$$

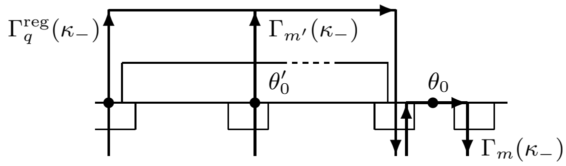
Let $\theta_0' = 2m'\pi/(n-p)$ such that $\rho_1(q) < \theta_0' < \rho_2(q)$. We have the following relation:

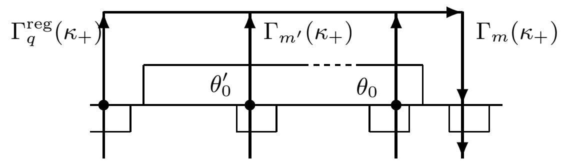
$$\Xi_{m,\Theta_0-\epsilon,\mathfrak{a}}(s) = \Xi_{m,\Theta_0+\epsilon,\mathfrak{a}}(s)$$

$$\Xi_{m',\Theta_0-\epsilon,\mathfrak{a}}(s) = \Xi_{m',\Theta_0+\epsilon,\mathfrak{a}}(s) + \Xi_{m,\Theta_0+\epsilon,\mathfrak{a}}(s)$$

$$\Xi^{\mathrm{reg}}_{q,\Theta_0-\epsilon,\mathfrak{a}}(s) = \Xi^{\mathrm{reg}}_{q,\Theta_0+\epsilon,\mathfrak{a}}(s) + \Xi_{m,\Theta_0+\epsilon,\mathfrak{a}}(s)$$

See the following pictures.





5. Meromorphic flat bundles on $\mathbb{P}^1$

5.1. **Construction of cycles.** Let (V, ∇) be a meromorphic flat bundle on $\mathbb{P}^1_t$. Let $\mathrm{Sing}(V)$ denote the set of the poles of (V, ∇). We will implicitly assume $\infty \in \mathrm{Sing}(V)$ in the following. We would like to construct flat sections of $\mathfrak{Four}(V)$ on small sectors around $\infty \in \mathbb{P}^1_\tau$. Let $z = \tau^{-1}$. Let $\psi : \Delta_\xi \longrightarrow \Delta_z$ be a ramified covering given by $z = \xi^\mu$ such that $\psi^*\mathfrak{Four}(V)$ is unramified. Let $\pi : \widetilde{\Delta}_\xi(0) \longrightarrow \Delta_\xi$ be the real blow up at $\xi = 0$. Let $e^{\sqrt{-1}\Theta} \in \pi^{-1}(0)$, and let $\mathcal{S}$ be a small sector around $e^{\sqrt{-1}\Theta}$. We will construct flat sections of $\mathfrak{Four}(V)^\vee_{|\mathcal{S}}$.

If μ is divisible by m, we have a factorization $\Delta_\xi \longrightarrow \Delta_y \longrightarrow \Delta_z$ such that $y = \xi^{\mu/m}$ and $z = y^m$. The point $e^{\sqrt{-1}\Theta}$ naturally induces a point of the fiber of the real blow up $\widetilde{\Delta}_y(0) \longrightarrow \Delta_y$ over 0, which is also denoted by $e^{\sqrt{-1}\Theta}$. The sector $\mathcal{S}$ naturally induces a sector in Δ_y^*, if it is sufficiently small. It is also denoted by $\mathcal{S}$.

5.1.1. *Meromorphic flat bundle on* $(\mathbb{P}^1, \{0, \infty\})$. Assume that (i) $\mathrm{Sing}(V, \nabla) = \{0, \infty\}$, (ii) (V, ∇) is regular singular at ∞. Let $\varphi : (\mathbb{P}^1_U, \{0, \infty\}) \longrightarrow (\mathbb{P}^1_t, \{0, \infty\})$ be a ramified covering such that $\varphi^*(V, \nabla)$ is unramified at 0. Let $\mathrm{Irr}(\varphi^*\nabla)$ denote the set of irregular values of $\varphi^*(V, \nabla)$ at 0. The Galois group $\mathrm{Gal}(\varphi)$ naturally acts on $\mathrm{Irr}(\varphi^*\nabla)$. The quotient is denoted by $\overline{\mathrm{Irr}}(\nabla)$. Let $\mathfrak{a} \in U^{-1}\mathbb{C}[U^{-1}]$ be a representative of an element of $\overline{\mathrm{Irr}}(\nabla)$. We have a factorization of φ,

$$\mathbb{P}^1_U \xrightarrow{\varphi_{1,\mathfrak{a}}} \mathbb{P}^1_u \xrightarrow{\varphi_{2,\mathfrak{a}}} \mathbb{P}^1_t$$

such that (i) $\mathfrak{a}$ is defined on $\mathbb{P}^1_u$, (ii) it is irreducible on $\mathbb{P}^1_u$, in the sense $g^*\mathfrak{a} \neq \mathfrak{a}$ for any $g \in \mathrm{Gal}(\varphi_{2,\mathfrak{a}}) \setminus \{1\}$. Let $R_{\mathfrak{a}}$ be a regular singular meromorphic flat bundle on $(\mathbb{P}^1_u, \{0,\infty\})$ such that $\varphi^*_{1,\mathfrak{a}}\big(L(\mathfrak{a}) \otimes R_{\mathfrak{a}}\big) \simeq \mathrm{Gr}_{\mathfrak{a}}\big(\varphi^*_{1,\mathfrak{a}}V\big)$.

Let p be the ramification index of $\varphi_{2,\mathfrak{a}}$, and put $n := -\mathrm{ord}_u(\mathfrak{a})$. We consider a factorization $\Delta_\xi \longrightarrow \Delta_\zeta \longrightarrow \Delta_z$ with $\zeta = \xi^{\mu/(n+p)}$, and we regard $\mathcal{S}$ as a sector in Δ^*_ζ. We use the notation in Subsection 3.2. Let s be a flat section of $\big(L(\mathfrak{a}) \otimes R_{\mathfrak{a}}\big)^\vee$ on $\mathcal{T}_{m,\Theta,\mathfrak{a}}$. We have constructed the family of cycles

$$\exp(u^p/\zeta^{n+p})\, s \otimes \widetilde{\Gamma}_{m,\Theta,\mathfrak{a}}(\zeta) \quad (\zeta \in \mathcal{S})$$

for $\big(L(\mathfrak{a}) \otimes R_{\mathfrak{a}}\big)^\vee \otimes L(-u^p/z)$. We obtain a family of cycles $\widetilde{\Delta}^{(0)}_{m,\Theta,\mathfrak{a}}(s;\zeta)$ for $V^\vee \otimes L(-t/z)$ on $\mathbb{P}^1_t$ by the following procedure:

- **Step 1:** Let q be the ramification index of $\varphi_{1,\mathfrak{a}}$. We obtain a family of cycles for $\mathrm{Gr}_{-\mathfrak{a}}(\varphi^*V^\vee) \otimes L(-U^{qp}/\zeta^{n+p})$ by (Lift) in Subsection 2.4.1.
- **Step 2:** If $\mathfrak{a} \neq 0$, let $\mathfrak{a}_0$ denote the image of $\mathrm{Irr}(\varphi^*\nabla) \longrightarrow \mathrm{Irr}(\varphi^*\nabla, -nq)$. We obtain a family of cycles for $\mathrm{Gr}^{(-nq)}_{-\mathfrak{a}_0}\big(\varphi^*V^\vee\big) \otimes L(-U^{qp}/\zeta^{n+p})$ by applying P1 in Subsection 2.4.2.
- **Step 3:** If $\mathfrak{a} \neq 0$, we obtain a family of cycles for $\mathrm{Gr}^{(-nq-1)}_0\big(\varphi^*V^\vee\big) \otimes L(-U^{qp}/\zeta^{n+p})$ by applying P1 and P3.
- **Step 4:** We obtain a family of cycles for $\varphi^*V^\vee \otimes L(-U^{qp}/\zeta^{n+p})$ by applying P2.
- **Step 5:** Applying (Descent), we obtain a family of cycles for $V^\vee \otimes L(-t/z)$.

