ACTA MATHEMATICA VIETNAMICA
Volume 35, Number 1, 2010, pp. 79–89

THE MODULI SPACE OF $\mathbb{Q}$-HOMOLOGY PROJECTIVE PLANES WITH 5 QUOTIENT SINGULAR POINTS

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ABSTRACT. We describe the moduli space of $\mathbb{Q}$-homology projective planes with 5 quotient singular points, the maximum possible case. In particular, we show that the moduli space has dimension 0.

We also present an Enriques surface having two different elliptic fibrations with a multi-section giving the same configuration of 9 smooth rational curves of Dynkin type $3A_1 \oplus 2A_3$.

1. INTRODUCTION

Throughout this paper, we work over the field $\mathbb{C}$ of complex numbers.

A normal projective complex surface is called a $\mathbb{Q}$-homology projective plane if it has the same $\mathbb{Q}$-homology groups as the complex projective plane $\mathbf{P}^2$, i.e., if it has Betti numbers $b_0 = b_2 = b_4 = 1$, $b_1 = b_3 = 0$. When a normal projective complex surface has only quotient singularities, it is a $\mathbb{Q}$-homology projective plane if and only if its second Betti number b_2 is equal to 1 ([13], p. 2). If a $\mathbb{Q}$-homology projective plane is smooth, then it is isomorphic to either $\mathbb{CP}^2$ or a fake projective plane, a smooth surface of general type with $p_g = q = 0$, $K^2 = 9$ (cf. [9], [10]). All possible fundamental groups of fake projective planes have been recently classified by Prasad and Yeung [21], but little has been known about geometric construction of them.

$\mathbb{Q}$-homology projective planes with nodes(conical double points) only were classified by Dolgachev, Mendes-Lopes and Pardini ([4], Theorem 3.3, Proposition 4.1, see also [11], Corollary 1.2). A complete list consists of a cone over a conic curve, $\mathbf{P}^2$, and fake projective planes.

It follows from the orbifold Bogomolov-Miyaoka-Yau inequality([22], [16], [15], [12]) that a $\mathbb{Q}$-homology projective plane with quotient singularities has at most 5 singular points ([6], Corollary 3.4). In a previous paper joint with DongSeon Hwang [6], we completely settled the classification problem of $\mathbb{Q}$-homology projective planes with 5 quotient singularities. Namely, we proved that the case

Received July 30, 2009; in revised form December 20, 2009.

2000 *Mathematics Subject Classification.* Primary 14J17, 14J28.

Key words and phrases. $\mathbb{Q}$-homology projective plane, quotient singularity, Enriques surface, Period domain.

Research supported by Basic Science Research Program through the National Research Foundation(NRF) of Korea funded by the Ministry of Education, Science and Technology (KRF-2007-C00002).

with 5 quotient singular points does occur, and must come from Enriques surfaces with a configuration of 9 smooth rational curves whose Dynkin diagram is of type $3A_1 \oplus 2A_3$. (By contracting the 9 curves we obtain a $\mathbb{Q}$-homology projective plane with 5 rational double points of type A_1, A_1, A_1, A_3, A_3, respectively.)

Theorem 1.1. [6] *Let S be a $\mathbb{Q}$-homology projective plane with quotient singularities. Assume that the canonical class K_S is nef. Then S has at most 4 singular points except the following case:*

S has 5 rational double points of type A_1, A_1, A_1, A_3, A_3, and its minimal resolution S' is an Enriques surface.

The case where $-K_S$ is ample was settled by G. B. Belousov [2]. He proved that Del Pezzo surfaces of Picard number 1 with quotient singularities have at most 4 singular points. Thus, Theorem 1.1 holds true without the nefness of K_S.

In this paper, we describe the moduli space of such Enriques surfaces, by using the description of the period domain for Enriques surfaces due to Horikawa [5]. In particular we prove that the moduli space has dimension 0 (Theorem 4.1). The moduli space seems to consist of a single point, but we do not know how to prove it.

We also present an Enriques surface having two different elliptic fibrations (one of them is given in [6]) with a multi-section which give the same configuration of 9 smooth rational curves of Dynkin type $3A_1 \oplus 2A_3$ (Example 4.3).

The problem of determining the maximum number of singular points on $\mathbb{Q}$-homology projective planes with quotient singularities is related to the algebraic Montgomery-Yang problem ([17], [13]). See [7] and [8] for recent progress on this problem.

We remark that if a $\mathbb{Q}$-homology projective plane S is allowed to have rational singularities, then there is no bound for the number of singular points. In fact, there are examples of $\mathbb{Q}$-homology projective planes with an arbitrary number of rational singularities (see e.g., [6], Introduction).