It induces a flat section of $\mathfrak{Four}(V)^\vee$ on $\mathcal{S}$, which is denoted by $\widetilde{\Xi}^{(0)}_{m,\Theta,\mathfrak{a}}(s)$. (See Subsection 2.5 for the flat section induced by a family of cycles.)

5.1.2. *Cycles and auxiliary* 1*-chains around* ∞. Assume (i) $\mathrm{Sing}(V,\nabla) = \{0,\infty\}$, (ii) (V,∇) is regular singular at 0. Let $x = t^{-1}$. Let $\varphi : (\mathbb{P}^1_U, \{0,\infty\}) \longrightarrow (\mathbb{P}^1_x, \{0,\infty\})$ be a ramified covering such that $\varphi^*(V,\nabla)$ is unramified at 0. We fix a representative $\mathfrak{a} \in U^{-1}\mathbb{C}[U^{-1}]$ for each element of $\overline{\mathrm{Irr}}(\nabla)$. We have the factorization of φ

$$\mathbb{P}^1_U \xrightarrow{\varphi_{1,\mathfrak{a}}} \mathbb{P}^1_u \xrightarrow{\varphi_{2,\mathfrak{a}}} \mathbb{P}^1_x$$

such that (i) $\mathfrak{a}$ is defined on $\mathbb{P}^1_u$, (ii) it is irreducible. Let $R^{(\infty)}_{\mathfrak{a}}$ be a regular singular meromorphic flat bundle on $(\mathbb{P}^1_u, \{0,\infty\})$ such that $\varphi^*_{1,\mathfrak{a}}\big(L(\mathfrak{a}) \otimes R^{(\infty)}_{\mathfrak{a}}\big) \simeq \mathrm{Gr}_{\mathfrak{a}}\big(\varphi^*V\big)$.

Let p be the ramification index of $\varphi_{2,\mathfrak{a}}$, and let n be the order of the pole of $\mathfrak{a}$. Assume $n > p$. We consider a factorization $\Delta_\xi \longrightarrow \Delta_\zeta \longrightarrow \Delta_z$ with $\zeta = \xi^{\mu/(n-p)}$, and we regard $\mathcal{S}$ as a sector in Δ^*_ζ. We use the notation in Subsection 4.2. Let s

be a flat section of $\big(L(\mathfrak{a})\otimes R^{(\infty)}_{\mathfrak{a}}\big)^{\vee}$ on $\mathcal{T}^{(\infty)}_{m,\Theta,\mathfrak{a}}$. We have constructed the family of cycles

$$\exp(u^{-p}/\zeta^{n-p})\, s\otimes\widetilde{\Gamma}^{(\infty)}_{m,\Theta,\mathfrak{a}}(\zeta)\quad(\zeta\in\mathcal{S})$$

for $\big(L(\mathfrak{a})\otimes R^{(\infty)}_{\mathfrak{a}}\big)^{\vee}\otimes L\big(-u^{-p}/z\big)$. We obtain a family of cycles $\widetilde{\Delta}^{(\infty)}_{m,\Theta,\mathfrak{a}}(s;\zeta)$ for $V^{\vee}\otimes L(-t/z)$ on $\mathbb{P}^1_t=\mathbb{P}^1_x$ by the following procedure:

Step 1: Let q be the ramification index of $\varphi_{1,\mathfrak{a}}$. We obtain a family of cycles for $\mathrm{Gr}_{-\mathfrak{a}}(\varphi^*V^{\vee})\otimes L(-U^{qp}/\zeta^{n-p})$ by (Lift) in Subsection 2.4.1.

Step 2 ($\cos(n\theta_0-\kappa)\geq 0$): Let $\mathfrak{a}_0$ denote the image of $\mathfrak{a}$ by the map $\mathrm{Irr}(\varphi^*\nabla)\longrightarrow\mathrm{Irr}(\varphi^*\nabla,-nq)$. We obtain a family of cycles for $\mathrm{Gr}^{(-nq)}_{-\mathfrak{a}_0}\big(\varphi^*V^{\vee}\big)\otimes L(-U^{qp}/\zeta^{n-p})$ by applying P1. Then, we lift it to a family of cycles for $\mathrm{Gr}^{(-nq-1)}_0\big(\varphi^*V^{\vee}\big)\otimes L(-U^{qp}/\zeta^{n-p})$ by applying P3.

Step 2 ($\cos(n\theta_0-\kappa)<0$): We obtain a family of cycles for $\mathrm{Gr}^{(-nq-1)}_0\big(\varphi^*V^{\vee}\big)\otimes L(-U^{qp}/\zeta^{n-p})$ by applying P4.

Step 3: We obtain a family of cycles for $\varphi^*V^{\vee}\otimes L(-U^{qp}/\zeta^{n-p})$ by applying P2.

Step 4: Applying (Descent), we obtain a family of cycles for $V^{\vee}\otimes L(-t/z)$.

It induces a flat section of $\mathfrak{Four}(V)^{\vee}$ on $\mathcal{S}$, which is denoted by $\widetilde{\Xi}^{(\infty)}_{m,\Theta,\mathfrak{a}}(s)$.

Let us give a complement on auxiliary chains. We take $x_{\Theta}\in\mathbb{C}^*_x$ as in Subsection 4.2.2. For each $(n,p)\in\mathbb{Z}^2_{>0}$, let $\mathcal{I}_{n,p}$ be the set of the elements $\mathfrak{b}$ of $\overline{\mathrm{Irr}}(\nabla)$ such that the ramification index of $\varphi_{2,\mathfrak{b}}$ is p, and $-\mathrm{ord}_u(\mathfrak{b})=n$. We put $q:=\mu/p$. Let $\varphi_{1,q}:\mathbb{P}^1_U\longrightarrow\mathbb{P}^1_u$ and $\varphi_{2,p}:\mathbb{P}^1_u\longrightarrow\mathbb{P}^1_x$ be the ramified covering given by $\varphi_{1,q}(U)=U^q$ and $\varphi_{2,p}(u)=u^p$. We take paths $\mathcal{P}^{(\infty)}_{j,\Theta,\mathfrak{b}}$ ($\mathfrak{b}\in\mathcal{I}_{n,p}, j=1,\ldots,p$), and sectors $\mathcal{U}^{(\infty)}_{j,\Theta,(n,p)}$ ($j=1,\ldots,p$) in $\mathbb{C}^*_u$, as in Subsection 4.2.2. Let $u_{j,\Theta,(n,p)}$ ($j=1,\ldots,p$) denote the fiber $\varphi^{-1}_{2,p}(x_{\Theta})$.

We take $U_{\Theta}\in\varphi^{-1}(x_{\Theta})$. Let S be a small sector in $\mathbb{C}^*_U$ such that $U_{\Theta}\in S$. We have a splitting of the full Stokes filtration $\widetilde{\mathcal{F}}^S$ such that the induced splitting of the Stokes filtration in the level qn can be extended to a splitting on $\varphi^{-1}_{1,q}(\mathcal{U}^{(\infty)}_{m,\Theta,(n,p)})$. The splitting induces the following isomorphism:

$$V^{\vee}_{|x_{\Theta}}\simeq\varphi^*V^{\vee}_{|U_{\Theta}}\simeq\bigoplus_{(n,p)}\bigoplus_{\mathfrak{b}\in\mathcal{I}_{n,p}}\bigoplus_{j}\big(L(\mathfrak{b})\otimes R^{(\infty)}_{\mathfrak{b}}\big)^{\vee}_{|u_{j,\Theta,(n,p)}}\tag{5.1}$$

Let $s_{x_{\Theta}}$ be an element of $V^{\vee}_{|x_{\Theta}}$. It induces $s_{j,\Theta,\mathfrak{b}}\in\big(L(\mathfrak{b})\otimes R_{\mathfrak{b}}\big)^{\vee}_{|u_{j,\Theta,(n,p)}}$ by the decomposition (5.1). We have the induced family of 1-chains $\exp(u^{-p}/z)\,s_{j,\Theta,\mathfrak{b}}\otimes\mathcal{P}^{(\infty)}_{j,\Theta,\mathfrak{b}}$ for $\big(L(\mathfrak{b})\otimes R_{\mathfrak{b}}\big)^{\vee}\otimes L(-u^{-p}/z)$. We obtain a family of 1-chains $I^{(\infty)}_{j,\Theta,\mathfrak{b}}(s_{x_{\Theta}})$ for $V^{\vee}\otimes L(-t/z)$ by the procedure applying (Lift), P1, P2 and (Descent).

5.1.3. *Meromorphic flat bundle on* $\mathbb{P}^1$. Let (V,∇) be a meromorphic flat bundle on $\mathbb{P}^1_t$. Let $\phi_c:\mathbb{P}^1_t\longrightarrow\mathbb{P}^1_t$ given by $\phi_c(t)=t-c$. For each $c\in\mathrm{Sing}(V)\setminus\{\infty\}$, we have a neighbourhood U_c and a meromorphic flat bundle (V_c,∇_c) on $(\mathbb{P}^1_t,\{0,\infty\})$

which is regular singular at ∞, such that $\phi_c^*(V_c,\nabla_c)_{|U_c}\simeq(V,\nabla)_{|U_c}$. We also have a neighbourhood U_∞ of ∞, and a meromorphic flat bundle (V_∞,∇_∞) on $(\mathbb{P}^1_t,\{0,\infty\})$ which is regular singular at 0, such that $(V_\infty,\nabla_\infty)_{|U_\infty}\simeq(V,\nabla)_{|U_\infty}$. We assume that $U_c\cap U_{c'}=\emptyset$ for distinct $c,c'\in\mathrm{Sing}(V)$.

Let $\widetilde{\Delta}^{(\infty)}_{m,\Theta,\mathfrak{a}}(s)$ be as in Subsection 5.1.2. If $|\zeta|$ is sufficiently small, the support of $\widetilde{\Delta}^{(\infty)}_{m,\Theta,\mathfrak{a}}(s)$ is contained in U_∞. Hence, it induces a family of cycles for $(V,\nabla)^\vee\otimes L(-t/z)$, denoted by $\overline{\Delta}^{(\infty)}_{m,\Theta,\mathfrak{a}}(s)$.

Assume $0\in\mathrm{Sing}(V)\setminus\{\infty\}$. Let s and $\widetilde{\Delta}^{(0)}_{m,\Theta,\mathfrak{a}}(s)$ be as in Subsection 5.1.1 for (V_0,∇_0). If $\widetilde{\Gamma}_{m,\Theta,\mathfrak{a}}(\zeta)$ does not contain a ray of the form $\mathrm{Ray}[Q(\zeta),\infty]$, the support of $\widetilde{\Delta}^{(0)}_{m,\Theta,\mathfrak{a}}(s)$ is contained in U_0 when $|\zeta|$ is sufficiently small. Hence, it naturally gives a family of cycles $\overline{\Delta}^{(0)}_{m,\Theta,\mathfrak{a}}(s)$ for $V^\vee\otimes L(-t/z)$. Let us consider the case that $\widetilde{\Gamma}_{m,\Theta,\mathfrak{a}}(\zeta)$ contains a ray of the form $\mathrm{Ray}[Q(\zeta),\infty]$. Let $P(\zeta):=\mathrm{Ray}[Q(\zeta),\infty]\cap\partial U_0$. Let x_Θ be a point in $\mathbb{C}_x^*$ such that (i) $\mathrm{Re}\big(x_\Theta^{-1}e^{-\sqrt{-1}\Theta}\big)<0$, $\mathrm{Im}\big(x_\Theta^{-1}e^{-\sqrt{-1}\Theta}\big)=0$, (ii) close to $x=0$. (See Subsection 4.2.2.) We connect $P(\zeta)$ to x_Θ by a path γ contained in $\big\{\mathrm{Re}(t/z)<0\big\}\setminus\bigcup_{c\in\mathrm{Sing}(V)\setminus\{\infty\}}U_c$. The flat section s naturally induces an element $s_{|x_\Theta}$ of $V^\vee_{|x_\Theta}$. By the procedure in Subsection 5.1.2, we obtain the 1-chains $I^{(\infty)}_{j,\Theta,\mathfrak{b}}(s_{|x_\Theta})$. Then, we cut $\mathrm{Ray}[Q(\zeta),\infty]$ at $P(\zeta)$, and add the 1-chains $\exp(t/z)\,s\otimes\gamma$ and $I^{(\infty)}_{j,\Theta,\mathfrak{b}}(s_{|x_\Theta})$ with appropriate signatures, and we obtain a family of cycles $\overline{\Delta}^{(0)}_{m,\Theta,\mathfrak{a}}(s)$ of $V^\vee\otimes L(-t/z)$.

For $c\in\mathrm{Sing}(V)\setminus\{\infty\}$, we take a ramified covering $\varphi:\mathbb{P}^1_U\longrightarrow\mathbb{P}^1_t$ for (V_c,∇_c) as in Subsection 5.1.1. Let $\mathfrak{a}\in\mathrm{Irr}(\varphi^*\nabla_c)$. We take $R^{(c)}_{\mathfrak{a}}$ for (V_c,∇_c) as in Subsection 5.1.1. Let s be a flat section of $\big(L(\mathfrak{a})\otimes R^{(c)}_{\mathfrak{a}}\big)^\vee$ on $\mathcal{T}_{m,\Theta,\mathfrak{a}}$. Applying the above procedures, we obtain $\overline{\Delta}^{(0)}_{m,\Theta,\mathfrak{a}}(s)$ for $\phi_{-c}^*(V,\nabla)^\vee\otimes L(-t/z)$. Applying the translation in Subsection 2.4.3, we obtain a family of cycles $\overline{\Delta}^{(c)}_{m,\Theta,\mathfrak{a}}(s)$ for $V^\vee\otimes L(-t/z)$.

The induced flat sections of $\mathfrak{Four}(V)^\vee_{|\mathcal{S}}$ are denoted by $\Xi^{(c)}_{m,\Theta,\mathfrak{a}}(s)$.

5.2. **Estimate of some pairings.**

5.2.1. *Preliminary.* Let $\mathcal{S}$ be a sector in Δ_z^*. Let $R(z)\in\mathbb{C}_t\setminus\{0\}$ such that $|R(z)|\leq C\,|z|^{1/(n+p)}$. Let $T(z)$ be the segment connecting $R(z)$ and 0.