Notation

K_Y the canonical class of Y

$b_i(Y)$ the i-th Betti number of Y

$q(X) := \dim H^1(X, \mathcal{O}_X)$ the irregularity of a smooth surface X

$p_g(X) := \dim H^2(X, \mathcal{O}_X)$ the geometric genus of a smooth surface X

$|G|$ the order of a finite group G

A $(-m)$-curve is a smooth rational curve on a surface with self-intersection $-m$.

A (-2)-curve is called a nodal curve.

2. Some basics from lattice theory

By a *lattice* L we mean a finitely generated free $\mathbb{Z}$-module L equipped with a symmetric bilinear form $\langle \cdot , \cdot \rangle$ taking values in $\mathbb{Z}$. By $\det(L)$ we denote the determinant of the symmetric matrix corresponding to the bilinear form with respect to a $\mathbb{Z}$-basis of L. A lattice with $|\det(L)| = 1$ is called *unimodular*. For a lattice L, we denote by $L(m)$ the lattice obtained from L by multiplying its bilinear form by m. We extend the bilinear form on L to the one on $L \otimes \mathbb{Q}$ taking values in $\mathbb{Q}$. Define

$$L^* := Hom(L, \mathbb{Z}).$$

Let L be a non-degenerate lattice. Then the canonical embedding

$$L \subset L^* \subset L \otimes \mathbb{Q}$$

defines a bilinear form on L^* taking values in $\mathbb{Q}$, and the factor group

$$disc(L) := L^*/L$$

is an abelian group of order $|\det(L)|$. We denote by $l(L)$ the minimum number of generators of $disc(L)$.

An element v of an indefinite lattice is said to be *isotropic* if it has self-intersection 0, i.e., $v^2 = \langle v, v \rangle = 0$.

If a lattice M has the same rank as L and contains L, then M is called an *over-lattice* of L.

Let L be a sublattice of a lattice M. The sublattice L is said to be *primitive* if the factor group M/L is torsion free. The minimal primitive sublattice of M containing L is called the *primitive closure* of L, and is denoted by $\overline{L}$. The orthogonal complement of L in M is denoted by $L^{\perp}_{M}$, or simply by $L^{\perp}$.

An *even* lattice is a lattice whose quadratic form induced from its bilinear form takes values in $2\mathbb{Z}$.

Let L be a non-degenerate even lattice. We define

$$q_L : disc(L) \to \mathbb{Q}/2\mathbb{Z}, \quad q_L(x + L) = x^2 + 2\mathbb{Z} \ (x \in L^*).$$

We call q_L the *discriminant quadratic form* of L. A subgroup A of $disc(L)$ is said to be *isotropic* if $q_L|_A \equiv 0$. The following is well known (see e.g. [19]).

Lemma 2.1. *Let L be a non-degenerate even lattice.*

(1) *If an even lattice M is an over-lattice of L, then the factor group $A := M/L$ is an isotropic subgroup of $disc(L)$, and $disc(M) \cong A^{\perp}/A$, where $A^{\perp}$ is the orthogonal complement of A in $disc(L)$.*
(2) *Conversely, every isotropic subgroup A of $disc(L)$ defines a unique over-lattice $M \subset L^*$ with $disc(M) \cong A^{\perp}/A$.*
(3) *If L is primitive in a unimodular even lattice, then*

$$(disc(L^{\perp}), q_{L^{\perp}}) \cong (disc(L), -q_L).$$

The hyperbolic unimodular lattice of rank 2 is denoted by

$$H := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

An element r of a negative definite even lattice is called a *root* if $r^2 = \langle r, r \rangle = -2$. A negative definite even lattice is called a *root lattice* if it is generated by its roots. By A_m $(m \geq 1)$, D_n $(n \geq 4)$, E_p $(p = 6, 7, 8)$, we denote the root lattices defined by the Dynkin diagrams of the corresponding type. Note that E_8 is the negative definite even unimodular lattice of rank 8.

Consider a root lattice of rank 9

$$R := A_1 \oplus A_1 \oplus A_1 \oplus A_3 \oplus A_3 = 3A_1 \oplus 2A_3.$$

Let r_1, r_2, r_3 be the roots corresponding to the sublattice $A_1 \oplus A_1 \oplus A_1$ of R. For $i = 1, 2, 3$, we denote by M_i the orthogonal complement of r_i in R,

$$M_i := r_i^{\perp} \cong A_1 \oplus A_1 \oplus A_3 \oplus A_3 = 2A_1 \oplus 2A_3.$$

Lemma 2.2. *Let $\Lambda := H \oplus E_8$ be the even unimodular lattice of signature $(1, 9)$.*

(1) *There is an embedding of R into Λ.*

(2) *Given an embedding $R \subset \Lambda$, regard R, M_1, M_2, M_3 as sublattices of Λ. Then the following hold true.*

   (a) $\overline{R}/R \cong (\mathbb{Z}/2) \times (\mathbb{Z}/4)$.