Lemma 5.1. *Let $n'/p'>n/p$ and $N\in\mathbb{R}$. We have the following estimate for any $C_1>0$:*

$$\int_{T(z)}\exp\big(t/z-\epsilon|t|^{-n'/p'}\big)\,|t|^{-N}\,dt=O\Big(\exp\big(-C_1|z|^{-n/(n+p)}\big)\Big)\tag{5.2}$$

Proof. Let $\zeta = z^{1/n+p}$ and $v = t^{1/p}/\zeta$. For any $C_2 > 0$, there exists $r_0 > 0$ such that the following holds for any $|\zeta| < r_0$:

$$|t/z| - \epsilon|t|^{-n'/p'} \leq -C_2|\zeta|^{-n}|v|^{-n}.$$

Then, it is easy to deduce (5.2). □

We set $H := \{\theta_1 \leq \arg(t) \leq \theta_2\}$ for some $\theta_i \in \mathbb{R}$ such that $\mathrm{Re}(t/z) < 0$ for any $t \in H$ and $z \in \mathcal{S}$. Let Γ be a bounded path contained in H. Let f be a holomorphic function around Γ. Then, there exists $C > 0$ such that the following holds:

$$\int_\Gamma \exp(t/z)\, f\, dt = O\Big(\exp(-C|z|^{-1})\Big) \tag{5.3}$$

5.2.2. *Cycles around c.* Let (V, ∇) be a meromorphic flat bundle on $\mathbb{P}^1$. We use the notation in Subsection 5.1. Let η be a section of V on $\mathbb{P}^1_t$, which induces a section Υ of $\mathfrak{Four}(V)$. Let $c \in \mathrm{Sing}(V) \setminus \{\infty\}$. We have the isomorphism:

$$\phi^*_{-c}(V)_{|\widehat{0}} \simeq V_{c|\widehat{0}} \simeq \bigoplus_{\mathfrak{a} \in \overline{\mathrm{Irr}}(\nabla_c)} \varphi_{2,\mathfrak{a}*}\big(L(\mathfrak{a}) \otimes R^{(c)}_{\mathfrak{a}}\big)_{|\widehat{0}}$$

We have the corresponding decomposition $\phi^*_{-c}(\eta)_{|\widehat{0}} = \sum \phi^*_{-c}(\eta)_{\widehat{0},\mathfrak{a}}$. For $\mathfrak{a} \in \overline{\mathrm{Irr}}(V_c, \nabla_c)$, let $\eta_{\mathfrak{a}}$ be a section of $\mathrm{Gr}_{\mathfrak{a}}(V_c) := \varphi_{2,\mathfrak{a}*}\big(L(\mathfrak{a}) \otimes R^{(c)}_{\mathfrak{a}}\big)$ on $\mathbb{P}^1$. The induced section of $\mathfrak{Four}(\mathrm{Gr}_{\mathfrak{a}}(V_c))$ is denoted by $\Upsilon_{\mathfrak{a},c}$. Let $\mathcal{V}_{\mathfrak{a},c}$ be the Deligne-Malgrange lattice of $\mathrm{Gr}_{\mathfrak{a}}(V_c)$.

Proposition 5.2. *For any $L > 0$, there exists $M > 0$ with the following property:*

- *Assume $\phi^*_{-c}(\eta)_{\widehat{0},\mathfrak{a}} - \eta_{\mathfrak{a}|\widehat{0}} \equiv 0$ modulo $t^M\mathcal{V}_{\mathfrak{a},c}$. Then, we have*

$$\big\langle \Xi^{(c)}_{m,\Theta,\mathfrak{a}}(s), \Upsilon\big\rangle - \exp(c/z)\,\big\langle \Xi_{m,\Theta,\mathfrak{a}}(s), \Upsilon_{\mathfrak{a},c}\big\rangle = \exp\big(c/z + \widehat{\mathfrak{a}}_{m,\Theta}(\zeta)\big)\, O\big(|\zeta|^L\big)$$

Proof. It can be reduced to the case $c = 0$. We can dominate the difference of the integrals over $\widetilde{\Gamma}_{m,\Theta,\mathfrak{a}}(\zeta) \cap U_0$ by using Lemma 3.5. We can deal with the integrals over some additional paths appeared in the procedures (P2) and (P3) by using Lemma 5.1 and (3.8), respectively. We can deal with the modification in Subsection 5.1.3 by using Lemma 4.3 and (5.3). □

Corollary 5.3. $\Xi^{(c)}_{m,\Theta,\mathfrak{a}}(s)$ *is a flat section of* $\widetilde{\mathcal{F}}^{\mathcal{S}}_{-c/z-\widehat{\mathfrak{a}}_{m,\Theta}}\big(\mathfrak{Four}(V)^\vee_{|\mathcal{S}}\big)$.

Proof. Let f be any section of $\mathfrak{Four}(V)$ on $\mathbb{P}^1$. From Proposition 3.6 and Proposition 5.2, we obtain

$$\big\langle \Xi^{(c)}_{m,\Theta,\mathfrak{a}}(s), f\big\rangle = \exp\big(c/z + \widehat{\mathfrak{a}}_{m,\Theta}(\zeta)\big)\, O\big(\zeta^{-N_1}\big)$$

for some $N_1 > 0$. Then, the claim of the corollary follows from the characterization **(A)** of the full Stokes filtration in Subsection 2.1.2. □

5.2.3. *Cycles around* ∞. Let η be a section of V on $\mathbb{P}^1_t$. Let $\varphi_{2,\mathfrak{a}}:\mathbb{P}^1_u\longrightarrow\mathbb{P}^1_x$ be as in Subsection 5.1.2. We denote it by $\varphi^{(\infty)}_{2,\mathfrak{a}}$, if we regard it as a morphism $\mathbb{P}^1_u\longrightarrow\mathbb{P}^1_t$ by $t=x^{-1}$. Under the isomorphism $V_{|\widehat{\infty}}\simeq\bigoplus\varphi^{(\infty)}_{2,\mathfrak{a}*}\big(L(\mathfrak{a})\otimes R^{(\infty)}_{\mathfrak{a}}\big)_{|\widehat{\infty}}$, we have the decomposition $\eta_{|\widehat{\infty}}=\sum\eta_{\widehat{\infty},\mathfrak{a}}$. Let $\mathfrak{a}\in\overline{\mathrm{Irr}}(\nabla_\infty)$. Let $(\mathcal{V}_{\mathfrak{a},\infty},\mathcal{W}_{\mathfrak{a},\infty})$ be the BDE-good lattice pair for $\mathrm{Gr}_{\mathfrak{a}}(V_\infty):=\varphi^{(\infty)}_{2,\mathfrak{a}*}\big(L(\mathfrak{a})\otimes R^{(\infty)}_{\mathfrak{a}}\big)$. The tensor product $\mathcal{W}_{\mathfrak{a},\infty}\otimes\mathcal{O}_{\mathbb{P}^1}(N\infty)$ is denoted by $\mathcal{W}_{\mathfrak{a},\infty}(N\infty)$. According to Proposition 4.5, there exists $C>0$ such that

$$\big\langle\Xi^{(\infty)}_{m,\Theta,\mathfrak{a}}(s),F\big\rangle=\exp\big(\widehat{\mathfrak{a}}^{(\infty)}_{m,\Theta}(\zeta)\big)\,O\big(|\zeta|^C\big),$$

where F denotes a section of $\mathfrak{Four}\big(\mathrm{Gr}_{\mathfrak{a}}(V_\infty)\big)$ induced by a section of $\mathcal{W}_{\mathfrak{a},\infty}(N\infty)$ on $\mathbb{P}^1$.

Proposition 5.4. *Let $M>N$. Let $\eta_{\mathfrak{a}}$ be a section of $\mathcal{W}_{\mathfrak{a},\infty}(M\infty)$ on $\mathbb{P}^1_t$. The induced section of $\mathfrak{Four}\big(\mathrm{Gr}_{\mathfrak{a}}(V_\infty)\big)$ is denoted by $\Upsilon_{\mathfrak{a},\infty}$. If $\eta_{\widehat{\infty},\mathfrak{a}}-\eta_{\mathfrak{a}}\equiv 0$ modulo $\mathcal{W}_{\mathfrak{a},\infty}(N\infty)_{|\widehat{\infty}}$, then we have*

$$\big\langle\overline{\Xi}^{(\infty)}_{m,\Theta,\mathfrak{a}}(s),\Upsilon\big\rangle=\big\langle\Xi^{(\infty)}_{m,\Theta,\mathfrak{a}}(s),\Upsilon_{\mathfrak{a},\infty}\big\rangle+\exp\big(\widehat{\mathfrak{a}}^{(\infty)}_{m,\Theta}(\zeta)\big)\,O\big(|\zeta|^C\big)\tag{5.4}$$

Proof. It follows from Proposition 4.5, the estimate in Subsection 4.1.3, the estimate (4.7), and Lemma 5.1. □

Corollary 5.5. *$\overline{\Xi}^{(\infty)}_{m,\Theta,\mathfrak{a}}(s)$ is a flat section of $\widetilde{\mathcal{F}}^{\mathcal{S}}_{-\widehat{\mathfrak{a}}^{(\infty)}_{m,\Theta}}\big(\mathfrak{Four}(V)^\vee_{|\mathcal{S}}\big)$.*

Proof. It follows from Proposition 4.5 and Proposition 5.4. □

5.3. The Stokes structure for the Fourier transform.

5.3.1. *Compatible frame.* For each $c\in\mathrm{Sing}(V)$ and for each $\mathfrak{a}\in\overline{\mathrm{Irr}}(V_c,\nabla_c)$, we take a flat frame $\boldsymbol{u}^{(c)}_{m,\Theta,\mathfrak{a}}=\big(u^{(c)}_{m,\Theta,\mathfrak{a},i}\,\big|\,i=1,\ldots,\mathrm{rank}\,R^{(c)}_{\mathfrak{a}}\big)$ of $\big(L(\mathfrak{a})\otimes R^{(c)}_{\mathfrak{a}}\big)^\vee$ on $\mathcal{T}_{m,\Theta,\mathfrak{a}}$. We obtain the flat sections $\overline{U}^{(c)}_{m,\Theta,\mathfrak{a},i}:=\overline{\Xi}^{(c)}_{m,\Theta,\mathfrak{a}}(u^{(c)}_{m,\Theta,\mathfrak{a},i})$ of $\mathfrak{Four}(V)^\vee_{|\mathcal{S}}$. We set $\overline{\boldsymbol{U}}^{(c)}_{m,\Theta,\mathfrak{a}}=\big(\overline{U}^{(c)}_{m,\Theta,\mathfrak{a},j}\big)$.

Theorem 5.6.

- *The tuple of flat sections $\overline{\boldsymbol{U}}_\Theta:=\bigcup_{\mathfrak{a},c,m}\overline{\boldsymbol{U}}^{(c)}_{m,\Theta,\mathfrak{a}}$ is a flat frame of $\mathfrak{Four}(V)^\vee_{|\mathcal{S}}$.*
- *Let $\mathfrak{Four}(V)^\vee_{c,\mathfrak{a},m,\mathcal{S}}$ denote the flat subbundle of $\mathfrak{Four}(V)^\vee_{|\mathcal{S}}$ generated by $\overline{\boldsymbol{U}}^{(c)}_{m,\Theta,\mathfrak{a}}$. Then, the decomposition*

$$\mathfrak{Four}(V)^\vee_{|\mathcal{S}}=\bigoplus_{\mathfrak{a},m,c}\mathfrak{Four}(V)^\vee_{c,\mathfrak{a},m,\mathcal{S}}$$

is a flat splitting of the full Stokes filtration of $\mathfrak{Four}(V)^\vee_{|\mathcal{S}}$. In particular, $\overline{\boldsymbol{U}}_\Theta$ is compatible with the full Stokes filtration.

Proof. The second claim follows from the first claim, Corollary 5.3, Corollary 5.5, and the comparison of the ranks given by the explicit stationary phase formula reviewed in Subsection 2.2.3.

Let us show the first claim. For $\mathfrak{a} \in \overline{\mathrm{Irr}}(\nabla_c)$, let $(\mathcal{V}^{(c)}_{\mathfrak{a}}, \mathcal{W}^{(c)}_{\mathfrak{a}})$ be the BDE-good lattice pair for $\mathrm{Gr}_{\mathfrak{a}}(V_c) := \varphi_{2,\mathfrak{a}*}\big(L(\mathfrak{a}) \otimes R^{(c)}_{\mathfrak{a}}\big)$ on $\mathbb{C}_t$. We take a tuple of algebraic sections ${}^0\boldsymbol{\eta}^{(c)}_{\mathfrak{a}} := \big({}^0\eta^{(c)}_{\mathfrak{a},j}\big)$ of $\mathcal{W}^{(c)}_{\mathfrak{a}}\, dt/t$, which induces a base of the $\mathbb{C}$-vector space $\mathrm{Cok}\big(\mathcal{V}^{(c)}_{\mathfrak{a}} \xrightarrow{dt} \mathcal{W}^{(c)}_{\mathfrak{a}} dt/t\big)$. It induces a frame ${}^0\boldsymbol{\Upsilon}^{(c)}_{\mathfrak{a}} = \big({}^0\Upsilon^{(c)}_{\mathfrak{a},j}\big)$ of the lattice of $\mathfrak{Four}\big(\mathrm{Gr}_{\mathfrak{a}}(V_c)\big)$ induced by $(\mathcal{V}^{(c)}_{\mathfrak{a}}, \mathcal{W}^{(c)}_{\mathfrak{a}})$ around ∞_τ. (See Subsection 2.2.5.)

Let $(\mathcal{V}, \mathcal{W})$ be the BDE-good lattice pair for V on $\mathbb{C}_t$. For each $c \in \mathrm{Sing}(V) \setminus \{\infty\}$, we take a tuple of sections $\boldsymbol{\eta}^{(c)}_{\mathfrak{a}} := \big(\eta^{(c)}_{\mathfrak{a},j}\big)$ of $\mathcal{W} \otimes \Omega^1_{\mathbb{C}_t}(\mathrm{Sing}(V))$, such that

$$\phi_c^*\big({}^0\eta^{(c)}_{\mathfrak{a},j}\big)_{|\widehat{c}} - \eta^{(c)}_{\mathfrak{a},j|\widehat{c}} \equiv 0 \quad \text{modulo } (t-c)^N \mathcal{W} \otimes \Omega^1\big(\mathrm{Sing}(V)\big)_{|\widehat{c}}$$

$$\eta^{(c)}_{\mathfrak{a},j|\widehat{c'}} \equiv 0 \quad \text{modulo } (t-c')^N \mathcal{W} \otimes \Omega^1\big(\mathrm{Sing}(V)\big)_{|\widehat{c'}} \quad (c' \neq c)$$

for a sufficiently large N. Let $\Upsilon^{(c)}_{\mathfrak{a},j}$ denote the section of $\mathfrak{Four}(V)$ induced by $\eta^{(c)}_{\mathfrak{a},j}$. The tuple $\big(\Upsilon^{(c)}_{\mathfrak{a},j}\big)$ is denoted by $\boldsymbol{\Upsilon}^{(c)}_{\mathfrak{a}}$.