   (b) *$M_i^{\perp}$ is generated by a primitive isotropic element v_i and r_i with*

$$\begin{pmatrix} v_i^2 & v_i r_i \\ v_i r_i & r_i^2 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 2 & -2 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 1 \\ 1 & -2 \end{pmatrix}$$

   and $\overline{M_i}/M_i \cong (\mathbb{Z}/4)$ in the first case, $\overline{M_i}/M_i \cong (\mathbb{Z}/2) \times (\mathbb{Z}/4)$ in the second.

   (c) *The second case of* (b) *occurs for some M_i.*

   (d) $\overline{R} \cong A_1 \oplus E_8$.

Proof. (1) One can give an arithmetic proof by writing an explicit embedding of $A_1 \oplus A_1 \oplus A_3 \oplus A_3$ into E_8 (see e.g. [20], p337) and an obvious embedding of A_1 into H. Also, a geometric proof is given by the existence of an Enriques surface yielding a $\mathbb{Q}$-homology projective plane with 5 singular points [6], since for an Enriques surface the Néron-Severi group modulo torsion has a lattice structure isomorphic to $H \oplus E_8$.

(2) Note that

$$disc(R) = \Big(\overset{3}{\underset{i=1}{\oplus}} (\mathbb{Z}/2)\langle e_i \rangle \Big) \oplus \Big((\mathbb{Z}/4)\langle w_1 \rangle \Big) \oplus \Big((\mathbb{Z}/4)\langle w_2 \rangle \Big),$$

where $\langle \cdot \rangle$ is the generator of the group, i.e., $e_i = \dfrac{r_i}{2}$ with $e_i^2 = -\dfrac{1}{2}$ and each w_i is a generator of $disc(A_3) \cong (\mathbb{Z}/4)$ with $w_i^2 = -\dfrac{3}{4}$.

(a) By Lemma 2.1(3),

$$l(\overline{R}) = l(R^{\perp}) = \text{rank} R^{\perp} = 1.$$

Now (a) follows from Lemma 2.1(1)(2).

(b) Since $|\det(M_i)|$ is a square number, so is $|\det(\overline{M_i})| = |\det(M_i^{\perp})|$. An integral quadratic form of signature $(1, 1)$ represents 0 if and only if the absolute value of its determinant is a square number. Thus $M_i^{\perp}$ contains an isotropic element. Let $v_i \in M_i^{\perp}$ be a primitive isotropic element. We know $r_i \in M_i^{\perp}$. Let N_i be the sublattice of $M_i^{\perp}$ generated by v_i and r_i. We omit the subscript i for simplicity. Let

$$c := vr.$$

We may assume that $c > 0$ by replacing v by $-v$ if necessary.

Assume that c is odd. We compute $disc(N)$ which is generated by two elements $v^*, r^* \in N^*$ satisfying $v^*v = r^*r = 1$ and $v^*r = r^*v = 0$. It is easy to see that

$$v^* = \frac{2v + cr}{c^2} \quad \text{and} \quad r^* = \frac{v}{c}$$

modulo N. Since v^* has order c^2, the order of the group $disc(N)$, $disc(N)$ is a cyclic group generated by v^*. (In fact, if $c = 2t + 1$, then $r^* = -2tr^* = -tcv^*$, a multiple of v^*.) Since $|disc(M)|$ is a power of 2, so does $|disc(\overline{M})| = |disc(M^\perp)|$. Considering the tower

$$N \subset M^\perp \subset (M^\perp)^* \subset N^*$$

we see that c divides the order of $M^\perp/N$. Thus $cv^* \in M^\perp$. Since $cv^* = \frac{2v}{c}$ and v is primitive, we must have $c = 1$. It follows that N is unimodular, hence $N = M^\perp$, giving the second case.