Let $(\overline{\mathcal{V}}, \overline{\mathcal{W}})$ be the BDE-good lattice pair for V on $\mathbb{P}^1_t$. We have $\overline{\mathcal{W}}_{|\mathbb{C}_t} = \mathcal{W}$. Let N_1 be a sufficiently large number such that ${}^0\eta^{(c)}_{\mathfrak{a},j}$ are sections of $\overline{\mathcal{W}}(N_1\infty) \otimes \Omega^1_{\mathbb{P}^1_t}\big(\mathrm{Sing}(V)\big)$.

Let $x = t^{-1}$ be the coordinate around $\infty \in \mathbb{P}^1_t$. For $\mathfrak{a} \in \overline{\mathrm{Irr}}(\nabla_\infty)$, let $(\mathcal{V}^{(\infty)}_{\mathfrak{a}}, \mathcal{W}^{(\infty)}_{\mathfrak{a}})$ be the BDE-good lattice pair for $\mathrm{Gr}_{\mathfrak{a}}(V_\infty) := \varphi^{(\infty)}_{2,\mathfrak{a}*}\big(L(\mathfrak{a}) \otimes R^{(\infty)}_{\mathfrak{a}}\big)$ on $\mathbb{C}_x$. Let $M_1 > 0$ be a large number. We take a tuple of sections ${}^0\boldsymbol{\eta}^{(\infty)}_{\mathfrak{a}} := \big({}^0\eta^{(\infty)}_{\mathfrak{a},j}\big)$ of $x^{-M_1}\mathcal{W}_{\mathfrak{a},\infty}\, dx/x$, which induces a base of the $\mathbb{C}$-vector space $\mathrm{Cok}\big(x^{-M_1}\mathcal{V}^{(\infty)}_{\mathfrak{a}} \xrightarrow{x^{-2}dx} x^{-M_1}\mathcal{W}^{(\infty)}_{\mathfrak{a}}\, dx/x\big)$. It induces a tuple of sections ${}^0\boldsymbol{\Upsilon}^{(\infty)}_{\mathfrak{a}} = \big({}^0\Upsilon^{(\infty)}_{\mathfrak{a},j}\big)$ of the lattice of $\mathfrak{Four}\big(\mathrm{Gr}_{\mathfrak{a}}(V_\infty)\big)$ induced by $(x^{-M_1}\mathcal{V}^{(\infty)}_{\mathfrak{a}}, x^{-M_1}\mathcal{W}^{(\infty)}_{\mathfrak{a}})$ around ∞_τ.

If M_1 is sufficiently large, we can take a tuple of sections $\boldsymbol{\eta}^{(\infty)}_{\mathfrak{a}} = \big(\eta^{(\infty)}_{\mathfrak{a},j}\big)$ of $\overline{\mathcal{W}}(M_1\infty) \otimes \Omega^1_{\mathbb{P}^1}(\mathrm{Sing}(V))$ such that

$$\eta^{(\infty)}_{\mathfrak{a},j|\widehat{\infty}} - {}^0\eta^{(\infty)}_{\mathfrak{a},j|\widehat{\infty}} \equiv 0 \quad \text{modulo } \overline{\mathcal{W}}(N_1\infty) \otimes \Omega^1_{\mathbb{P}^1}\big(\mathrm{Sing}(V)\big)_{|\widehat{\infty}}$$

$$\eta^{(\infty)}_{\mathfrak{a},j|\widehat{c}} \equiv 0 \quad \text{modulo } (t-c)^N \overline{\mathcal{W}} \otimes \Omega^1_{\mathbb{P}^1}(\mathrm{Sing}(V))_{|\widehat{c}} \qquad (c \in \mathrm{Sing}(V) \setminus \{\infty\})$$

Let $\Upsilon^{(\infty)}_{\mathfrak{a},j}$ denote the section of $\mathfrak{Four}(V)$ around ∞_τ induced by $\eta^{(\infty)}_{\mathfrak{a},j}$. The tuple $\big(\Upsilon^{(\infty)}_{\mathfrak{a},j}\big)$ is denoted by $\boldsymbol{\Upsilon}^{(\infty)}_{\mathfrak{a}}$.

For simplicity, we assume that $\boldsymbol{u}^{(c)}_{m,\Theta,\mathfrak{a}}$ and ${}^0\boldsymbol{\eta}^{(c)}_{\mathfrak{a}}$ $(c \in \mathrm{Sing}(V))$ are compatible with the generalized eigen decomposition of the monodromy.

For a fixed $(c,\mathfrak{a},j)$, let us consider the following vector valued function on $\mathcal{S}$:

$$V(c',m',\mathfrak{a}') := \Bigl(\exp\bigl(-c'/z-\widehat{\mathfrak{a}}'_{m',\Theta}\bigr)\,\langle U^{(c')}_{m',\mathfrak{a}',\Theta,i},\ \Upsilon^{(c)}_{\mathfrak{a},j}\rangle\,\Bigm|\,i=1,\ldots,\operatorname{rank}R^{(c')}_{\mathfrak{a}'}\Bigr)$$

According to Proposition 3.6, Lemma 3.8, Proposition 4.5, and the estimates in Subsection 5.2, we have the following for some large L:

$$\bigl|V(c',m',\mathfrak{a}')\bigr| \leq \bigl|V(c,m,\mathfrak{a})\bigr|\,|\zeta|^{L}\quad (c',\mathfrak{a}')\neq(c,\mathfrak{a})$$

Let E denote the lattice of $\mathfrak{Four}(V)$ at ∞_τ induced by $\bigl(\overline{\mathcal{V}}(M_1\infty),\overline{\mathcal{W}}(M_1\infty)\bigr)$. Under the isomorphism of local Fourier transform, we have $\boldsymbol{\Upsilon}^{(c)}_{\mathfrak{a}}\equiv{}^{0}\boldsymbol{\Upsilon}^{(c)}_{\mathfrak{a}}$ modulo $z^{L}E$ for some large L. Hence, the tuple $\boldsymbol{\Upsilon}=\bigcup_{c\in\operatorname{Sing}(V)}\bigcup_{\mathfrak{a}}\boldsymbol{\Upsilon}^{(c)}_{\mathfrak{a}}$ gives a frame of E around ∞_τ. We take orderings of the tuples $\overline{\boldsymbol{U}}_{\Theta}$ and $\boldsymbol{\Upsilon}$. Let $\det\langle\overline{\boldsymbol{U}}_{\Theta},\boldsymbol{\Upsilon}\rangle$ be the determinant of the matrix valued function whose (i,j)-entries are the pairings between the i-th member of $\overline{\boldsymbol{U}}_{\Theta}$ and the j-th member of $\boldsymbol{\Upsilon}$ with respect to the above orderings. It is well defined up to signatures. Similarly, we obtain $\det\langle\boldsymbol{U}^{(c)}_{\Theta,\mathfrak{a}},\,{}^{0}\boldsymbol{\Upsilon}^{(c)}_{\mathfrak{a}}\rangle$. Recall the following elementary lemma.