Assume that c is even, say $c = 2^t d$ for some $t \geq 1$ and some odd d. In this case $disc(N)$ is generated by two elements

$$v^* = \frac{v + 2^{t-1}dr}{2^{2t-1}d^2}, \quad r^* = \frac{v}{2^t d}$$

modulo N. Note that v^* has order $2^{2t-1}d^2$. Since $2^t dv^* = 2r^*$, the element $2^{t-1}dv^* - r^* = \frac{r}{2}$ is of order 2. Thus

$$disc(N) =< v^* > \oplus < \frac{r}{2} >=< 2^{2t-1}v^* > \oplus < d^2v^* > \oplus < \frac{r}{2} > .$$

The first factor $< 2^{2t-1}v^* >$ is a cyclic group of order d^2, so by the same reason as before

$$d2^{2t-1}v^* = \frac{v}{d} \in disc(M^\perp),$$

so $d = 1$ by the primitivity of v. Now

$$disc(N) =< v^* > \oplus < \frac{r}{2} >$$

where the factor $< v^* >$ is a cyclic group of order 2^{2t-1}. If $t \geq 2$, then $disc(N)$ contains an element of order ≥ 8. Since $disc(M^\perp)$ contains no element of order > 4, it follows from the tower above that $N \neq M^\perp$, i.e., 2 divides the order of the factor group $M^\perp/N$, so the order 2 element $2^{2t-2}v^* \in M^\perp$. Since $2^{2t-2}v^* = \frac{v}{2}$, this contradicts the primitivity of v. Hence $t \leq 1$.

If $t = 0$, then N is unimodular, hence $N = M^\perp$.

If $t = 1$, then $v^* = \frac{v+r}{2}$ and

$$disc(N) =< \frac{v+r}{2} > \oplus < \frac{r}{2} >\cong (\mathbb{Z}/2) \oplus (\mathbb{Z}/2).$$

This group has a unique isotropic element $\frac{v}{2}$. But $\frac{v}{2} \notin M^\perp$ by the primitivity of v. Hence $N = M^\perp$, giving the first case.

It is obvious that in the second case, $M^\perp$ is unimodular, hence $\overline{M_i}/M_i \cong (\mathbb{Z}/2) \times (\mathbb{Z}/4)$. In the first case, $\det(M^\perp) = -4$, hence $\overline{M_i}/M_i \cong (\mathbb{Z}/2) \times (\mathbb{Z}/2)$

or $(\mathbb{Z}/4)$. To prove the second assertion of (b) we need to show that

$$\overline{M_i}/M_i \not\cong (\mathbb{Z}/2) \times (\mathbb{Z}/2)$$

for any M_i. Suppose that $\overline{M_j}/M_j \cong (\mathbb{Z}/2)\times(\mathbb{Z}/2)$ for some M_j. We set $M := M_j$ and omit the subscript j for simplicity. Let $A := \overline{M}/M \subset disc(M)$. Let

$$disc(M) \cong (\mathbb{Z}/2) \oplus (\mathbb{Z}/2) \oplus (\mathbb{Z}/4) \oplus (\mathbb{Z}/4)$$

be generated by e_1, e_2, w_1, w_2 with $e_1^2 = e_2^2 = -\frac{1}{2}$, $w_1^2 = w_2^2 = -\frac{3}{4}$. It has a unique isotropic subgroup isomorphic to $(\mathbb{Z}/2)^2$, which is generated by $e_1+e_2+2w_1$ and $e_1+e_2+2w_2$. Thus $A =< e_1+e_2+2w_1, e_1+e_2+2w_2 >$. By Lemma 2.1(1), it is easy to see that

$$disc(\overline{M}) = A^{\perp}/A =< e_1+w_1+w_2 = e_2-w_1+w_2, e_1+e_2 = 2w_1 = 2w_2 >,$$

hence

$$disc(\overline{M}) \cong disc(H(2)),$$

the even type quadratic space of dimension 2 over $\mathbb{Z}/2$. Thus $disc(\overline{M})$ is not isomorphic to

$$-disc\begin{pmatrix} 0 & 2 \\ 2 & -2 \end{pmatrix},$$

a contradiction to Lemma 2.1(3).

(c) By (a), $\overline{R}/R \cong (\mathbb{Z}/2) \times (\mathbb{Z}/4)$. An isotropic element of order 4 of $disc(R)$ is of the form $e_i \pm v_1 \pm v_2$, hence is unique up to a choice of e_i and sign of v_j. Let us assume that

$$x := e_1 + v_1 + v_2 \in \overline{R}/R.$$

There are exactly two isotropic subgroups, isomorphic to $(\mathbb{Z}/2)\times(\mathbb{Z}/4)$, of $disc(R)$ containing x. They are $< x, y >$ and $< x, z >$ where

$$y := e_1 + e_2 + 2v_1, \quad z := e_1 + e_3 + 2v_1.$$

We may assume that $\overline{R}/R =< x, y >$. Then it follows that $\overline{M_3}/M_3 \cong (\mathbb{Z}/2) \times (\mathbb{Z}/4)$. This proves (c).