Lemma 5.7. *Let A be an ℓ-square matrix valued function on $\mathcal{S}$ divided into blocks $A_{i,j}$, where $A_{i,j}$ are $(\ell_i\times\ell_j)$-matrix valued functions $(\ell=\sum\ell_i)$. Assume that the entries of $A_{i,i}$ are bounded up to log order, and the entries of $A_{i,j}$ $(i\neq j)$ are $O(|z|^{\delta})$ for some $\delta>0$. Then,* $\det A=\prod\det A_{i,i}+O(|z|^{\delta/2})$. □

Then, we obtain the following estimate for some $\delta>0$, by Proposition 3.6, Lemma 3.8, and Proposition 4.5 and the estimates in Subsection 5.2:

$$\det\langle\overline{\boldsymbol{U}}_{\Theta},\,\boldsymbol{\Upsilon}\rangle=\pm\prod_{c\in\operatorname{Sing}(V)}\prod_{\mathfrak{a}}\Bigl[\exp(c/z)^{\operatorname{rank}V}\det\langle\boldsymbol{U}^{(c)}_{\Theta,\mathfrak{a}},\,{}^{0}\boldsymbol{\Upsilon}^{(c)}_{\mathfrak{a}}\rangle\Bigr]\,\Bigl(1+O\bigl(|z|^{\delta}\bigr)\Bigr)$$

In particular, it is non-zero. Hence, we obtain that $\overline{\boldsymbol{U}}_{\Theta}$ is a frame. □

5.3.2. *The induced map on the associated graded bundles.* For $\mathfrak{a}\in u^{-1}\mathbb{C}[u^{-1}]$, let $\widehat{\mathfrak{a}}$ (resp. $\widehat{\mathfrak{a}}^{(\infty)}$) denote an irregular value of $\psi^{*}\mathfrak{Four}\bigl(\varphi_{2\mathfrak{a}*}L(\mathfrak{a})\bigr)$ (resp. $\psi^{*}\mathfrak{Four}(\varphi^{(\infty)}_{2\mathfrak{a}*}L(\mathfrak{a}))$). We use the symbols to denote the induced elements in the quotient of $\mathbb{C}(\!(\xi)\!)/\mathbb{C}(\!(z)\!)$ by the Galois group $\operatorname{Gal}(\psi)$ of ψ. Note the transitivity of the $\operatorname{Gal}(\psi)$-action on the sets of the irregular values of $\psi^{*}\mathfrak{Four}\bigl(\varphi_{2\mathfrak{a}*}L(\mathfrak{a})\bigr)$ and $\psi^{*}\mathfrak{Four}\bigl(\varphi^{(\infty)}_{2\mathfrak{a}*}L(\mathfrak{a})\bigr)$. Let us observe that the following isomorphisms are induced for $\mathfrak{a}\in\overline{\operatorname{Irr}}(\nabla_c)$ or $\mathfrak{a}\in\overline{\operatorname{Irr}}(\nabla_\infty)$, from the above description of the Stokes structure:

$$\Phi^{(c)}_{\mathfrak{a}}:\operatorname{Gr}_{c/z+\widehat{\mathfrak{a}}}\Bigl(\mathfrak{Four}\bigl(\phi_c^{*}\operatorname{Gr}_{\mathfrak{a}}(V_c)\bigr)\Bigr)\longrightarrow\operatorname{Gr}_{c/z+\widehat{\mathfrak{a}}}\bigl(\mathfrak{Four}(V)\bigr)\tag{5.5}$$

$$\Phi^{(\infty)}_{\mathfrak{a}}:\operatorname{Gr}_{\widehat{\mathfrak{a}}^{(\infty)}}\Bigl(\mathfrak{Four}\bigl(\operatorname{Gr}_{\mathfrak{a}}(V_\infty)\bigr)\Bigr)\longrightarrow\operatorname{Gr}_{\widehat{\mathfrak{a}}^{(\infty)}}\bigl(\mathfrak{Four}(V)\bigr)\tag{5.6}$$

Here, the meaning of Gr is as in the last paragraph of Subsection 2.1.2.

By taking the dual of the correspondence $\overline{\Xi}^{(c)}_{m,\Theta,\mathfrak{a}}(s) \longleftrightarrow \Xi^{(c)}_{m,\Theta,\mathfrak{a}}(s)$, we obtain the map

$$\Phi^{(c)}_{\mathfrak{a},\mathcal{S}} : \bigoplus_m \mathrm{Gr}^{\widetilde{\mathcal{F}}^{\mathcal{S}}}_{c/z+\widehat{\mathfrak{a}}_{m,\Theta}}\Big(\psi^*\mathfrak{Four}\big(\phi_c^*\,\mathrm{Gr}_{\mathfrak{a}}(V_c)\big)_{|\mathcal{S}}\Big) \longrightarrow \bigoplus_m \mathrm{Gr}^{\widetilde{\mathcal{F}}^{\mathcal{S}}}_{c/z+\widehat{\mathfrak{a}}_{m,\Theta}}\big(\psi^*\mathfrak{Four}(V)_{|\mathcal{S}}\big).$$

Lemma 5.8. $\Phi^{(c)}_{\mathfrak{a},\mathcal{S}}$ *is well defined.*

Proof. Let $\overline{\Xi}^{(c)\prime}_{m,\Theta,\mathfrak{a}}(s)$ be obtained by other choices of splittings of Stokes filtrations in Subsections 5.1.1–5.1.2 and other choice of paths in Subsection 5.1.3 connecting $P(\zeta)$ and x_Θ. We can easily obtain the following estimate for any meromorphic section f of $\mathfrak{Four}(V)$ around ∞_τ:

$$\big\langle \overline{\Xi}^{(c)}_{m,\Theta,\mathfrak{a}}(s) - \overline{\Xi}^{(c)\prime}_{m,\Theta,\mathfrak{a}}(s),\ f\big\rangle = O\Big(\exp(c/z+\widehat{\mathfrak{a}}_{m,\Theta}-\epsilon|z|^{-\delta})\Big), \quad \epsilon,\delta>0$$

Then, the claim of the lemma follows from the characterization of the full Stokes filtration (Subsection 2.1.2). □

The paths for integration of $\Delta^{(c)}_{m,\Theta,\mathfrak{a}}(s)$ are changed for variation of Θ as in Subsections 3.2.2 and 4.4. By using $\Phi^{(c)}_{\mathfrak{a},\mathcal{S}}$ with an argument used in the proof of Lemma 5.8, we obtain a flat morphism for any $\mathfrak{a}\in\mathrm{Irr}(V_c)$:

$$\Phi^{(c)}_{\mathfrak{a}} : \mathrm{Gr}_{c/z+\widehat{\mathfrak{a}}}\Big(\mathfrak{Four}\big(\phi_c^*\,\mathrm{Gr}_{\mathfrak{a}}(V_c)\big)_{|\Delta_z^*}\Big) \longrightarrow \mathrm{Gr}_{c/z+\widehat{\mathfrak{a}}}\big(\mathfrak{Four}(V)_{|\Delta_z^*}\big)$$

It is naturally extended to the morphism (5.5) on Δ_z. Similarly, we obtain (5.6) for $\mathfrak{a}\in\mathrm{Irr}(V_\infty,\nabla_\infty)$.

We have the isomorphism $\mathrm{Gr}_{\mathfrak{a}}\big(\phi_{-c}^*(V)_{|\widehat{O}}\big)\simeq\mathrm{Gr}_{\mathfrak{a}}(V_{c|\widehat{O}})$, where the meaning of $\mathrm{Gr}_{\mathfrak{a}}$ is as in (2.2). It induces the isomorphism through the local Fourier transform:

$$\widehat{\Phi}^{(c)}_{\mathfrak{a}} : \mathrm{Gr}_{c/z+\widehat{\mathfrak{a}}}\Big(\mathfrak{Four}\big(\phi_c^*\,\mathrm{Gr}_{\mathfrak{a}}(V_c)\big)\Big)_{|\widehat{z=0}} \longrightarrow \mathrm{Gr}_{c/z+\widehat{\mathfrak{a}}}\big(\mathfrak{Four}(V)\big)_{|\widehat{z=0}} \tag{5.7}$$

Similarly, we have

$$\widehat{\Phi}^{(\infty)}_{\mathfrak{a}} : \mathrm{Gr}_{\widehat{\mathfrak{a}}^{(\infty)}}\Big(\mathfrak{Four}\big(\mathrm{Gr}_{\mathfrak{a}}(V_\infty)\big)\Big)_{|\widehat{z=0}} \longrightarrow \mathrm{Gr}_{\widehat{\mathfrak{a}}^{(\infty)}}\big(\mathfrak{Four}(V)\big)_{|\widehat{z=0}} \tag{5.8}$$

Proposition 5.9. *We have* $\big(\Phi^{(c)}_{\mathfrak{a}}\big)_{|\widehat{z=0}}=\widehat{\Phi}^{(c)}_{\mathfrak{a}}$ *for any* $c\in\mathrm{Sing}(V)$.