(d) For any M_i satisfying $\overline{M_i}/M_i \cong (\mathbb{Z}/2) \times (\mathbb{Z}/4)$, $\overline{M_i}$ is unimodular, hence is isomorphic to E_8. □

3. Period domain for Enriques surfaces

Let $L := L_{K3}$ be the K3 lattice, that is,

$$L = H \oplus H \oplus H \oplus E_8 \oplus E_8.$$

We define an involution $\theta : L \to L$ by

$$\theta(x_1, x_2, x_3, y, z) = (-x_1, x_3, x_2, z, y).$$

Its $\mathbb{C}$-linear extension to $L \otimes \mathbb{C}$ is also denoted by θ. The θ-invariant sublattice and θ-anti-invariant sublattice of L are

$$L^+ = \{v \in L \,|\, \theta(v) = v\}, \quad L^- = \{v \in L \,|\, \theta(v) = -v\}.$$

Then

$$L^+ \cong H(2) \oplus E_8(2), \quad L^- \cong H \oplus H(2) \oplus E_8(2).$$

The unimodular lattice $L^+(\frac{1}{2})$ is isomorphic to $\Lambda = H \oplus E_8$, the cohomology lattice of an Enriques surface. We put

$$\Omega^- = \{[\omega] \in \mathbf{P}(L^- \otimes \mathbb{C}) \mid \omega^2 = 0,\ \omega\overline{\omega} > 0\}$$
$$\Gamma = \{g \in O(L) \mid g\theta = \theta g\}, \quad \Gamma^- = \Gamma\,|_{L^-}$$
$$\mathcal{E} = \Omega^-/\Gamma^-.$$

Lemma 3.1. [5] *Let $\pi : X \to Y$ be the universal cover of an Enriques surface Y, and let $\tau : X \to X$ be the covering involution. Then there exists an isometry*

$$\phi : H^2(X, \mathbb{Z}) \to L$$

such that

$$\phi\tau^* = \theta\phi.$$

In particular ϕ induces an isomorphism

$$\phi^+ : H^2(X, \mathbb{Z})^{\tau^*} = \pi^* H^2(Y, \mathbb{Z}) = \pi^* Pic(Y) \to L^+.$$

A *marked Enriques surface* is a pair (Y, ϕ) with Y an Enriques surface and an isometry $\phi : H^2(X, \mathbb{Z}) \to L$ such that $\phi\tau^* = \theta\phi$, as in Lemma 3.1. Let ω_X be a non-zero holomorphic 2-form on X. Then $\tau^*\omega_X = -\omega_X$ (there is no holomorphic 2-form on Y). The period point $[\omega_X]$ of the marked K3 surface belongs to Ω^-. We call it the *period point* of (Y, ϕ).

The choice of ϕ, as in Lemma 3.1, is unique modulo Γ. Thus the assignment

$$Y \longmapsto [\omega_X] \in \mathcal{E}$$

is well defined and called the *period map for Enriques surfaces*. Global Torelli theorem for Enriques surfaces [5] says that this map is injective. We put

$$\Omega_0^- := \{[\omega] \in \Omega^- \mid \omega d \neq 0 \text{ for any } d \in L^- \text{ with } d^2 = -2\}$$
$$\mathcal{E}_0 := \Omega_0^-/\Gamma^-.$$

E. Horikawa [5] showed that every point of $\mathcal{E}_0$ is the period point of an Enriques surface, or equivalently, every point of Ω_0^- is the period point of a marked Enriques surface.

4. Enriques surfaces with 9 nodal curves of Dynkin type $3A_1 \oplus 2A_3$

Let Y be an Enriques surface with 9 nodal curves of Dynkin type $3A_1 \oplus 2A_3$, and let

$$C_1 \quad C_2 \quad C_3 \quad C_4 — C_5 — C_6 \quad C_7 — C_8 — C_9$$

be the dual graph of the 9 nodal curves C_i. Let $\pi : X \to Y$ be the universal cover of Y, $\tau : X \to X$ be the covering involution, and $\phi : H^2(X, \mathbb{Z}) \to L$ be an isometry such that $\phi\tau^* = \theta\phi$. Let

$$H^2(X, \mathbb{Z})^+, \quad H^2(X, \mathbb{Z})^-$$

be the τ^*-invariant, the τ^*-anti-invariant sublattices of $H^2(X, \mathbb{Z})$. These sublattices are isomorphic to L^+ and L^- respectively.

Let C_{i1} and C_{i2} $(1 \leq i \leq 9)$ be the two nodal curves on the K3 surface X lying over C_i. The 9 divisors

$$C_{i1} + C_{i2} \quad (1 \leq i \leq 9)$$

generate a sublattice of $H^2(X,\mathbb{Z})^+$ isomorphic to $3A_1(2) \oplus 2A_3(2)$, and by Lemma 2.2(2-d) its primitive closure is isomorphic to $A_1(2) \oplus E_8(2)$. Thus the 9 divisors

$$C_{i1} - C_{i2} \quad (1 \leq i \leq 9)$$

generate a sublattice of $H^2(X,\mathbb{Z})^-$ whose primitive closure is isomorphic to $A_1(2) \oplus E_8(2)$ by Lemma 2.2(2-d), and hence its orthogonal complement T in L^- is isomorphic to the rank 3 even lattice $H \oplus < 4 >$, i.e.

$$T := H \oplus < 4 > \subset L^- = H \oplus H(2) \oplus E_8(2),$$

where the rank one lattice $< 4 >$ is contained in the second factor $H(2)$ of L^- in an obvious way. The period ω_X of the K3 surface X is orthogonal to the 18 classes $[C_{ij}]$. Thus

$$\omega_X \in T \otimes \mathbb{C}.$$

We put

$$\Omega^T := \{[\omega] \in \mathbf{P}(T \otimes \mathbb{C}) \,|\, \omega^2 = 0,\ \omega\overline{\omega} > 0\} \subset \Omega^-.$$

Let

$$\Omega_0^T := \Omega^T \cap \Omega_0^-$$

and $\mathcal{M}$ be the image of Ω_0^T under the map $\Omega_0^- \to \mathcal{E}_0 = \Omega_0^-/\Gamma^-$. Then the period $[\omega_X]$ of the marked Enriques surface (Y, ϕ) belongs to $\mathcal{M}$, i.e.

$$[\omega_X] \in \mathcal{M}.$$

We note that $\mathcal{M}$ also contains the period point of an Enriques surface with 9 nodal curves of Dynkin type $A_1 \oplus E_8$.

Theorem 4.1. *Let $\mathcal{M}'$ be the moduli space of Enriques surfaces with* 9 *nodal curves of Dynkin type* $3A_1 \oplus 2A_3$. *Then the following hold true.*

(1) $\mathcal{M}' \subset \mathcal{M}$.
(2) $\mathcal{M}$ *is irreducible and* $\dim \mathcal{M} = 1$.
(3) *A general point of $\mathcal{M}$ corresponds to an Enriques surface with* 9 *nodal curves of Dynkin type* $A_1 \oplus E_8$.
(4) $\dim \mathcal{M}' = 0$.

Proof. (1) The assertion follows from the above argument.

(2) Since $\mathrm{rank} T = 3$, $\dim \Omega^T = 1$. The orthogonal complement of T in L^- is isomorphic to $A_1(2) \oplus E_8(2)$, hence contains no (-2)-vector. This means that Ω^T is not contained in the hyperplane in Ω^- orthogonal to any (-2)-vector $d \in L^-$. Hence $\dim \Omega_0^T = 1$ and the second assertion follows.
An explicit calculation shows that Ω^T consists of two components obtained by removing a real plane conic curve (topologically a circle) defined by the condition $\omega\overline{\omega} \leq 0$ from a plane conic curve (topologically a sphere) defined by the equation $\omega^2 = 0$. The two components are switched by the isometry $\theta|_{L^-} \in \Gamma^-$. This proves the first assertion.

(3) Consider the 1-dimensional family of Enriques surfaces of type I in [14]. These Enriques surfaces have only finitely many automorphisms [3] and have exactly 12 nodal curves, which contains 9 nodal curves of Dynkin type $A_1 \oplus E_8$. (These surfaces have an elliptic pencil with a singular fibre of type $II^* = \tilde{E}_8$ and a 2-section.) Thus their period points are contained in $\mathcal{M}$.