Proof. Let us argue the case of $\Phi^{(0)}_{\mathfrak{a}}$. The other cases can be argued in similar ways. We give an argument to use the induced lattice of the Fourier transform. We have the lattices ${}^0E^{(0)}_{\widehat{\mathfrak{a}}}\subset\mathrm{Gr}_{\widehat{\mathfrak{a}}}\Big(\mathfrak{Four}\big(\mathrm{Gr}_{\mathfrak{a}}(V_0)\big)\Big)_{|\widehat{z=0}}$ and $E^{(0)}_{\widehat{\mathfrak{a}}}\subset\mathrm{Gr}_{\widehat{\mathfrak{a}}}\big(\mathfrak{Four}(V)\big)_{|\widehat{z=0}}$ induced by the BDE-good lattice pairs for $\bigoplus_{\mathfrak{c}\in\overline{\mathrm{Irr}}(\nabla_0)}\mathrm{Gr}_{\mathfrak{c}}(V_0)_{|\widehat{0}}$ and $V_{|\widehat{0}}$. By Lemma 2.6, $\Phi^{(0)}_{\mathfrak{a}|\widehat{z=0}}$ and $\widehat{\Phi}^{(0)}_{\mathfrak{a}}$ preserve the lattices ${}^0E^{(0)}_{\mathfrak{a}}$ and $E^{(0)}_{\mathfrak{a}}$.

Let ${}^0\boldsymbol{\Upsilon}^{(0)}_{\mathfrak{a}}$ and $\boldsymbol{\Upsilon}_{\mathfrak{a}}$ be as in the proof of Theorem 5.6. Note ${}^0\boldsymbol{\Upsilon}^{(0)}_{\mathfrak{a}|z=0}=\boldsymbol{\Upsilon}^{(0)}_{\mathfrak{a}|z=0}$ under the identification ${}^0E^{(0)}_{\widehat{\mathfrak{a}}|z=0}=E^{(0)}_{\widehat{\mathfrak{a}}|z=0}$ induced by (5.7). By the construction of $\Phi^{(0)}_{\mathfrak{a}}$ and Proposition 5.2, $\Phi^{(0)}_{\mathfrak{a}}$ and $\widehat{\Phi}^{(0)}_{\mathfrak{a}}$ induce the same linear map ${}^{0}E^{(0)}_{\widehat{\mathfrak{a}}|z=0} \longrightarrow E^{(0)}_{\widehat{\mathfrak{a}}|z=0}$. Then, the claim of the proposition follows from Lemma 2.3 and Lemma 2.6. □

Thus, the study of $\mathrm{Gr}_{c/z+\widehat{\mathfrak{a}}}(\mathfrak{Four}(V))$ and $\mathrm{Gr}_{\widehat{\mathfrak{a}}(\infty)}(\mathfrak{Four}(V))$ are reduced to the case that V is elementary.

5.3.3. *Comparison of the induced* (K,k)*-structures.* Let K and k be subfields of $\mathbb{C}$. We assume that they are algebraically closed, for simplicity. Let (V,∇) be a meromorphic flat bundle on $\mathbb{P}^1$ such that (i) it is defined over k, (ii) the associated local system $\mathrm{Loc}(V,\nabla)$ on $\mathbb{P}^1\setminus\mathrm{Sing}(V)$ has a K-structure, compatible with the Stokes structure. Then, $\mathfrak{Four}(V)$ is also defined over k, which induces the k-structure of $\mathfrak{Four}(V)_{|\widehat{\infty}}$. It is also obtained through the local Fourier transform. We consider a K-structure of the flat bundle $\mathcal{O}_{\mathbb{C}_t\times\mathbb{C}_\tau}\,e$ with $\nabla e = e\,d(t\tau)$ given by $\exp(-t\tau)\longleftrightarrow 1$. Then, the associated local system $\mathrm{Loc}\big(\mathfrak{Four}(V)\big)$ has the induced K-structure. The construction of flat sections in Subsection 5.1 can be done in a way compatible with the K-structure. Hence, the induced K-structure of $\mathfrak{Four}(V)$ is compatible with the Stokes structure, according to the description in Theorem 5.6.

Corollary 5.10. *The isomorphisms* (5.5) *and* (5.6) *preserve the* K*-structures of the associated local systems and the* k*-structures of the completion at* $z=0$*.*

Proof. Let us consider the case $c=0$. The other cases can be shown similarly. By the description of the Stokes structure, the isomorphism $(\Phi^{(0)}_{\mathfrak{a}})_{|\Delta^*_z}$ preserves the induced K-structures. Hence, (5.7) preserves the K-structure. Since Proposition 5.9 implies that $(\Phi^{(0)}_{\mathfrak{a}})_{|\widehat{z=0}}$ preserves the k-structure, (5.7) preserves the k-structure. □

We obtain the (K,k)-structure of $\psi\big(\mathrm{Gr}\,\mathfrak{Four}(V)\big)$ with respect to the coordinate z as in Subsection 2.6. For any $c\in\mathrm{Sing}(V)\setminus\{\infty\}$ and $\mathfrak{a}\in\overline{\mathrm{Irr}}(\nabla_c)$, we have the isomorphism

$$\psi\big(\mathrm{Gr}_{c/z+\widehat{\mathfrak{a}}}\,\mathfrak{Four}\big(\phi_c^*\,\mathrm{Gr}_{\mathfrak{a}}(V_c)\big)\big)\simeq\psi\big(\mathrm{Gr}_{c/z+\widehat{\mathfrak{a}}}\,\mathfrak{Four}(V)\big) \tag{5.9}$$

induced by $\Phi^{(c)}_{\mathfrak{a}}$. Similarly, for any $\mathfrak{a}\in\overline{\mathrm{Irr}}(\nabla_\infty)$, we obtain

$$\psi\big(\mathrm{Gr}_{\widehat{\mathfrak{a}}(\infty)}\,\mathfrak{Four}\big(\mathrm{Gr}_{\mathfrak{a}}(V_\infty)\big)\big)\simeq\psi\big(\mathrm{Gr}_{\widehat{\mathfrak{a}}(\infty)}\,\mathfrak{Four}(V)\big). \tag{5.10}$$

Corollary 5.11. *The isomorphisms* (5.9) *and* (5.10) *preserve the* (K,k)*-structures.* □

By taking the determinant, we obtain the isomorphism

$$\det\big(\mathfrak{Four}(V)\big)\simeq \bigotimes_{\mathfrak{a}\in\overline{\mathrm{Irr}}(\nabla_\infty)}\det\Big(\mathfrak{Four}\big(\mathrm{Gr}_{\mathfrak{a}}(V_\infty)\big)\Big)\otimes\bigotimes_{\substack{c\in\mathrm{Sing}(V)\setminus\{\infty\}\\ \mathfrak{a}\in\overline{\mathrm{Irr}}(\nabla_c)}}\det\Big(\mathfrak{Four}\big(\phi_c^*\,\mathrm{Gr}_{\mathfrak{a}}(V_c)\big)\Big) \tag{5.11}$$

It induces an isomorphism between the one-dimensional vector spaces of their multi-valued flat sections. According to Corollary 5.11, it preserves their (K, k)-structures.

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