(4) By [6], Example 7.3, $\mathcal{M}$ is not empty. It is enough to show that none of the Enriques surfaces considered in (3) contains 9 nodal curves of Dynkin type $3A_1 \oplus 2A_3$. It is easy to see that the configuration of the 12 nodal curves (Figure 1.4, [14]) on such an Enriques surface contains no configuration of 9 nodal curves of Dynkin type $3A_1 \oplus 2A_3$. This proves (4) □

Corollary 4.2. *The moduli space of* $\mathbb{Q}$*-homology projective planes with* 5 *quotient singularities has dimension* 0*.*

Proof. Given an Enriques surface Y, the set $\mathcal{N}$ of all nodal curves on Y is discrete (no nodal curve can move), so is the set $\mathcal{C}$ of configurations of 9 nodal curves on Y of Dynkin type $3A_1 \oplus 2A_3$. Now the assertion follows from Theorem 4.1. □

Question: Is $\mathcal{C}/\mathrm{Aut}(Y)$ finite?
Note that $\mathcal{C} \subset \mathcal{N}^9$, so the finiteness would follow from the finiteness of $\mathcal{N}^9/\mathrm{Aut}(Y)$. We know that $\mathcal{N}/\mathrm{Aut}(Y)$ is finite by [18], but this does not necessarily imply the finiteness of $\mathcal{N}^9/\mathrm{Aut}(Y)$.

Let us discuss an explicit example of Enriques surface with 9 nodal curves of Dynkin type $3A_1 \oplus 2A_3$.

Example 4.3. Let Y_{III} be the Enriques surface of type III in [14]. It admits an elliptic pencil with 2 double fibres of type I_4, 2 fibres of type I_2, and a special 2-section intersecting only one component in each fibre. See [1] for Kodaira's notation for types of singular fibres. One can easily find 8 nodal curves among the components of singular fibres not meeting the 2-section, so can get a configuration of 9 nodal curves with Dynkin type $3A_1 \oplus 2A_3$. This was mentioned in [6], Example 7.3. For example, using the notation there, we take the elliptic pencil

$$|2(E_1 + E_2 + E_3 + E_9)| = |2(E_5 + E_6 + E_7 + E_{12})| = |E_{14} + E_{18}| = |E_{13} + E_{17}|$$

and a 2-section E_4. The 9 curves

$$E_2 — E_1 — E_9 \quad E_6 — E_7 — E_{12} \quad E_{14} \quad E_{13} \quad E_4$$

gives the configuration. If M is the sublattice of the cohomology lattice of Y_{III} generated by the first 8 curves, then $M^{\perp}$ is generated by the class of the half elliptic pencil and the 2-section E_4, hence belongs to the second case of Lemma 2.2(2-b).

There is another elliptic pencil: an elliptic pencil with a fibre of type $I_2^* = \tilde{D}_6$ and 2 double fibres of type I_2, e.g.,

$$|E_2 + E_9 + 2(E_1 + E_{10} + E_7) + E_6 + E_{12}| = |2(E_{11} + E_{13})| = |2(E_4 + E_{18}|$$

and a 4-section E_{14}. This gives the configuration

$$E_2—E_1—E_9 \quad E_6—E_7—E_{12} \quad E_{13} \quad E_4 \quad E_{14}$$

the same as above. This shows that different elliptic pencils with a multi-section may give the same configuration. In the latter case, if M is the sublattice of the cohomology lattice of Y_{III} generated by the first 8 curves, then $M^\perp$ is generated by the class of the half elliptic pencil and the 4-section E_{14}, hence belongs to the first case of Lemma 2.2(2-b).

Also, one can check that all configurations on Y_{III} of 9 nodal curves with Dynkin type $3A_1 \oplus 2A_3$ are conjugate under the automorphism group $\mathrm{Aut}(Y_{III})$.

Proposition 4.4. *Let Y be an Enriques surface with 9 nodal curves of Dynkin type $3A_1 \oplus 2A_3$. Assume that there is an elliptic pencil on Y containing in its singular fibres eight of the 9 nodal curves of Dynkin type $2A_1 \oplus 2A_3$. Then the elliptic pencil must have singular fibres of type*

$$I_4 + I_4 + I_2 + I_2 \text{ or } I_2^* + I_2 + I_2 \text{ or } I_1^* + I_4,$$

where some fibres may be double fibres.

Proof. An elliptic pencil contains in its singular fibres a configuration of 8 nodal curves of Dynkin type $2A_1 \oplus 2A_3$ if and only if its singular fibres are of type, one of the 3 above or $III^* + I_2$ or $III^* + III$.

We will rule out the latter two cases by showing that there is no multi-section meeting none of the 8 nodal curves. Suppose that there is a multi-section S meeting none of the 8 nodal curves. The seven non-central components of the singular fibre of type III^* form a configuration of Dynkin type $A_1 \oplus 2A_3$. Thus S cannot meet any component other than the central component of the singular fibre of type III^*. Denote by B the central component. We know that B has multiplicity 4. Thus, by Lemma 2.2(2-b), $SB = 1$. Hence S is a 4-section. On the K3 cover X of Y, the 4-section S splits to give two 2-sections, both meeting the central component of each of the two fibres of type III^*. A contradiction. □

Each of the first two cases from Proposition 4.4 occurs, as we have seen in Example 4.3. But we do not know if the third case actually occurs.

Question: Does the moduli space $\mathcal{M}'$ contain a point different from the point corresponding to the example given in 4.3? How many points does it have?

Acknowledgments

I thank Igor Dolgachev and Shigeyuki Kondō for helpful discussions, and the referee for helpful comments. I also thank the organizers of the conference "Complex Geometry" for inviting me to participate in this fruitful meeting and contribute this paper to the proceedings.

References

1. W. Barth, K. Hulek, Ch. Peters and A. Van de Ven, *Compact Complex Surfaces*, second ed. Springer 2004.
2. G. N. Belousov, Del-Pezzo surfaces with log-terminal singularities, *Mat. Zametki* **83** (2008), no. 2, 170-180.
3. I. Dolgachev, On automorphisms of Enriques surfaces, Invent. Math. **76** (1984), 163-177.
4. I. Dolgachev, M Mendes Lopes and R. Pardini, Rational surfaces with many nodes, *Compositio Math.* **132** (2002), 349-363.
5. E. Horikawa, On the periods of Enriques surfaces, I, II, *Math. Ann.***234** (1978), 73-88, *Math. Ann.* **235** (1978), 217-246.
6. D. Hwang and J. Keum, The maximum number of singular points on rational homology projective planes, arXiv:0801.3021, to appear in *J. Algebraic Geometry*
7. D. Hwang and J. Keum, Algebraic Montgomery-Yang Problem: the noncyclic case, arXiv:0904.2975
8. D. Hwang and J. Keum, Algebraic Montgomery-Yang Problem: the non-rational case and the log del Pezzo case, arXiv:0906.0633
9. J. Keum, A fake projective plane with an order 7 automorphism, *Topology* **45** (2006), 919-927.
10. J. Keum, Quotients of fake projective planes, *Geometry & Topology* **12** (2008), 2497-2515.
11. J. Keum, Projective surfaces with many nodes, arXiv:0908.4372, Proc. of the conference Algebraic Geometry in East Asia (KIAS, Seoul, 2008), to appear in *Advanced Studies in Pure Mathematics*, Math. Soc. Japan, Kinokuniya, Tokyo.
12. R. Kobayashi, S. Nakamura, and F. Sakai, A numerical characterization of ball quotients for normal surfaces with branch loci, *Proc. Japan Acad. Ser. A, Math. Sci.* **65** (1989), no. 7, 238-241.
13. J. Kollár, Is there a topological Bogomolov-Miyaoka-Yau inequality? *Pure Appl. Math. Q.* **4** (2008), no. 2, part 1, 203-236.
14. S. Kondō, Enriques surfaces with finite automorphism groups, *Japanese J. Math.***12** (1986), 191-282.
15. G. Megyesi, Generalisation of the Bogomolov-Miyaoka-Yau inequality to singular surfaces, *Proc. London Math. Soc.* (2) **78** (1999), 241-282.
16. Y. Miyaoka, The maximal number of quotient singularities on surfaces with given numerical invariants, *Math. Ann.* **268** (1984), 159-171.
17. D. Montgomery and C. T. Yang, Differentiable pseudo-free circle actions on homotopy seven spheres, *Proc. of the Second Conference on Compact Transformation Groups* (Univ. Massachusetts, Amherst, Mass., 1971), Part I, Springer, 1972, pp. 41-101. *Lecture Notes in Math.*, Vol. 298
18. Y. Namikawa, Periods of Enriques surfaces, *Math. Ann.* **270** (1985), 201-222.
19. V. V. Nikulin, Integral symmetric bilinear forms and its applications, *Izv. Akad. Nauk SSSR Ser. Mat.* **43** (1979), no. 1, 111-177; English translation: *Math. USSR Izv.* **14** (1979), no. 1, 103-167 (1980)
20. K. Nishiyama, The Jacobian fibrations on some K3 surfaces and their Mordell-Weil group, *Japanese J. Math.* **22** (1996), no. 2, 293-347.
21. G. Prasad, S.-K. Yeung, Fake projective planes, *Invent. Math.* **168** (2007), no. 2, 321–370.
22. F. Sakai, Semistable curves on algebraic surfaces and logarithmic pluricanonical maps, *Math. Ann.* **254** (1980), no. 2, 89-120.

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